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Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem

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1 The Miller's Son

In a 14th century hall at the University of Cambridge, within the hallowed confines of Gonville and Caius College, there is a peculiar stained glass window. Instead of a typical religious scene, this window features a diagram used for setting up calculations involving Green's Theorem.[1]

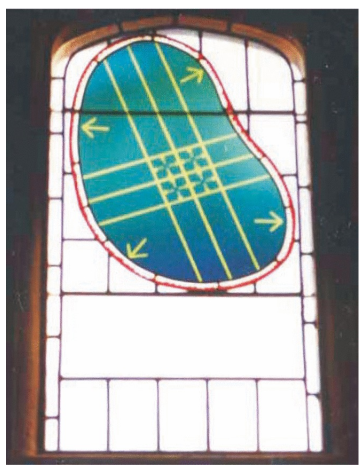

Figure 1: Green's Theorem in Stained Glass

George Green (1793–1841) was an Englishman who lived during the Industrial Revolution. Largely self-taught, his formal schooling amounted to one year around the age of eight. Starting at the age of fourteen, he was responsible for maintenance of his father's windmill in Nottinghamshire[2], a job he would perform for the next twenty years [Cannell, 1999].

Green became interested in mathematics at a young age, and throughout his adult life he kept abreast of the latest developments happening on the European continent. These were the halcyon days when Laplace, Fourier, Poisson, Cauchy, and so many others were doing groundbreaking work in mathematical physics. In 1828, at the age of 35, George Green published *An Essay on the*

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[1]Caius College has no shortage of candidates for such honors. Indeed, visitors today may also view stained glass windows dedicated to the work of Sir Ronald Fisher, John Venn, Sir James Chadwick, and Francis Crick.

[2]The windmill is still standing, and is now a science museum.

Application of Mathematical Analysis to the Theories of Electricity and Magnetism [Green, 1828]. Instead of submitting the paper to an established journal or professional society, Green had it printed himself, and sold it privately to a grand total of fifty-one people. Green feared that no one would read his work, but he was so convinced of the importance of his ideas that he could not keep them to himself. As he explained:

> Indeed, the trifle would never have appeared before the public as an independent work if I had then possessed the means of making its contents known in any other way. But as I thought it contained something new, and feared that coming from an unknown individual it might not be deemed worthy of notice of a learned society, I ventured to publish it at my own risk feeling conscious at the time that this would be attended with certain loss.
>
> Considering how desirable it was that a power of universal agency, like electricity should, as far as possible, be submitted to calculation, and reflecting on the advantages that arise in the solution of many difficult problems, from dispensing altogether with a particular examination of each of the forces which actuate the various bodies in any system, by confining the attention solely to that peculiar function on whose differentials they all depend, I was induced to try whether it would be possible to discover any general relations, existing between this function and the quantities of electricity in the bodies producing it.

Task 1

(a) Why didn't Green think that a "learned society" would take his work seriously?

(b) What do you think Green meant by a "power of universal agency"? Give another example of such a phenomenon in our world.

Green wrote that he was interested in knowing what we can discover about the forces in a system by looking at "a function on whose differentials they all depend." The basic idea is that if we are given a force $\mathbf{F}$, we may be able to find a function $f(x, y)$ or $f(x, y, z)$ whose partial derivatives describe the components of the force. We call such a function a potential function. You have likely encountered potential functions in your study of line integrals and conservative vector fields.[3] Although the concept of a potential function seems to have originated with Leonhard Euler (1707–1783), it was Green who first articulated it in connection with the conservation of energy principle [Cannell and Lord, 1993].

Task 2 Suppose we have a vector field given by $\mathbf{F} = (2x^3y^4 + x)\mathbf{i} + (2x^4y^3 + y)\mathbf{j}$. Show that this vector field is conservative, and find a potential function for the vector field.

Potential functions became incredibly important in the nineteenth century with the discoveries by Charles-Augustin de Coulomb (1736–1806) and André-Marie Ampère (1775–1836) that electrostatic forces obey an inverse-square law. Green, who must have spent his days at the windmill reading such stuff, positioned his own essay as an important step in applying calculus to the study of electricity and magnetism. In making this connection, Green was, in the words of Einstein, "twenty years ahead of his time" [Cannell and Lord, 1993].

[3] A vector field $\mathbf{F}$ is called a conservative vector field is it the gradient of some scalar function.

The hypotheses on which the received theory of magnetism is founded, are by no means so certain as the facts on which the electrical theory rests; it is however not the less necessary to have the means of submitting them to calculation, for the only way that appears oppen [sic] to us in the investigation of these subjects, which seem as it were desirous to conceal themselves from our view, is to form the most probable hypotheses we can, to deduce rigorously the consequences which flow from them, and to examine whether such consequences agree numerically with accurate experiments.

The applications of analysis to the Physical Sciences, have the double advantage of manifesting the extraordinary powers of this wonderful instrument of thought, and at the same time of serving to increase them; numberless are the instances of the truth of this assertion.

Task 3 Green suggested that, when it comes to the less-understood theory of magnetism, we should do the following: (1) Form the best hypotheses we can, (2) Deduce the consequences of those hypotheses, and (3) Check to see if the consequences agree with the results of accurate experiments. Is this process similar to, or different from, the "scientific method" you learned in school? When it comes to determining the accuracy of a mathematical theory, is it enough to see if it agrees with experimental results? Why or why not?

Finally, Green wrapped up his introduction by challenging *analysts* (that is, those who study calculus) to apply their skills to new fields:

It must certainly be regarded as a pleasing prospect to analists [sic], that at a time when astronomy, from the state of perfection to which it has attained, leaves little room for farther applications of their art, the rest of the physical sciences should show themselves daily more and more willing to submit to it.

Task 4 According to Green, astronomy had achieved "state of perfection" in the early 19th century, so it was time for mathematicians to apply calculus to other branches of science, such as electricity and magnetism. What is another branch of science where calculus is important today? How are the methods of calculus (e.g., differentiation, integration) used in that field?

Green was not the first to suggest that calculus might be applied to the study of magnetism. In 1826, Siméon-Dénis Poisson (1781–1840) published a paper in which he converted triple integrals over the volume of a magnetic body into double integrals over its surface. Poisson seems to have viewed this merely as a technique for simplifying certain formulas related to magnetism, but it caught Green's attention [Grattan-Guinness, 2005]. He realized there was something deeper going on behind this simplification. In short, it might be possible to understand properties related to the <u>interior</u> of an object simply by examining its <u>exterior</u>. In his *Essay*, Green provided the following theorem:

Before proceeding to make known some relations which exist between the density of the electric fluid at the surface of bodies, and the corresponding values of the potential function within. . .those surfaces,. . .we shall in the first place lay down a general theorem which will afterwards be very useful to us. This theorem may be thus enunciated: Let U and V be two continuous functions of rectangular co-ordinates x, y, z ... then will

$$\int dx\,dy\,dz\,U\delta V + \int d\sigma U\Big(\frac{dV}{dw}\Big) = \int dx\,dy\,dz\,V\delta U + \int d\sigma V\Big(\frac{dU}{dw}\Big).$$

The theorem above relates a surface S to the volume V which it encloses. This may be difficult to see, in part because we are not very familiar with the notation Green used. For one, he used a single integral sign regardless of what type of integral he was discussing. In addition, Green's work involved surface integrals, Laplace operators (which he represented with the δ symbol), and other concepts that lie ahead in your studies. Nevertheless, we can dig into this equation slightly to see the genesis of Green's Theorem, at least to the point of seeing that he is making a correspondence between two different types of integrals.

Task 5

(a) Based on the number of differential terms (e.g. dx) in the first integral on each side of the equation, what type of integrals must these be?

(b) The second integral on each side of the equation contains the $d\sigma$ term. This represented an element of the surface area of the body. What other kind of integral have you worked with where the differential element is a small area of a region?

(c) Notice that the expression $U\delta V$ on the left becomes $V\delta U$ on the right. Likewise, $U\left(\frac{dV}{dw}\right)$ on the left becomes $V\left(\frac{dU}{dw}\right)$ on the right. Based on the way the variables and differentials switch, what technique of integration do you think Green would have used to prove this relationship?

One of the most important ideas that Green discussed in his *Essay* is the connection between what happens within a body and the properties of that body's surface. He realized that, because the boundary of an object is one dimension lower than the interior, the connection can be represented by transforming $k-$dimensional integrals to simpler integrals involving a $(k-1)-$dimensional object. In fact, the origin of Green's Theorem goes to the very heart of Calculus: the Fundamental Theorem.

Task 6 Write out the Fundamental Theorem of Calculus. How might we see this as reducing a $k-$dimensional integral to a $(k-1)$ dimensional integral?

Green's *Essay* represented an important contribution to the mathematical analysis of electricity and magnetism. He developed an approach to studying these phenomena that was based on fundamental mathematical structures rather than on any physical model. Green did not have, nor did he need, a microscopic picture of how electric or magnetic fields operate. Instead, he derived important properties that are independent of any physical model.

Few scholars recognized the importance of Green's ideas when his *Essay* was published. Perhaps discouraged by the lack of sales, Green continued at the windmill for another five years. Around the age of forty, the mathematician Edward Bromhead[4] (1789–1859) encouraged Green to obtain a university education at Cambridge. This presented several difficulties, including the fact that Green knew neither Latin nor Greek, a requirement for a Cambridge education at the time. But Green was a determined man. On a scrap of paper found among his papers dealing with business at the windmill he had written, *"Caesar scribere et legere simul dictate et audire solebat"* (Caesar was accustomed at the same time to write, read, dictate, and listen), evidence that he was beginning to teach himself classical languages [Cannell, 1999]. With help from Bromhead, the middle-aged Green enrolled as an undergraduate at Gonville and Caius College within Cambridge University. He studied mathematics, published several papers in the Transactions of the Cambridge Philosophical Society, and was eventually elected a fellow of the college. Sadly, after being a fellow for less than a year, Green became ill and returned to Nottinghamshire. Within another year he was dead, aged forty-one.

Thankfully, Green's *Essay* lived on. Thanks to the the enthusiasm (and marketing skill) of William Thompson (later known as Lord Kelvin), other scientists became convinced of the importance of Green's work to their understanding of electricity and magnetism.[5] After Green's death, his *"Essay"*, and its new ideas, were eventually read and appreciated by the likes of Kelvin, Liouville, Stokes, Maxwell, and many others. Today, Green's work is foundational to our understanding of electromagnetism, elasticity, nuclear physics, fiber optics, and electrodynamics [Cannell and Lord, 1993].

Of the dozen or so results that Green presented in his 1828 essay, the one we are interested in today is his connection between different types of integrals. While Green wrote of transforming a volume integral into a surface (double) integral, Green's Theorem (as it has become known) actually connects two different types of integrals. To see this connection, we must move forward to 1846, and travel to the European continent where Augustin-Louis Cauchy (1789–1857) was developing the theory of complex functions.

2 The Cauchy Connection

Cauchy was a titan of 19th century mathematics. He is famous for putting calculus on a solid, rigorous footing and for founding the field of complex analysis. The fact that you learned limits in first semester calculus, before you studied derivatives or integrals, is due to Cauchy. In many ways, Cauchy was the opposite to George Green. Green's father was a baker, living in rural England. Cauchy's father held a high position under Napoleon in France. As a child, Green attended a small school for one year, but Cauchy attended the best schools in Paris. While Green published only a handful of papers, which were hardly read in his lifetime, Cauchy published over eight hundred papers, authored several textbooks, and rivals Euler for having the most results named after him. But one thing unites mathematicians, regardless of their background or reputation: a love for discovering and proving beautiful new ideas.

[4] A member of the Analytical Society founded by Charles Babbage, William Hershel, and George Peacock.

[5] For a thorough account of how Green's ideas became known after his death, see Grattan-Guinness, I. (1995) "Why did George Green Write His Essay of 1828 on Electricity and Magnetism," *The American Mathematical Monthly* 102(5), pp.387-396, or Cannell D., and Lord, N. (1993) "George Green, Mathematician and Physicist 1793–1841," *The Mathematical Gazette*, 77(478), pp. 26–51.

In his 1846 paper "Sur les intégrales qui s'étendent à tous les points d'une courbe fermée (On the integreation" ("On integrals that extend over all of the points of a closed curve") [Cauchy, 1846], Cauchy wrote the following:[6]

The two memoirs that I have the honor to present to the Academy relate, one to analytical geometry, the other to integral calculus. As I intend to successively publish these two memoirs in the *Exercices d'Analyse et de Physique mathématique* (*Exercises of Analysis and Mathematical Physics*), I will confine myself to stating here, in a few words, some of the results I have achieved ...

Now let us talk about integrals that relate to closed curves. They enjoy a large number of remarkable properties, among which we must point out those which are stated in the following theorems:

First Theorem. The position of a mobile point P being determined in space using rectilinear, or polar coordinates, or of any other nature, let us name $x, y, z, \ldots$ quantities which vary in a way that varies with the position of that point. Let S also be an area which is measured in a given plane, or on a given surface, and which limits a single closed curve on all sides. Let us conceive then that the mobile point P is subjected to traverse this curve while turning around the area S in a determined direction.

Let us name s the arc of the same curve, measured positively in the direction in question, from a fixed origin, or at least a variable which constantly increases with this arc. Finally, let k be a function of the variables $x, y, z, \ldots$ and of their derivatives relating to s; and denote by (S) the value acquired by the integral

$$\int k\, ds$$

when the mobile point P, having traversed the entire contour of the area S, returns to its original position.

There's a lot to unpack here, so let's begin by drawing some pictures. Cauchy referred to some important objects: a "mobile point" P, a region S, and the boundary of S which is "a single closed curve". In the tasks that follow, we'll refer to this boundary as C_0.

Task 7 Draw a picture of a region S such as Cauchy describes. You may wish to use the general shape depicted in stained glass in Figure 1. Label S, C_0, and also include "the mobile point P subjected to traverse this curve while turning around the area S in a determined direction." Think about how you might indicate the "determined direction" taken by P.

Some mathematicians refer to the boundary, C_0, as a *simple, closed* curve. That is, C_0 is a continuous closed curve that does not cross itself. This terminology was first used by the mathematician Camille Jordan in 1887, and eventually we'll see later that Green's Theorem can apply to other types

[6]All translations of Cantor excerpts in this section were prepared by the project author, 2020.

of regions. For now, we'll follow Cauchy's lead and think more deeply about integration over regions bounded by simple closed curves.

Cauchy asked us to think about "s, the arc of the same curve" and then he set up an integral $\int k\,ds$. This might look like a normal single integral, but there's a bit more going on here.

Task 8 (a) What does it mean for k to "be a function of the variables $x, y, z, \ldots$ and of their derivatives relating to s"?

(b) What type of integral have you seen where we integrate a function with respect to arc length?

Cauchy was about to make a connection between this integral, and another mathematical object with which you are familiar. This connection is known today as Green's Theorem. Once we establish this connection, we can use it to translate tricky problems involving line integrals into more straightforward problems involving...well, something else. But let's not open all the presents too early. Cauchy continued ...

> If, using several straight or curved lines, drawn on the plane or on the given surface, we divide the area S into several other $A, B, C, \ldots$, then, by naming $(A), (B), (C), \ldots$ what becomes (S) when the contour [boundary] of area S is substituted for the contour [boundary] of area A, or B, or $C, \ldots$, we will have, not only $S = A + B + C + \ldots$, but also $(S) = (A) + (B) + (C) + \ldots$, provided that the function k remains finite and continuous at each point of each contour.

Task 9 Let's try to make sense of Cauchy's ideas. First, he used S to refer to the entire enclosed region, and (S) to refer to the line integral $\int k\,ds$, taken along the boundary C_0 of region S.

(a) In Cauchy's notation, what is the difference between $A, B, C, \ldots$ and $(A), (B), (C), \ldots$?

(b) What does Cauchy mean that $S = A + B + C + \ldots$?

It's not too hard to imagine a region S broken down into sub-regions $A, B, C, \ldots$ and so forth. As long as these regions don't overlap, then when we combine the sub-regions, we end up with the entire region S. But Cauchy went further. He claimed that "not only $S = A + B + C + \ldots$, but also $(S) = (A) + (B) + (C) + \ldots$, provided that the function k remains finite and continuous at each point of each contour." This is a bold claim, but with a little thinking, we might see why it could be true:

Task 10 Subdivide your region S from Task 7 into two subregions, a left-hand side and a right-hand side by placing a vertical line segment in between. Call the curves that comprise the boundaries of these regions C_1 and C_2 respectively. Make sure that C_1 and C_2 are both simple closed curves in and of themselves. Cauchy claimed that $(S) = (A) + (B)$, that is, $\oint_{C_0} f(x, y)\,ds = \oint_{C_1} f(x, y)\,ds + \oint_{C_2} f(x, y)\,ds$. Is this true? What property of line integrals allows us to know this?

Task 11 Now, place a horizontal segment to subdivide your two regions each into a top and bottom, creating four non-overlapping subregions A, B, C, D under the same conditions. That is, your overall region S should be cut into four sub-regions, each of which is bounded by a simple closed curve, C_1, C_2, C_3 and C_4 respectively. Although the regions should not overlap, the boundaries of the simple closed curves may. Is there a way to traverse all the boundaries of each subregion so that the line integral (S) is equivalent to the sum of the line integrals $(A)+(B)+(C)+(D)$? If so, describe the path by which this might be done. Why must we be careful about the direction taken by P along the borders between any two adjacent regions?

In theory, we could continue dividing S into greater and greater numbers of every tinier regions. To obtain the line integral around the entire boundary C_0, we could add the line integrals around each of the many sub-regions (provided we are careful about describing the path taken by P so as to obtain the correct cancellations due to direction). Although this may seem like a strange thing to do, Cauchy's approach anticipated the proof of Green's Theorem that Bernhard Riemann (1826–1866) would later supply.[7]

It suffices to say that all this splitting and summing gave Cauchy another way to think about the line integrals around each of those small little regions. Suppose R is one of the tiny regions that make up S. If the function k is of a certain type, we might dispense with the idea of the line integral around the boundary of R and instead think about the influence of k across the *area* of R. Summing over all the areas of the subregions would represent a different type of integral altogether. Cauchy was ready to make his connection:

When, the surface S being plane, x, y are reduced to two rectilinear coordinates, or polar, or of any other nature, suitable for determining the position of a point in the plane of the surface S, then, denoting by $\mathcal{X}, \mathcal{Y}$ two continuous functions of the variables x, y, and assuming

$$k = \mathcal{X} D_s\, x + \mathcal{Y} D_s\, y$$

we have

$$(S) = \pm \iint \Big(D_y \mathcal{X} - D_x \mathcal{Y} \Big)\, dx\, dy$$

the double integral extending to all the points of the surface S.

Let's try to make Cauchy's connection a bit more tangible. First, $\mathcal{X}$ and $\mathcal{Y}$ are just two functions of x, y that are continuous. The notation $D_s x$ is captured in twenty-first century notation by $\frac{dx}{ds}$. Likewise, we would write $D_s y$ as $\frac{dy}{ds}$.

Task 12
(a) Re-write the equation $k = \mathcal{X} D_s\, x + \mathcal{Y} D_s\, y$ using modern notation.
(b) Express the line integral $(S) = \int k\, ds$ using modern notation (canceling where appropriate).

To continue, $D_y \mathcal{X}$ refers to the partial derivative of a function $\mathcal{X}$ with respect to y, and $D_x \mathcal{Y}$ refers to the partial derivative of a function $\mathcal{Y}$ with respect to x.

[7] If you have already studied the idea of *curl* then you may also understand why this circulation argument is important

Task 13 (a) Re-write Cauchy's double integral above using modern notation.

(b) Set your equation for the line integral (S) in Task 12 equal to your double integral. What you have is an explicit connection between the line integral around the boundary of S, and a double integral across that region.

There are still some things to clean up. First, Cauchy needed to address his use of the $\pm$ sign. As he wrote in another part of this same source,

…the calculation…depends on the direction attributed to a certain rotational movement, and is reduced to $+1$ or to -1, depending on whether this movement is direct or retrograde.

Task 14 Cauchy used the terms "direct" and "retrograde" where we would say "clockwise" or "counter-clockwise." Why does the direction taken by the point P matter in this calculation?

Today we use the term "positive orientation" to refer to the case where the mobile point P travels counterclockwise around C_0. Said more clearly, positive orientation means movement in a direction such that the region S is always to your left.

A modern text might present Green's Theorem as follows[8]:

Theorem 2.1 (Green's Theorem) *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let T be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains T, then*

$$\int_C P\,dx + Q\,dy = \iint_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Another way to write the left side of Theorem 2.1 is $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ where the curve C is parameterized as $\mathbf{r}(t)$.[9]

Task 15 How does the modern expression of Green's Theorem compare and contrast with Cauchy's version from 1846?

To illustrate the usefulness of Green's Theorem, let's solve a problem using line integrals, which we'll then re-solve using Green's Theorem.

[8] Although the variable P is often used in modern statements of Green's Theorem, it is not the same as Cauchy's "mobile point P"

[9] Or, as $\int_C \begin{bmatrix} P \\ Q \end{bmatrix} d\mathbf{r}$.

Task 16 Evaluate the line integral

$$\oint_C x^4\,dx + xy\,dy$$

where C is the triangle with vertices $(0,0),(1,0)$ and $(0,1)$ traversed counterclockwise.

(a) Draw a sketch of the region enclosed by C.

(b) Because there is no single formula we can use to describe the boundary C, we need to compute three separate line integrals corresponding to the three sides of the triangle. Describe these three sides in vector form. For example, the side corresponding to the segment on the $x-$axis from $x=0$ to $x=1$ may be parameterized by $\langle t,0\rangle$, $0 \le t \le 1$.

(c) Using these parameterizations and the corresponding bounds on t, set up the three requisite integrals corresponding to $\oint_C x^4\,dx + xy\,dy$.

(d) Evaluate the integrals and sum the results.

Task 17 In the problem you just solved, none of the integrations should have been that difficult but the problem is tedious because of the need to set up three separate integrals. Now, let's rework the same problem using Green's Theorem. Evaluate the line integral

$$\oint_C x^4\,dx + xy\,dy$$

where C is the triangle with vertices $(0,0),(1,0)$ and $(0,1)$ traversed counterclockwise.

(a) As with the previous task, begin with a sketch of this region. This time, use your sketch to write the appropriate bounds for a double integral in the form $a \le x \le b$, $g_1(x) \le y \le g_2(x)$.

(b) Find the partial derivatives $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$.

(c) Use Green's Theorem to write the appropriate double integral that corresponds to the given line integral.

(d) Evaluate the double integral.

(e) Verify that the answer you obtained matches that from the previous task.

While Green's Theorem applies to any region bounded by a simple closed curve, Cauchy noted that some boundaries give us certain advantages:

Note, moreover, that the theorems stated remain, whatever the shape of the line which contains the area S, such as the case where this line becomes the perimeter of a rectilinear or curvilinear polygon, or where the area of S is that of a circular sector.

Let's explore a situation where we have a region enclosed by a circle, to see what benefit such a boundary might provide:

Task 18 Evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F}(x, y) = \langle 3y - e^{\sin x}, 7x + \sqrt{y^4+1} \rangle$ and C is the circle $x^2 + y^2 = 9$ oriented counterclockwise. *Hint*: Once you have set up the correct double integral using Green's Theorem, do not evaluate it from the inside-out as you normally would. Instead, think about what the double integral represents in terms of the region.

By now, you should be getting a sense for why Green's Theorem is important. For a mathematician, however, it's not enough to say that something works – we'd like to know *why* it works. For example, the informal argument about dividing S into many subregions doesn't help us understand why we see $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ in the double integral. As we read earlier, Cauchy intended to publish two memoirs in *Exercises of Analysis and Mathematical Physics*, a journal he personally edited. His proofs, however, would never appear. The proof was eventually provided by another 19th century mathematician, Bernhard Riemann.

3 Riemann's Proof

In 1846, nineteen-year-old Bernhard Riemann was a student at the University of Göttingen. Although his father intended him to pursue the study of theology, Riemann took a mathematics course taught by the "prince of mathematicians," Carl Friedrich Gauss. Riemann quickly switched majors to mathematics, and five years later presented his doctoral dissertation in mathematics. Riemann's dissertation was a wide-ranging paper on the foundations of complex analysis. In section seven, he provided a proof of Green's Theorem. Let's explore a basic idea in this proof, with an eye toward developing a better understanding of why Green's Theorem is true. Note that all of the actors in this drama used different variables in their work. For the purposes of clarity in what follows, we slightly change Riemann's original variables.

Here is how Riemann started his discussion of Green's Theorem in his doctoral dissertation, *Grundalgen für eine allegemeine Theorie der Functionen einer veränderlichen complexen Grösse (Foundations for a General Theory of Functions of a Complex Variable)* [Riemann, 1851].

Let P and Q be two functions of x, y continuous at all points of the surface T spread out over A. With the surface integral extended over all elements dT, we have:

$$\int \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dT = -\int (Q\cos\zeta + P\cos\eta)\, ds$$

Here, at each point of the boundary, ζ denotes the inclination of the interior normal to the $x-$axis, and η denotes its inclination to the $y-$axis. On the right side, the integral extends over all elements ds of the boundary line.

Task 19 At this point, you should be used to the nineteenth century custom of using single integral signs while allowing the reader to determine the type of integral based on context. Therefore, we can tell that the integral on the left side of Riemann's equation is a double integral, while the integral on the right side is a line integral. What aspects of the equation allow us to deduce this?

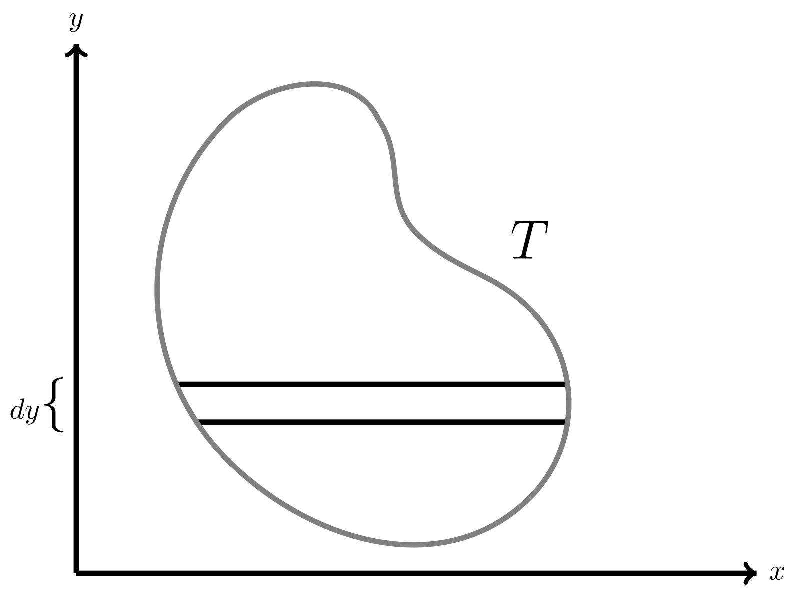
Riemann's version of Green's Theorem looks different from Cauchy's, and indeed from the modern version in Theorem 2.1. Later in his dissertation, Riemann provided the helpful substitutions:

$$\cos\zeta = -\frac{dy}{ds}, \cos\eta = \frac{dx}{ds}$$

and we can re-state the relationship in a way that is similar to Theorem 2.1.[10] Riemann continues:

In order to transform the integral $\int \frac{\partial Q}{\partial x}\, dT$, we divide the part of the plane A covered by T into strips, via a system of lines parallel to the $x-$axis, in such a way that each branch point of T falls on one of these lines. With this assumption, each part of the surface corresponding to one of these strips is formed by one or more trapezoidal pieces.

Think about a region T in the $xy-$plane, that has been cut into slices parallel to the $x-$axis. Each strip would have a very small vertical component, given by dy. You might be able to see why Riemann described these as "trapezoidal pieces."



In order to demonstrate the truth of his formula, Riemann's began by proving half of the it. That is, he started with a piece from the left side of his formula and matched it up with a corresponding piece on the right side. Thus, he stated his intention to "transform the integral $\int \frac{\partial Q}{\partial x}\, dT$." To see which piece on the right side this equates to, we read ahead to the end of his proof.

[10]You should verify that making the substitutions provided by Riemann results in a version of Green's Theorem similar to Theorem 2.1.

By integration over all the elements dy that occur, it is clear that all elements of the surface T and all elements of the boundary will be exhausted. Accordingly, we obtain, with the integration taken over the perimeter,

$$\int \frac{\partial Q}{\partial x}\, dT = -\int Q \cos\zeta\, ds.$$

Let's translate this into modern notation, so that it will correspond with the way Green's Theorem appears in Theorem 2.1.

Task 20

(a) Remember that dT is an element of the area of the region T. Start by re-writing $\int \frac{\partial Q}{\partial x}\, dT$ using modern integral signs.

(b) Use Riemann's substitution $\cos\zeta = -\frac{dy}{ds}$ to re-write $-\int Q \cos\zeta\, ds$.

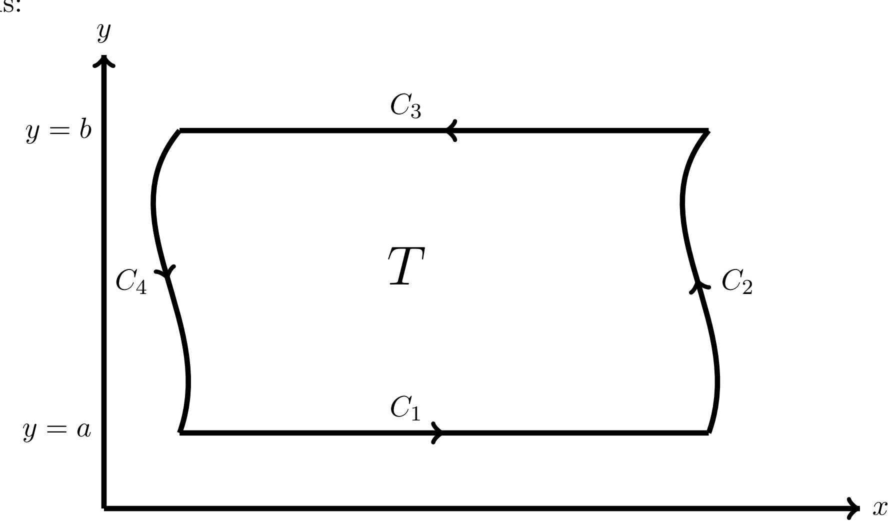
(c) What do you think Riemann meant by the phrase "with the integration taken over the perimeter"? Which of your two integrals in this task is he referring to?

(d) Re-write Green's Theorem (the way it appears in Theorem 2.1) and highlight the two parts that Riemann is trying to equate.

Let's follow the spirit of Riemann's proof and see why this equality holds. Once we have done so, we'll have proved half of Green's Theorem. For this project, we restrict ourselves to a particularly simple type of region, but the proof can extend to regions of other types. First, a preliminary step that might seem trivial, but will become important shortly!

Task 21 Suppose that Q is a function of x and y. We will write it as $Q(x, y)$ for emphasis. If you take the partial derivative of $Q(x, y)$ with respect to x, and then integrate that result with respect to x, what would you obtain? That is, evaluate $\int \frac{\partial Q}{\partial x}\, dx$.

Begin with a region T in the $xy-$plane whose boundary is a positively oriented, simple closed curve, and whose top and base are determined by "lines parallel to the x-axis". We note that the boundary of the region T can depicted as four individual segments. Such a region might look like this:



The curve C_2 may be written as a function, $x = g_2(y)$, and the curve C_4 may be written as a function $x = g_1(y)$. Under this setup, the region T can be described as follows:

$$T = \{(x, y) | a \leq y \leq b, g_1(y) \leq x \leq g_2(y)\}.$$

Consider the integral

$$\iint \frac{\partial Q}{\partial x}\, dT$$

where Q is a function of x and y. This can be written more explicitly as:

$$\iint \frac{\partial Q}{\partial x}\, dT = \int_{y=a}^{y=b} \left[\int_{x_1=g_1(y)}^{x_2=g_2(y)} \frac{\partial Q}{\partial x}\, dx \right] dy \tag{1}$$

Task 22

(a) Based on your work in Task 21, evaluate the definite integral

$$\int_{x_1=g_1(y)}^{x_2=g_2(y)} \frac{\partial Q}{\partial x}\, dx.$$

In doing so, apply the Fundamental Theorem of Calculus, substituting bounds as appropriate.

(b) Re-write Equation 1 by replacing the inside integral with what you obtained in the previous step.

Now that we have an expression for $\iint \frac{\partial Q}{\partial x}\, dT$, we turn our attention to the corresponding integral on the right side of Green's Theorem, $\int_C Q\, dy$. Using the diagram above, we break up C as the union of four curves, C_1, C_2, C_3 and C_4. That is,

$$\int_C Q\, dy = \int_{C_1} Q\, dy + \int_{C_2} Q\, dy + \int_{C_3} Q\, dy + \int_{C_4} Q\, dy.$$

Where the notation $\displaystyle\int_{C_2} Q\, dy$, for example, means $\displaystyle\int_{y=a}^{y=b} Q(g_2(y), y)\, dy$

Task 23 Of the four separate integrals $\int_{C_1} Q\, dy$, $\int_{C_2} Q\, dy$, $\int_{C_3} Q\, dy$, and $\int_{C_4} Q\, dy$, two of them are simply equal to zero. Which two, and why?

Task 24

(a) Re-write $\int_{C_2} Q\, dy$ with explicit bounds, substituting $Q(x, y)$ for Q.

(b) Re-write $\int_{C_4} Q\, dy$ with explicit bounds, paying careful attention to direction, and substituting $Q(x, y)$ for Q.

(c) Finally, re-write $\int_C Q\, dy$ a single integral that sums your two separate integrals above. To combine them into a single integral, you'll need to carefully reverse the bounds on one of your integrals (and insert a negative sign as appropriate).

(d) Compare you resulting equation with your result at the end of Task 22. If they're not the same, either Riemann is wrong, or we are. (So we probably want to check our work!)

At this point we've only showed that

$$\iint \frac{\partial Q}{\partial x}\,dT = \int_C Q\,dy.$$

Riemann then claimed:

By entirely analogous reasoning, we obtain

$$\int \frac{\partial P}{\partial y} dT = -\int P\cos\eta\,ds$$

That is, we can follow the same line of reasoning to equate the remaining portions of Green's Theorem.

Task 25 Re-work the proof above to show that

$$\iint \frac{\partial P}{\partial y}\,dT = -\int_C P\,dx$$

In particular, begin by drawing a region of the form

$$T = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

In setting up your diagram for this part of the proof, you'll want to think about inserting lines that are parallel to the $y-$axis.

4 Applications of Green's Theorem

In 1857, Riemann drew on some of many of the same ideas to write *The Theory of Abelian Functions* [Riemann, 1851]. After discussing some general assumptions, Riemann presented various theorems which he considered indispensable:

Let T be a given surface spread simply or multiply over the (x,y) plane, and let P,Q be continuous functions of position on this surface, such that $P\,dx + Q\,dy$ is a complete differential. Thus

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0.$$

It is well-known that the contour integral

$$\int \Big(P\,dx + Q\,dy\Big),$$

is zero when taken in a positive or negative sense around part of T. That is, the integral is taken over the entire boundary in the same direction (positive or negative) in relation to the outward normal. For this integral is equal, in the former case, to the integral

$$\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dT$$

over this part of T.

You're likely unfamiliar with terms such as simply connected, multiply connected or contour integral. Informally, Riemann is telling us that what he's about to describe works even on regions that have holes in them (e.g. they are *multiply connected*). Let's use a simple example (e.g. a region with no holes in it) to illustrate the first indispensable theorem, which allows us to quickly evaluate certain line integrals (or, applying Green's Theorem, their equivalent double integrals) if we know that $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$.

Task 26 Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field on an open region T, bounded by a simple closed curve C.

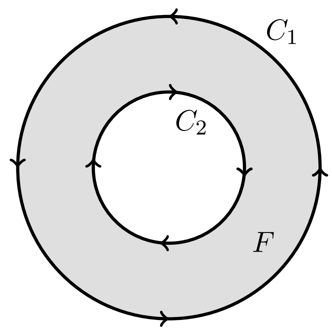
(a) **Scenario 1**: Let $P = 2xe^{xy} + x^2ye^{xy}$ and $Q = x^3e^{xy} + 2y$. Can we know, without actually computing any integrals, that $\int_C P\,dx + Q\,dy = 0$? Explain.

(b) **Scenario 2**: Let $P = x^2 - yx$ and $Q = y^2 - xy$. Can we know, without actually computing any integrals, that $\int_C P\,dx + Q\,dy = 0$? Explain.

Up until this point we have only seen Green's Theorem applied to regions enclosed by simple closed curves. Both Riemann and Cauchy, however, extended the theorem to more general cases. As Cauchy noted earlier, the results hold "whatever the shape of the line which contains the area S." Riemann also discussed this above when he wrote about the surface T being spread "simply or multiply" over the plane. Indeed, it turns out that the important connection between line integrals and double integrals exists even when the region we're working with fails to be bounded by a simple closed curve. Riemann continued his discussion in *The Theory of Abelian Functions* [Riemann, 1851], and again, we need to wade through some vocabulary to see the essential idea: if we're clever, we can always find a way to take a region with holes in it, and cut it up into several simply-connected (e.g. no holes) regions.

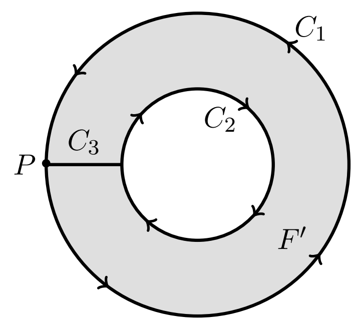
This gives rise to a classification of surfaces into simply connected surfaces, in which every closed curve forms a complete boundary of a part of the surface (for example a disc), and multiply connected surfaces for which this is not true (for example the annular surface bounded by two concentric circles). A multiply connected surface can always be cut up into simply connected pieces...An $(n+1)$-times connected surface F can be converted into an $n-$times connected surface F' by a transverse cut, that is a line starting from a point on the boundary of the surface, ending on another point of the boundary, and lying interior to the surface.

To make this more clear, consider a region with one hole in it. In Riemann's terminology, this is a *2-times connected surface*, but we can carefully define a boundary that turns it into a region with no holes, a *1-times connected surface*. Does it seem like magic that we can remove a hole by cleverly defining a new boundary? Consider the region F enclosed by circles of radius 2 and radius 1 centered at the origin with positive orientation. We'd like to evaluate the line integral $\int_C P\,dx + Q\,dy$ where the boundary C is the union of C_1 and C_2, shown below. Note that in this situation, positive orientation means that we must traverse C_1 counterclockwise, but C_2 clockwise to keep the region F on our left. At first glance, Green's Theorem doesn't seem to apply to such a region, because curves C_1 (the outside circle) and C_2 (the inside circle) are not a single, simply connected, closed loop.



However, if we are careful, it is possible to describe a positively oriented path that would satisfy the hypotheses of Green's Theorem.

Task 27 Consider the region F', which is the region F with the addition of a "transverse cut." This new segment allows us to move from the outside boundary to the inside boundary, and vice versa.



(a) Describe the route the "moveable point P" must travel such that it covers a single closed loop, positively orientated, that contains both the outer and inner circles, and *most importantly*, the value of the line integral $\int_C P\,dx + Q\,dy$ is unaffected by the new segment C_3. Once we establish that such a path is possible, it suggests we can apply Green's Theorem to regions such as this.

(b) Apply Green's Theorem to evaluate

$$\int_C y^3\,dx - x^3\,dy$$

where C is defined by circles of radius 2 and 1 centered at the origin with positive orientation. While you solve this problem, remember Cauchy's comment that "The position of a mobile point P being determined in space using rectilinear, or polar coordinates, or of any other nature."

We often use Green's Theorem to evaluate a double integral that is more straightforward than an equivalent line integral (as in Task 18), but there are cases in which it's easier to evaluate the line integral instead of an equivalent, but more complicated double integral. One important application of Green's Theorem "in reverse" is that it can be used to determine the area of a shape, solely given information about its perimeter. This is the principle behind a device called the planimeter[11] [Gatterdam, 1981]. You may recall that the area of a region D is described by

$$\iint_D 1\, dA.$$

To use Green's Theorem (Theorem 2.1) to compute the area of D, the expression $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ must equal 1.

Task 28

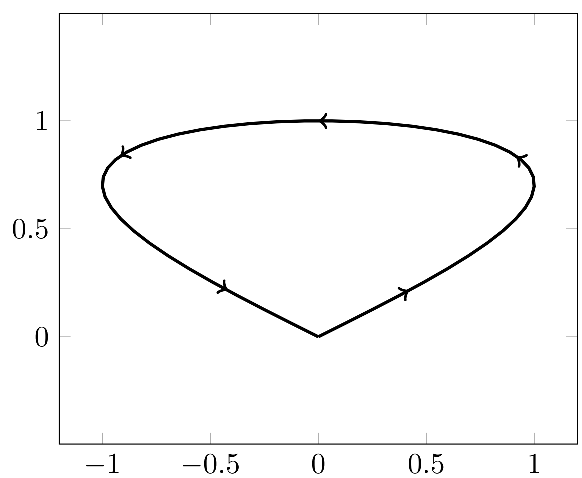
(a) One way to satisfy the differential equation $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ is if $P(x, y) = 0$ and $Q(x, y) = x$. Verify this, and find two other pairs of expressions for P and Q that satisfy this equation.

(b) Using the expressions $P(x, y) = 0$ and $Q(x, y) = x$, Green's Theorem simplifies to

$$\oint_C x\, dy = \iint_D 1\, dA.$$

where C is a positively oriented boundary of D and the right-hand side represents the area of the enclosed region D. Use your results from the previous step to write two additional formulas for the area of D.

(c) Use one of your area formulas to determine the area of the region D bounded by the curve C parameterized by $\mathbf{r}(t) = \sin 2t\, \mathbf{i} + \sin t\, \mathbf{j}$ for $0 \le t \le \pi$. A sketch of this region is shown below:



[11] A planimeter is a mechanical device used for measuring the area of any region bounded by a smooth curve. That is, we can determine the area of a shape simply by measuring its perimeter and applying Green's Theorem. Students who are familiar with the concept of *curl* might be interested in studying this device further, because it is a clever application of line integrals over vector fields that have constant curl.

5 Ghosts of the Past, Present, and Future

There are many more applications of Green's Theorem, and we cannot hope to discuss them all in this project. Green's hope was to see scientists applying calculus to further our understanding of electricity and magnetism, and he provided some key ideas related to how this might be done, including the notion that we can understand properties inside an object by examining its outside. As we've seen, Cauchy and Riemann (among others) wrote extensively about Green's ideas. They formally developed his theorem, and eventually would go on to apply it to the world of complex numbers (which themselves have important applications in electronics). Yet, as we read Green's Essay, it becomes clear that Green himself also saw the potential of his ideas to influence our understanding of a wide range of diverse phenomena:

> Should the present *Essay* tend in any way to facilitate the application of analysis to the most interesting physical sciences, the author will deem himself amply repaid for any labour he may have bestowed upon it ... All those to whom the practice of analysis is familiar, will readily perceive that ...the solution of the question originally proposed has been rendered much easier by what has preceded... The simplification in question, seems worthy of the attention of analysts, and may be of use in other researches where equations of this nature are employed.

Task 29 Throughout this project, we've seen how Green's Theorem allows us to unite three core ideas: (1) Reducing a $k-$dimensional integral to a $(k-1)-$dimensional integral, (2) Shifting our focus from the domain of a region to the region's boundary, and (3) Transforming a given integrand to some other function whose derivatives give you the integrand. In a couple of sentences, discuss how these ideas are present in Green's Theorem.

In the introduction to his *Essay*, Green hinted that his work at the windmill was a hindrance to his mathematical studies:

> It is hoped the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man, who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means, as other indispensable avocations which offer but few opportunities of mental improvement, afforded.

Ironically, despite Green's lament that the windmill "offered few opportunities for mental improvement", scientists today use Green's Theorem (along with related ideas such as Stokes' Theorem, the Divergence Theorem, and so forth) to better understand and design renewable energy sources – such as wind power [Jeon and Lee, 2014]. Furthermore, the idea that we can measure the boundary of a closed surface, and use this to know things about the interior of the surface has greatly influenced our understanding of many physical processes[12].

The more we learn about our Universe, the more places we see Green's ideas appearing. Theoretical physicist Donald Marloff is one person who believes that the Universe itself may have a boundary. If true, Marloff writes, "Quantities that one might think are *a priori* independent are in fact locked together. Thus one can have boundary observables that are identically equal to bulk observables." [Musser, 2016]. Said otherwise, if space has a boundary, then physical processes on that boundary are identical to the processes going on within the Universe itself. An observer looking at the boundary alone would have all the information about the *bulk*, or interior of our Universe. One implication is that the entire Universe could be collapsed onto its boundary, making the volume redundant. Whereas we *think* we live within the bulk of the Universe, it could well be that this is an illusion and we all exist at a lower dimension on a boundary.

Task 30 How is Marloff's idea about the structure of the Universe related to Green's Theorem?

Task 31 Look closely at the stained glass window pictured in Figure 1 of this project. Describe how the artist captures the essential aspects of Green's Theorem.

[12] As another example, it means we can measure the electric flux through a closed surface and determine the amount of enclosed charge

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Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is intended to give students in a multivariable calculus course a thorough introduction to Green's Theorem. The project begins with an overview of a main idea: transforming a $k-$dimensional integral into a $(k-1)-$degree integral, and invites the reader to think about the history of physics in the early nineteenth century that prompted Green to write his *Essay*. Students are then taken through Cauchy's explanation of Green's Theorem where they see the first explicit connection between a double integral across a region and a line integral over the boundary of that region. The third part of the project explores Riemann's proof of the theorem and students re-create the proof for a specific type of region. The project includes an example of Green's Theorem over regions with discontinuities (such as the region bounded by two circles), the connection between Green's Theorem and line integrals over conservative vector fields, and the use of Green's Theorem to calculate areas. Finally, the project hints at potential present-day applications of Green's ideas to theories about the nature of our Universe.

Student Prerequisites

To succeed with this project, students should be familiar with vector fields, and know how to compute multiple integrals and line integrals. This PSP does not directly reference curl or divergence in order to let instructors present Green's Theorem before teaching those topics. However, the ideas (particularly curl) are just below the surface, and instructors may wish to adapt certain tasks to specifically bring those ideas out.

PSP Design, and Task Commentary

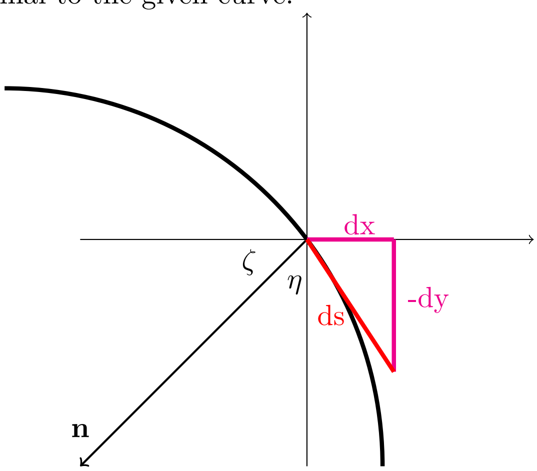
The PSP begins with a historical introduction to George Green, and includes selections from his 1828 essay, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Although the theorem which now bears his name does not appear in this essay, Green introduces the reader to the importance of being able to connect properties within the interior of a region with properties along the boundary of that region, and provides a reciprocity theorem that connects volume integrals with surface integrals. Some of the tasks in the first section, including Tasks 1, 3, and 4 are meant to get students thinking about the state of science more broadly in 19th century Europe and provide context for understanding Green's motives in writing his Essay. Task 2, connected with Green's ideas about potential functions, is meant to be a review problem and a mathematical warm-up for the computations to come. Task 5 lets students investigate one of the theorems that appears in Green's Essay, but the mathematics involved in truly understanding that theorem requires knowledge of surface integrals, the Laplace operator, and other topics that most students have yet to see.

Cauchy takes up the thread in 1846 with his paper *Sur les intégrales qui s'étendent à tous les points d'une courbe fermée* (*On integrals that extend over all of the points of a closed curve*). In Tasks 7 through 14, students think more carefully about line integrals, regions enclosed by simple closed curves, and work through Cauchy's argument related to dividing a region into multiple non-overlapping regions. Importantly, students should understand that the line integral around the entire boundary may be written as the sum of line integrals around the sub-regions. Instructors may wish to take the opportunity to reinforce the idea that line integrals are (usually) path dependent, and that reversing the direction along a path reverses the value of that line integral. This idea underlies

Cauchy's explanation. Cauchy provides the first (known) explicit statement of Green's Theorem, although students will have to put a few pieces together for it to resemble the version presented in most modern textbooks. Cauchy does not provide a proof, although students should begin to see a reason why the line integral – double integral connection might be legitimate. Students also grapple with the importance of giving a curve an *orientation*.

Tasks 16 and 17 serve to verify the truth of Green's Theorem for a specific situation, and also give students additional practice computing line integrals, taking partial derivatives, and computing double integrals. Task 18 reenforces the vector form of line integrals, and takes a comment from Cauchy about shapes of boundaries to give students additional practice applying Green's Theorem. (Toward the end of this task students should recognize that the double integral they are working with simply describes the area of a circle.)

In the third section of this project, students read selections from *Foundations for a general theory of functions of a complex variable*, taken from Bernhard Riemann's 1851 doctoral dissertation. In the selections taken for this PSP, Riemann outlines the basic idea behind his proof, which is similar to the modern proof. As written, the project passes lightly over Riemann's use of trigonometric functions. You may wish to give more advanced students the challenge of *deriving* Riemann's substitutions for $\cos\zeta$ and $\cos\eta$ based on his remarks about the "inclination of the interior normal". This may be done by considering a differential triangle based on the sketch below, where **n** represents the interior normal to the given curve.



The appropriate substitutions are:

$$\cos\zeta = -\frac{dy}{ds}, \cos\eta = \frac{dx}{ds}$$

Students then follow the spirit of Riemann's proof to construct their own proof of Green's Theorem over simplified regions. Tasks 21 and 22 are where the students will need to really understand the Fundamental Theorem of Calculus. Note that in order to see the magic, students should not evaluate the double integral in Task 22 completely, but stop once they have evaluated the inside integral. In Tasks 23 and 24, students think about properties of integrals along four boundary lines and eventually re-write a line integral that matches their result from Task 22. Students complete the second half of the proof in Task 25

The fourth section considers applications of Green's Theorem, such as to regions with holes, or those enclosing conservative vector fields. Task 28 reminds students that double integrals may be used to find areas, and extends this to find the area of a region given information about its perimeter.

The final section returns to Green's 1828 essay, and prompts students to reflect on their learning. It also leaves students with a connection between Green's Theorem and the structure of the Universe. This is merely intended to spark student curiosity and potentially lead to further independent reading.

Suggestions for Classroom Implementation

This PSP would fit well near the end of a course in multivariable calculus. It connects ideas such as conservative vector fields, line integrals, double integrals, partial derivatives, and much more. It can be taught before students are introduced to curl and divergence, but could also be taught after students are familiar with those topics. Students will need to draw on ideas from throughout the semester, which makes this a nice project for wrapping up the course.

Most of the tasks lend themselves to small groups of students working together. Even the strongest students will benefit from collaborating as they try to interpret 19th century notation, and wade through challenging vocabulary. Encourage the students to draw many pictures. In fact, some tasks specifically ask students to compare their sketches with one another to verify the generality of certain results.

The project includes some tasks, especially in the first section, that serve as easy onramps for students regardless of their mathematical strength.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Possible Modifications of the PSP

Few of us have the luxury to spend four days on Green's Theorem. On the other hand, this project covers much more than Green's Theorem, and could serve as a great way to review man of the main ideas in a multivariable calculus course. Instructors who wish wish to spend less than four class periods on Green's Theorem may shorten this PSP in several ways. First, they could skip the proof of Green's Theorem, which comprises all of section three. Instructors may also wish to skip this proof if they intend to develop student intuition by teaching the circulation form of Green's Theorem, and making a connection with curl.

Another way to shorted this PSP would be to pass quickly over some of the introductory material in section one and begin with Cauchy's connection in section two. Or, you may choose to skip Tasks 8 through 11, and just discuss Cauchy's statement of Green's Theorem without getting into his arguments about why a connection exists between line and double integrals. Some applications of Green's Theorem, while important, could also be eliminated.

Sample Implementation Schedule (based on a 50-minute class period)

The full PSP can be implemented over four (50-minute) class periods, along with some preparatory work by the students. It would fit well after instruction on line integrals and conservative vector fields. The following schedule is suggested:

- Day Zero. At the culmination of your lesson(s) on line integrals, introduce the project and ask students to read Section 1 before the next class period.

- Day One. In groups of 3 or 4, students should work through Tasks 1 through 6. You may wish to have students share their answers to with the entire class. Take time to review conservative vector fields and the Fundamental Theorem of Calculus, if necessary.

- Day Two. Students should work through section two, in which they will become familiar with Green's Theorem, and verify its usefulness, but not prove it.

- Day Three. Students should work through section three, which follows Riemann's proof of the theorem. His proof is actually quite similar to that found in most modern texts. Students will likely require some significant direction (i.e. hints) to complete the proof. Drawing pictures helps!

- Day Four. Students should work through sections four and five, in which they apply Green's Theorem to compute line integrals over conservative vector fields, analyze regions with holes, and use Green's Theorem to find the area of a shape bounded by a parameterized curve. If they don't finish the tasks in section five, those can be assigned for homework, but the "wow factor" of Task 30 works best in class.

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students. Estimates on the high end of the range assume most PSP work is completed by students working in small groups during class time.

Connections to other Primary Source Projects

There are two additional projects based on primary sources for the multivariable calculus classroom. The PSP author name of each is given, along with approximate implementation time. Classroom ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_calculus`.

- Braess' Paradox in City Planning: An Application of Multivariable Optimization, Kenneth M Monks (2 class periods)

- The Radius of Curvature According to Christiaan Huygens, Jerry Lodder (6 class periods)

Recommendations for Further Reading

Those interested in reading more about Green's ideas related to the calculus of electricity and magnetism, as well as some social commentary on 19th century science, will appreciate Green's *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* (1828), which is readily available online. Cauchy's 1846 paper *Sur les intégrales qui s'étendent à tous les points d'une courbe fermée* is available (in French) through Internet Archive `https://archive.org/details/ComptesRendusAcademieDesSciences0023`. Riemann's dissertation was translated into English, along with many other papers, by Roger Baker, Charles Christenson and Henry Orde.

Several other papers were useful in the writing of this PSP, and make excellent reading for those interested in the further study of this topic. Among those not specifically listed in the references:

- "The History of Stokes' Theorem" by Victor Katz (Mathematics Magazine, May 1979)

- "Why did George Green Write His Essay of 1828 on Electricity and Magnetism?" by I. Grattan-Guiness (The American Mathematical Monthly, May 1995)

- "The Real and the Complex: A History of Analysis in the 19th Century" by Jeremy Gray (Springer, 2005)

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