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Topology From Analysis

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1 Introduction

Topology is often described as having no notion of distance, but a notion of nearness. How can such a thing be possible? Isn’t this just a distinction without a difference? In this project, we will discover the notion of nearness without distance by studying the work of Georg Cantor\(^1\) and a problem he was investigating involving Fourier series. We will see that it is the relationship of points to each other, and not their distances per se, that is a proper view. We will see the roots of topology organically springing from analysis.

2 Some background

In a calculus course covering sequences and series, you were introduced to a power series; that is, a function \( f(x) \) may be written as

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]

where \( a_n \) is some coefficient for each \( n \). This transforms what could be a fairly complex function into a polynomial (albeit an infinite one) which allows you to approximate the

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\(^1\) Cantor (1845 - 1918) was a German mathematician best known for his work in set theory. He also made contributions to number theory, philosophy, and as we will see in this module, topology.
function. Another reason such a form is desirable is because under reasonable hypotheses, one can integrate and differentiate the function term by term.

In the 19th century another kind of representation of a function was discovered, the so-called Fourier series. A function $f(x)$ has Fourier series of the form

$$f(x) = \frac{1}{2}b_0 + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx)$$

where $b_0, a_n, b_n$ are coefficients. This is also a very important representation of a function, one often used in physics. The German mathematician Georg Cantor (1845 - 1918), who is best known for his work in set theory, studied Fourier series in the 1870s. In an 1872 paper [1], he writes:

In the following, I will announce a certain extension of a theorem on trigonometric series representations. [It is a fact] that two trigonometric representations

$$\frac{1}{2}b_0 + \sum (a_n \sin nx + b_n \cos nx) \quad \text{and} \quad \frac{1}{2}b'_0 + \sum (a'_n \sin nx + b'_n \cos nx)$$

which converges for every value of $x$ have the same coefficients...I have also shown that the theorem holds if we give up either convergence or the representation for a finite number of values of $x$.

Exercise 2.1. Give a precise statement of what Cantor has “also shown.”

Cantor continues:

The extension under discussion here asserts that the theorem remains valid even when the assumption of the convergence of the series or the value of the limit is eliminated for an infinite number of values of $x$ in the interval $[0, 2\pi]$. 
Cantor is interested in showing that a Fourier series is unique even if we give up convergence or $f$ is not defined on an infinite number of points in $[0, 2\pi]$. But what kinds of infinite sets? We will let $P \subseteq [0, 2\pi]$ be a set of points where $f$ either gives up convergence or is not defined. Given the function $f$, Cantor uniquely constructed a new function $F$.

**Proposition 2.1.** (Cantor) Let $f$ be a function on $[0, 2\pi]$. Then there is a function $F$, based on $f$, which is continuous on $[0, 2\pi]$. Furthermore, if $F$ is linear on all of $[0, 2\pi]$, then the Fourier series for $f$ is unique.

Now showing $f$ has a unique Fourier series has been kicked back to showing this mysterious $F$ is linear. Fortunately, Cantor gives us a practical way to show $F$ is linear.

(A) *If there is an interval $(p, q)$ in which only a finite number of points of the set $P$ lie, then $F(x)$ is linear in this interval.*

This result will be key below.

**Exercise 2.2.** Give two examples of non-empty sets $P$ for which, if $f$ is not defined or gives up convergence on $P$, then the Fourier Series for $f$ is unique.

### 3 Limit points and Derived sets

How can we use (A) to show that $F$ is linear even when we give up convergence or are not defined on an infinite set?

**Exercise 3.1.** Suppose $f$ gives up convergence on $P := \{ \frac{1}{n} + 1 : n = 1, 2, 3, \ldots \} \cup \{ \frac{1}{n} + 2 : n = 1, 2, 3, \ldots \}$.

(a) Use result (A) to show that $F$ is linear on all but a finite number of points of $[0, 2\pi]$.

(b) Denote by $P'$ the set of points of $[0, 2\pi]$ for which we can’t (yet) conclude that $F$ is linear. Compute $P'$.
(c) Use the fact that $F$ is continuous on all of $[0, 2\pi]$ along with (A) to conclude that $F$ is linear on all of $[0, 2\pi]$.\footnote{You may assume that $F$ is the same linear function on each interval, a fact that Cantor himself proves in general.}

The result you showed in Exercise 3.1 is the basic idea behind Cantor’s main result. Even though $P$ was infinite, it was relatively easy to apply result (A) to prove that $F$ is linear on all of $[0, 2\pi]$. But what exactly was the property of $P$ which made it a “well-behaved” kind of infinity? Cantor abstracts away the particulars of the example and defines the essence of what it is that makes such an infinite set $P$ susceptible to this kind of argument.

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For the sake of brevity I call a finite or infinite number of points on the line a point set... If a point set is given in a finite interval, a second point set is generally given along with it, and with the latter a third, etc., which are essential to understanding the first set. In order to define these point sets, we must begin with the concept of a limit point of a set. I define a “limit point of a point set $P$” to be a point of the line situated in such a way that each neighborhood of it contains infinitely many points of $P$, and it may happen that the point itself belongs to the set. By a “neighborhood of a point” I mean any interval that has the point in its interior. It is easy to prove that a bounded point set consisting of an infinite number of points has at least one limit point.

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Exercise 3.2. Prove that a bounded subset of $\mathbb{R}$ consisting of an infinite number of points has at least one limit point.

Cantor will now define these other point sets “which are essential to understanding the first set.”
Every point of the line is now in a definite relation to a given set $P$, either being a limit point of $P$ or not, and thereby along with the point set $P$ the set of limit points of $P$ is a set which I wish to denote by $P'$ and call the *first derived point set of* $P$.

Unless the point set $P'$ contains only a finite number of points, it also has a derived set $P''$, which I call the *second derived point set of* $P$. By $v$ such transitions one obtains the concept of the $v$th derived set $P^{(v)}$ of $P$.

It may happen - and this is the case we are exclusively interested in at present - that after $v$ transitions the set $P^{(v)}$ consists of a finite number of points, and hence has no derived set; in this case we wish to call the original point set $P$ a set of type $v$, so that $P', P'', \ldots$ are of types $v - 1, v - 2 \ldots$.

Exercise 3.3. Let us practice computing the derived set. Find all derived sets of $P$ when

(a) $P = [0, 1]$
(b) $P = (0, 1]$
(c) $P = (0, 1)$
(d) $P = \{.2, .3\}$
(e) $P = \mathbb{Q} \cap [0, 1]$
(f) $P = \{\frac{1}{n} : n = 1, 2, 3 \ldots\}$
(g) $P = \{m + \frac{1}{n+1} : m, n \in \mathbb{Z}^+\}$

Exercise 3.4. If $P$ has finitely many elements, show that $P' = \emptyset$. 

4 Main result

In general, we let $P \subseteq [0, 2\pi]$ be a point-set of the $v^{th}$ kind so that $P^{(v)}$ is finite and $P^{(v+1)} = \emptyset$ and suppose that we give up convergence on $P$. We will show that $F$ is linear on $(0, 2\pi)$ by induction on $k$, where $k$ is the $k^{th}$ derived set of $P$. For $k = 0$, we have that $(p, q)$ contain a finite number of points of $P$, so by $(A)$, $F$ is linear on $(p, q)$. Cantor shows the case $k = 1$; that is, Cantor shows that

\[(A') \text{ If } (p', q') \text{ is any interval in which only a finite number of points of the set set } P' \text{ lie, then } F(x) \text{ is linear in this interval.}\]

By definition $P'$ is finite, so consider any subinterval $(p', q')$ of $(p, q)$ which contains a finite number of points $x'_0, x'_1, \ldots, x'_v \in P'$, where $x'_0 < x'_1 < \ldots < x'_v$. We quote Cantor’s argument:

\[
\text{“Each of these subintervals generally contains infinitely many points of } P \text{ so that result } (A) \text{ does not directly apply; however each interval } (s, t) \text{ that falls within } (x'_0, \ldots, x'_1) \text{ contains only a finite number of points from } P \text{ (otherwise another point of the set } P' \text{ would fall between } x'_0 \text{ and } x'_1 \text{), and the function is also linear on } (s, t) \text{ because of } (A). \text{ The endpoints } s \text{ and } t \text{ can be made arbitrarily close to the points } x'_0 \text{ and } x'_1 \text{ so that the continuous function } F(x) \text{ is also linear in } (x'_0, \ldots, x'_1).”
\]

Cantor illustrates this situation with the following picture:

\[
\begin{array}{c}
\text{---|---|---|---|---|---|---|---|---|---}
\text{o | p' | x'_0 | x'_1 | q'}
\end{array}
\]

He then notes that it follows that $F(x)$ is linear over all of $(p', q')$ and since $F$ is continuous on all of $[0, 2\pi]$, $F$ must be linear over all of $[0, 2\pi]$, and hence result $(A)$ applies and the Fourier series for $f$ is unique.
derived point set $P^{(k)}$ containing $P$, it follows as in the (A) to (A') case that $F(x)$ is a linear function on every interval $(p^{(k+1)}, q^{(k+1)})$ which contains only a finite number of points of the $(k + 1)^{th}$ derived point set $P^{(k+1)}$.

**Exercise 4.1.** Using an argument similar to Cantor’s for the $k = 1$ case, assume the inductive hypothesis and prove the inductive step.

We have thus shown that

$$(A^{(n)}) \quad “\text{If } (p^{(n)}, q^{(n)}) \text{ is any interval in which only a finite number of points of the set set } P^{(n)} \text{ lie, then } F(x) \text{ is linear in this interval.}”$$

Combining $(A^{(n)})$ with Cantor’s previous result that $F(x)$ linear on $[0, 2\pi]$, we immediately see that $f(x)$ has a unique trigonometric representation when convergence is given up on a point set $P$ of the $n^{th}$ kind.

**5 Conclusion**

In this short project, we have seen how a problem in Fourier series led Cantor to define topological concepts which controlled the kind of infinite set he was dealing with. In order to apply a result that only worked for finite sets, Cantor had to control how an infinite set behaved. The points could not be too “bunched up” or “close together.” This notion was made precise in the definition of the limit point and the derived set, two concepts which, although having their origins in Fourier series, are now fundamental in topology. This transition from distance to a more general “closeness” or “nearness” was just the beginnings of point-set topology.

**References**

Notes to the Instructor

This project is intended as a transition from more known mathematics into the ideas found in topology. In that sense, it is best utilized as a project to complete on either the first or second day of an introduction to topology course. Part of the idea is that the students can see in Fourier series a kind of math that is familiar to them, even if they have never worked with Fourier series per se. The question of uniqueness and Fourier series is again, something that they should be able to appreciate. By the end of the PSP, from this natural question arises concepts like limit points and derived sets which are purely point set notions. In this way, it is hoped that the students will appreciate where these more abstract definitions come from.

The key exercise to help draw out what is happening in this PSP is Exercise [3.1]. This exercise has the students reduce the question of the uniqueness of a Fourier series which is not defined on an infinite set or gives up convergence to a finite set. It is recommended to have the students work on this in groups, followed by a sharing on answers together as a class. The instructor can ensure that the students stay on track, and are guided to the correct answer. If this goes according to plan, the natural question is “how do we quantify or control the infiniteness of such a set? What exactly is that property of \( P \) that makes its infiniteness much more manageable than other infinite sets?” This last question is very important for students to appreciate. Even if the entire class has seen, for example, Cantor’s theory of infinite numbers, most students will still be stuck in the mindset that “infinity is infinity. If a set is infinite, than its infinite.” Part of the night in grasping the import of point-set topology (and in particular, the content contained in this PSP) is getting students to realize that not all infinite sets are created equal. Some infinite sets are nicer, and better to work with than others. Can we describe such “nice” properties of indefinite sets? Once a student is appreciating and even asking that question himself, he has made it past a large mental barrier and is ready to work with the concepts in topology.

The Latex source file is available for modification from the author upon request (nscoville@ursinus.edu).
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