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Understanding Compactness Through Primary Sources: Early Work Uniform Continuity to the Heine-Borel Theorem

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1 Introduction

When you hear the word "compact," you most likely think of something that is small and takes up little space, or perhaps a collection of things that are closely packed together. Usually in mathematics, the simplest case in examining certain properties of a set is when the set is finite. It is also the case in mathematics, that we can extend such properties of a finite set to more general sets also satisfying similar properties, but under certain constraints placed on the sets.

Compactness gives us one notion of "finiteness" for sets (and specifically for topological spaces), in a somewhat unusual way. Because of this, the concept of compactness can be quite challenging to grasp. Like many concepts in analysis (and topology), the modern notions of compactness emerged out of the work of early 19th-century analysts like Bolzano, Dirichlet, and Weierstrass, then evolved over many decades starting with Borel and Lebesgue in the latter part of the 19th century, to Frechét and Hausdorff in the early 1900s, then Alexandroff and Urysohn in the 1930s, and finally to formulations of the concept involving more abstract notions known as nets and filters. An overview of this fascinating historical development is given in [Raman-Sundström, 2015].

There are two important theorems underlying the development of compactness, the first concerning the uniform continuity of functions, and the second being the Extreme Value Theorem. On one hand, the various modern notions of compactness evolved in order to first prove these two theorems on closed intervals $[a, b]$ in $\mathbb{R}$ and then to extend them to the most general setting possible. On the other hand, the study of the topological properties of the real line $\mathbb{R}$ led to the extension of those ideas to an abstract setting. Such a study in itself was influenced by the proof of these theorems. The aim of the activities in the "Understanding Compactness Through Primary Sources" series of projects is to explore how the ideas behind these theorems developed. And by exploring the historical development through *primary sources*, we can hopefully come to appreciate the intricacies of and motivation behind this important concept.

Let's state the two theorems of concern below, restricting ourselves to a closed bounded interval $[a, b]$ in $\mathbb{R}$.

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Theorem 1.1 (Uniform Continuity)**.** *Every continuous function* $f : [a, b] \longrightarrow \mathbb{R}$ *is uniformly continuous.*

Theorem 1.2 (Extreme Value Theorem)**.** *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous function. Then* f *attains its maximum and minimum value on* $[a, b]$*. That is, there exists* c *and* d *in* $[a, b]$ *such that for all* x *in* $[a, b]$*,*

$$f(x) \leq f(c), \quad f(x) \geq f(d).$$ [1]

Historically, there were two concurrent ideas — one related to sequences and the other related to open sets — involved in the evolution of these theorems. The collection of projects in the series "Understanding Compactness Through Primary Sources" follows the evolution of these theorems and their various proofs to explore these two ideas and see how they can be unified under certain assumptions about the underlying spaces. In this particular project (the first in the series), we focus on early proofs of the Uniform Continuity Theorem to see how questions about integration led to a first formulation of the topological property that eventually came to be called 'compactness.'

2 Uniform Continuity: Some History and Basic Results

In this section, we will review some basic concepts regarding uniformly continuous functions and highlight some aspects of the history of that concept.

Although calculus itself dates back to the work of Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), the concept of a continuous function became an important object of mathematical investigation only during the early part of the 19th century. It was also at that time that mathematicians began to use the algebra of inequalities to work with calculus concepts. Two mathematicians are credited with giving the earliest such definitions for continuity: Bernhard Bolzano (1781–1848) and Augustin-Louis Cauchy (1789–1857). Both men also put their definition of continuity to use in the proof of theorems like the Intermediate Value Theorem, which until that time had been accepted on the basis of geometric intuition.[2]

Theorem 2.1 (Intermediate Value Theorem)**.** *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous function. Given any* L *between* $f(a)$ *and* $f(b)$*, there exists some* c *in* $[a, b]$ *such that* $f(c) = L$*.*[3]

[1]Note that the points c and d in the Extreme Value Theorem need not be unique, since a function may take on its maximum and minimum values multiple times on $[a, b]$. The key fact is that such points *exist*.

[2]Many 19th-century mathematicians, including Bolzano and Cauchy, were concerned that the use of geometrical intuition to justify theorems in analysis was a methodological error. This concern became one of the major motivations behind the changes that occurred in that century, during which proofs in analysis became increasingly precise and more rigorous. For more about this and other motivating factors behind those changes, see the project "Why be so Critical? 19th Century Mathematics and the Origins of Analysis" (author Janet Heine Barnett), available at `https://digitalcommons.ursinus.edu/triumphs_analysis/1/`. Concise biographies of Bolzano and Cauchy can be found at [O'Connor and Robertson, 2005] and [O'Connor and Robertson, 1997].

[3]Note that the key fact in the Intermediate Value Theorem is again that such a point c *exists*, where that point c need not be unique.

Cauchy, who was to be the more influential of these two early analysts,[4] also used his definition of continuity to give a new definition of the definite integral, based on the idea of approximating the area under a curve using sums of rectangles.[5] He did this by defining the definite integral of a function as the limit of finite sums, $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$, provided this limit exists. Cauchy then proved that this limit does in fact exist in the case of continuous functions; that is, he proved that all continuous functions are integrable. The crux of this proof (in Cauchy's time and today) relies on what we today call the Uniform Continuity Theorem — a theorem that expresses a fact about continuous functions that Cauchy neglected to mention or even seemed to be aware of.[6]

While some scholars have suggested that Cauchy simply made an easily-remedied mistake in his proof that continuous functions are integrable, a careful reading of how he actually used his definition of continuity in this and other proofs suggests that Cauchy's notion of continuity is equivalent to what we today call uniform continuity.

The renowned German mathematician Karl Weierstrass was the first to give the definition of pointwise continuity that we use today, which makes it possible to then give the following definition of continuity on an interval:

> **Definition 2.1** (Continuity)**.** Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a function defined on I. We say f is (*pointwise*) *continuous* on I if and only if for all $y \in I$ and all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in I$,
>
> $$|x - y| < \delta \implies |f(x) - f(y)|) < \epsilon.$$

Let's compare this definition with the one we give today for uniform continuity:

> **Definition 2.2** (Uniform Continuity)**.** Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ a function defined on I. We say f is *uniformly continuous* on I if and only if for all $x, y \in I$ and all $\epsilon > 0$, there exists a $\delta > 0$ such that
>
> $$|x - y| < \delta \implies |f(x) - f(y)|) < \epsilon.$$

Notice that the key point in the definition of uniform continuity is that given an $\epsilon > 0$, we can pick a *single* $\delta > 0$ that works (i.e., $|f(x) - f(y)| < \epsilon$) for *all* points x, y satisfying $|x - y| < \delta$. In other words, the choice of δ depends only on ϵ and not on the points x and y.

[4]Cauchy's greater influence was not due to the quality of their mathematics, but rather the result of social and political factors. Bolzano was a mathematician, philosopher, theologian and Catholic priest who spent most of his life in Prague, far removed from the center of European mathematical activity at the time. Cauchy, on the other hand, spent most of his working life at the heart of that activity in Paris, where he was a respected professor at one of its most important school. For additional biographical information about the two, see [O'Connor and Robertson, 1997, 2005].

[5]Before Cauchy, the integral was simply viewed as an antiderivative. See footnote 19 for a brief discussion of how the definition of integration changed after Cauchy.

[6]While we won't look at the proof that continuous functions are integrable in this project, you can find it in any current introductory analysis textbook (in the chapter on Riemann integration). Unlike Cauchy's version, the proof found in today's textbook will make explicit use of the fact that integration takes place over closed, bounded intervals, where the function's continuity is necessarily uniform.

For example, consider a linear function $f(x) = ax + b$ on $\mathbb{R}$ with $a \neq 0$. Then, for any two points x, y we have $f(x) - f(y) = a(x - y)$. Given an $\epsilon > 0$, we can pick $\delta = \epsilon/|a|$ so that for $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. This makes $f(x)$ uniformly continuous on $\mathbb{R}$. Although uniform continuity implies pointwise continuity, the converse statement is not true in general. We explore some examples in Task 1.

Task 1

(a) Consider the interval (0, 1]. Give an example of a continuous function on (0, 1] that is not uniformly continuous. Prove that your example works.

(b) Consider the function $f(x) = \sin\left(\frac{1}{1-x}\right)$ on [0, 1). State why $f(x)$ is a *bounded continuous* function on this interval.

(c) Consider the previous example in part (b). Describe what happens to $f(x)$ as x approaches 1. How many zeros does $f(x)$ have? Is $f(x)$ uniformly continuous on [0, 1)? Explain why or why not.

(d) Consider the unbounded interval $[0, \infty)$. Sketch an example of a *bounded continuous* function $f(x)$ that is not uniformly continuous on $[0, \infty)$. You do not have to prove your claim, just a sketch of such a function and an explanation would suffice.

In the next section of this project, we will look at some of the earliest work done on uniform continuity, by the German mathematician Gustav Lejeune Dirichlet (1805–1859).

Task 2 In his work, Dirichlet gave an example of a bounded continuous function on $(-\infty, \infty)$ that is nevertheless not uniformly continuous.[7] Let's preview this example here:

$$f(x) = \sin(x^2).$$

(a) Give an intuitive reason why you think that $f(x)$ is not uniformly continuous on $(-\infty, \infty)$. Hint: Think about the zeros and peaks of the function; it will help to make a rough sketch of the graph of the function.

(b) Prove that the function is not uniformly continuous. Hint: Consider where the peaks occur, and use contradiction.

[7]See Pages 7–8 of [Dirichlet, 1904].

3 An Early Precursor of Compactness: Dirichlet and the Uniform Continuity Theorem

We begin our story of compactness with some work done by Gustav Lejeune Dirichlet (1805–1859), a prominent German mathematician of the first half of the nineteenth century. He is remembered for his work on Fourier series, a topic in which integration plays a key role, and the application of analytical techniques to solve problems in number theory.[8]

Dirichlet was known as a great expositor of mathematics and gave carefully prepared and captivating lectures without notes.[9] After his death, several of his students published his lectures as books. Our concern is with Dirichlet's 1854 lectures on definite integrals, which were only published much later as *Vorlesungen Über die Lehre von den einfachen und mehrfachen bestimmten Integralen* (*Lectures on the theory of simple and multiple definite integrals*) [Dirichlet, 1904]. In these lectures, he proved what he called "a fundamental property of continuous functions:"[10]

> Let $y = f(x)$ be a function of x, continuous on the finite interval that goes from a to b; by *sub-interval*, we mean the difference between two arbitrary values of x, that is any part of the x-axis between a and b. It is then always possible to find, for every arbitrarily small positive quantity ϱ, a second quantity σ, proportional to it, that has the property that on any sub-interval [of $[a, b]$ of length] $\leq \sigma$, the function y does not vary by more than ϱ.

While Dirichlet used ϱ–σ (the Greek letters rho and sigmaa) here — instead of the now-standard ϵ–δ notation that we use today — reading his description carefully and comparing to today's terminology, we see that this is just a statement of the Uniform Continuity Theorem: a continuous function on a closed and bounded interval $[a, b]$ is uniformly continuous on that interval. *(Re-read Dirichlet's statement as needed to be sure you see this!)* Dirichlet himself stated that, although this theorem may seem obvious *(do you think it is?)*, it is in fact quite non-trivial because it is not exhibited, for example, by unbounded functions in $\mathbb{R}$. Moreover, Dirichlet went on to give the example of a bounded function in $\mathbb{R}$ that is not uniformly continuous which you looked at in Task 2: $f(x) = \sin(x^2)$.

Dirichlet's proof of the Uniform Continuity Theorem is interesting and worth examining for a few reasons. Firstly, the proof involves taking certain open intervals on the y-axis and looking at corresponding intervals on the x-axis produced by the preimages. Central to his argument is the fact that the number of corresponding intervals on the x-axis will be finite. This is an idea that essentially extends to a more general setting by using open sets to "cover" the image set (range) and then looking at the preimages of those open sets (in the domain). We shall see this important idea appear repeatedly later on. Secondly, his argument also has a sequential component to it. In some sense, his proof thus combines the two key ideas behind compactness.

[8] Dirichlet's life and work has been studied in detail by Jürgen Elstrodt in [Elstrodt, 2007]. For a more concise biography, see [O'Connor and Robertson, 2000].

[9] See pp. 15–18 of [Elstrodt, 2007].

[10] All translations of primary source excerpts used in this project were prepared by the author.

We shall work through Dirichlet's proof of the Uniform Continuity Theorem[11] in Task 3. Following Dirichlet, we'll use the notation ϱ–σ. Unlike Dirichlet, we will explicitly fill in certain details that would have been taken for granted at the time; this occurs, for example, in part (b) of Task 3 where we explicitly state where Dirichlet made implicit use of the Intermediate Value Theorem.

Task 3 In this task, we examine Dirichlet's proof of the "fundamental property of continuous functions" that is captured in the Uniform Continuity Theorem.

Let $f : [a,\, b] \longrightarrow \mathbb{R}$ be continuous. Fix a $\varrho > 0$. Dirichlet's objective is to pick a $\sigma > 0$ such that *for all* x, x' in $[a,\, b]$ such that $|x - x'| < \sigma$, we have $|f(x) - f(x')| < \varrho$. Here is how he began.

To prove this theorem, we use the more convenient way of representing the geometrical concept. Go from the initial value a to the side where b lies and, without leaving a value of $f(x)$ unexamined, end the first subinterval in the value c_1 of x for which for the first time the function differs from its initial value $f(a)$ by exactly the given absolute ϱ, so that one has:

$$f(c_1) - f(a) = \pm\varrho,$$

where the sign of ϱ is not of our choosing, but results from the behavior of the continuous, but arbitrary, curve, for which the ordinates can sometimes increase, sometimes decrease.

(a) Dirichlet claimed above that his proof is intended to represent a geometrical concept. Before starting on the proof, sketch a continuous function on an interval $[a,\, b]$ and for some fixed ϱ draw two horizontal lines $f(a) + \varrho$ and $f(a) - \varrho$. Mark the point c_1 on the x-axis where your graph *first* cuts one of these lines. As Dirichlet noted at the end of this quote, depending on your sketch, the value of $f(c_1) - f(a)$ may be ϱ or $-\varrho$.

You may (and should!) be wondering what allowed Dirichlet to make the claim that there is "a value c_1 of x for which for the first time ... one has $f(c_1) - f(a) = \pm\varrho$." Based on your sketch from part (a), you probably agree with Dirichlet that c_1 does exist for at least some values of ϱ. More specifically, as long as ϱ is small enough that $|f(x)-f(a)| \geq \varrho$ (i.e., $f(x) \geq f(a)+\varrho$ or $f(x) \leq f(a)-\varrho$) for at least some values of x in $(a, b]$, it should seem reasonable to expect that the **equality** $|f(x)-f(a)| = \varrho$ holds for some value of x and that we can then pick the *smallest* such x in $(a, b]$ and simply label

[11] The proof is presented in Pages 3–7 of [Dirichlet, 1904].

that point at c_1.[12] At the time that Dirichlet worked, most mathematicians would have implicitly assumed statements like this were correct and used them without comment within their proofs.[13]

Today, mathematicians expect to see a more formal justification for the existence of c_1, using the Axiom of Completeness or one of its equivalencies:[14]

> **Axiom 3.1** (Completeness). *Every non-empty set that is bounded above has a supremum (least upper bound).*

(b) Use the infiumum equivalent of the Axiom of Completeness to provide a detailed justification for the existence of c_1 by considering the set

$$A = \{x \in (a, b] \mid |f(x) - f(a)| = \varrho\},$$

where ϱ is assumed to be such that $|f(x)-f(a)| \geq \varrho$ for some values of x in $(a, b]$. Begin by showing that A is non-empty using the Intermediate Value Theorem. Also explain why A is bounded below. Then let c_1 denote the infimum of A. Is it possible that $c_1 = a$? Argue it is not by using the continuity of $f(x)$. Conclude that we have $c_1 \in (a,\ b]$

[12]Of course, if $f(x)$ is a constant function, then we will never have $|f(x)-f(a)| \geq \varrho$ for any value of ϱ; since constant functions are clearly uniformly continuous on their domain, however, this case is not a concern. Values of ϱ that are large enough that $|f(x) - f(a)| < \varrho$ for all $x \in [a, b]$ also aren't a problem — why not?

[13]An exception was Bolzano, who explicitly assumed the convergence of Cauchy sequences and used that assumption to prove a somewhat complicated least upper bound version of completeness. For details about Bolzano's ideas about completeness, see the project "Investigations into Bolzano's Bounded Set Theorem" (author Dave Ruch), available at `https://digitalcommons.ursinus.edu/triumphs_analysis/14/`.

[14] Recall that the basic idea of completeness is that the real number line has no gaps. There are other ways to characterize this idea, including the statements given in the following:

> **Theorem 3.1** (Axiom of Completeness Equivalencies). *Each of the following is equivalent to the Axiom of Completeness:*
>
> *(a) Every non-empty set that is bounded below has a infimum (greatest lower bound).*
> *(b) Every Cauchy sequence $\{x_n\}$ in $\mathbb{R}$ converges.*
> *(c) Every bounded sequence $\{x_n\}$ in $\mathbb{R}$ has a least upper bound.*
> *(d) Every bounded monotonic sequence $\{x_n\}$ in $\mathbb{R}$ converges.*
> *(e) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (Bolzano-Weierstrass Theorem, sequence version)*
> *(f) Every bounded infinite set in $\mathbb{R}$ has a limit (accumulation) point. (Bolzano-Weierstrass Theorem, topological version)*

Although Bolzano's name is attached to the Bolzano-Weierstrass Theorem (equivalencies (e) and (f) in the list above), this appears to be due to a series of misunderstanding rather than Bolzano's early recognition of the concept of completeness [Moore, 2000, pp. 171–172]. The topological version of the Bolzano-Weierstrass Theorem was first published by the famous German mathematician George Cantor (1845–1918) [Moore, 2000, p. 176]. As early as 1868, however, Weierstrass' lectures at the University of Berlin included statements of a theorem similar to that of Cantor. Although Weierstrass (like Dirichlet) never published his own lectures, lots of his students' notes did become public and the attachment of his name to this theorem was well-known among mathematicians. See [Moore, 2000] for a nice discussion of this and other aspects of the history of the theorem.

Dirichlet continued his proof as follows:

For each value of x between a and c_1, we have $|f(x) - f(a)| < \varrho$; because if for such a value of x, the difference were equal to ϱ, we would have had to end the first sub-interval earlier [i.e., before c_1], at this point x; the difference [i.e., $|f(x) - f(a)|$] can also not be greater than ϱ, because by virtue of the continuity of the function $f(x)$, the difference would have assumed the value ϱ at a value anterior to x, and with the same sign.

(c) Read the quote above again, and decide if you agree with Dirichlet that for all x in (a, c_1), we have $|f(x) - f(a)| < \varrho$. Justify this claim.

Dirichlet next repeated the procedure starting from the new point $x = c_1$.

Proceed in exactly the same way: starting from c_1, denote by c_2 that value of x, for which $f(x)$ now differs for the first time by $\pm\varrho$ from the initial value $f(c_1)$, it follows that one has in turn: $f(c_2) - f(c_1) = \pm\varrho$, but for each x within c_1 and c_2: $|f(x) - f(c_1)| < \varrho$; etc.

In this way a series of values is obtained:

$$a, c_1, c_2, c_3, \ldots$$

of the nature just described, ...

(d) Repeat the previous steps on the interval $(c_1, b]$. If $|f(x) - f(c_1)| \geq \varrho$ for some x on $(c_1, b]$, then would you be able to find a c_2 with similar properties to c_1 ? That is, do you agree with Dirichlet that there exists c_2 for which $f(c_2) - f(c_1) = \pm\varrho$, but for each x within c_1 and c_2, we have $|f(x) - f(c_1)| < \varrho$?

(e) Continue repeating these steps to produce a sequence

$$a, c_1, c_2, \ldots$$

which may or may not be finite. For what possible reasons would the sequence (or the algorithm we outlined) terminate?

Next we claim the sequence $a, c_1, c_2, \ldots$ does indeed terminate, and so did Dirichlet!

> The question arises, however, whether the entire interval from a to b will also be filled with a finite number of these values, i.e. whether one finally arrives at a last c_μ, which either equals b, or is so close to b that between it and b, the function $f(x)$ differs from $f(c_\mu)$ by less than ϱ.
>
> If this were not the case, one would have an infinite series of intermediate values c, which, however, converge towards a fixed, finite value C located between a and b or coinciding with b, ...

(f) Let's follow Dirichlet's reasoning and show that the sequence $\{c_\mu\}$ does terminate using contradiction.

(i) Presume, by contradiction, that we have an infinite sequence $\{c_\mu\}$. Justify why c_μ converges to some point C in $\mathbb{R}$. Hint: you will need some form of completeness from the Axiom of Completeness Equivalencies Theorem in footnote 14 here.

(ii) Show that C is in fact in $(a, b]$, hence $f(C)$ is defined. Justify why there exists a positive integer N such that $|f(C) - f(c_\mu)| < \varrho/2$ for all $\mu > N$.

(iii) Show that the previous step implies that $|f(c_{\mu+1}) - f(c_\mu)| < \varrho$ for any $\mu > N$. Conclude why this results in a contradiction.

We now know that we have a finite sequence $a < c_1 < \cdots < c_n \leq b$. We partition the interval $[a,\, b]$ accordingly, and use it (as did Dirichlet) to find a uniform bound on $|f(x) - f(x')|$ for any two points x, x' in $[a,\, b]$ that are sufficiently close to each other. To begin, pick

$$\sigma = \min\{|c_1 - a|, |c_2 - c_1|, |c_3 - c_2|, \ldots, |c_n - c_{n-1}|\}.$$

Suppose x and x' are points in $[a,\, b]$ such that $|x - x'| < \sigma$. Dirichlet next claimed that one of the following two cases would occur:

Case 1. Both x and x' occur *within* some interval $[c_\mu,\, c_{\mu+1}]$ for μ in $\{1, 2, \ldots, n-1\}$ or in $[c_n,\, b]$ (assuming $c_n < b$). This is shown in Figure 1 below.

Case 2. The points x and x' occur within the interval spanned by three consecutive c_μ's; that is, in $[c_{\mu-1},\, c_{\mu+1}]$ for $1 < \mu < n$ or in $[c_{n-1},\, b]$. This is shown in Figure 2 below.[15]

[15] These drawings appear in Pages 6–7 of [Dirichlet, 1904].

Fig. 1.

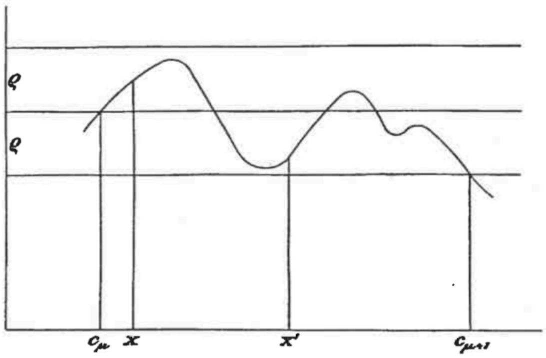
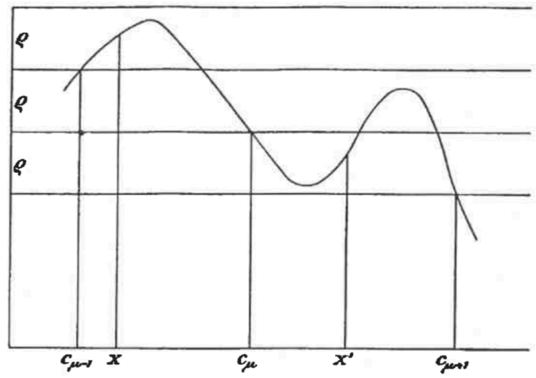


Fig. 2.



(g) Justify Dirichlet's claim that these are indeed the only two possible cases for our choice of x and x'. Moreover, for both the cases presented above, show (as did Dirichlet) that we have $|f(x) - f(x')| < 3\varrho$ for $|x - x'| < \sigma$. (You will have to apply the triangle inequality twice in one case.)

(h) Go back to the step where we picked σ. Why do you think it is important that we only had a finite sequence? Could we not have simply picked σ as the infimum of a possibly infinite sequence?

In the words of Dirichlet,[16]

> This completely proves our theorem. Because one replaces the given arbitrarily small size ϱ by $\frac{1}{3}\varrho \ldots$, ...

(i) Dirichlet claimed we are done and gave a suggestion as to why this is the case. Complete his explanation that we have now proved the theorem.

Now that we have completed the task of working through Dirichlet's proof, let's pause to reflect on it.

Task 4 Dirichlet's proof of the Uniform Continuity Theorem has many important ideas worth remembering when tackling proofs in analysis. What are some ideas that you think are worth remembering?

Task 5 Use the ideas presented in Task 3, to show that a continuous function $f(x)$ on $[a, b]$ is bounded. Hint: Let $\varrho = 1$, and show that there exists a finite sequence $c_1, \ldots, c_n$ with the given properties.

[16] See page 7 of [Dirichlet, 1904].

Task 6 Dirichlet's proof requires the fact that the interval $[a, b]$ is closed and bounded.

(a) In which parts of the proof did we assume a bounded interval?

(b) In which parts of the proof did we assume closure?

(c) Revisit the examples in Task 1, and reflect on how the procedure of the proof given in Task 3 fails in these cases.

Note that Dirichlet's proof also required the Intermediate Value Theorem, which is a consequence of the fact that the interval $[a, b]$ has a topological property that we call today as connectedness.[17] Therefore, Dirichlet's proof would fail for an arbitrary closed and bounded set that is not connected. However, the property of uniform continuity still holds in this case. Here is an example.

Task 7 Let A be the set of integers $\{1, 2, 3, 4, 5\}$. Define the function $f : A \to \mathbb{R}$ by $f(i) = i$ for $i \in A$.

(a) State why A a closed and bounded subset of $\mathbb{R}$.

(b) Show that f is uniformly continuous *on* A.

In a later project in the series "Understanding Compactness Through Primary Sources," we shall extend the Uniform Continuity Theorem to more general sets other than $[a, b]$. In the next section of this project, we turn instead to some early 20th-century work that gave us the tools to do that.

4 Borel's Theorem and Lebesgue's Extension

We switch now from arguments using sequences of real numbers (e.g., the sequence $\{c_\mu\}$ in Dirichlet's proof) to ideas involving the use of collections of open sets. This type of shift in the level of abstraction (i.e., from collections of real numbers to collections of *sets* of real numbers) was typical of mathematical thinking in the late-19th and early-20th century. In this section, we focus on the work of two contemporary French mathematicians of that period: Émile Borel (1871–1956) and Henri Lebesgue (1875–1941).[18] Both Borel and Lebesgue are considered to be the pioneers of modern measure theory, an area of mathematics that investigates one way of characterizing the size of sets, based on a generalization of concepts like length, area and volume. Both men also made other important contributions to the theory of real functions, including Lebesgue's highly influential generalization of the Riemann integral.[19]

[17] Intuitively, a connected set is one that "hangs together" without having any gaps. For example, all intervals in $\mathbb{R}$ are connected (but these are the only sets in $\mathbb{R}$ that are), while sets like $\mathbb{N}$, $\mathbb{Q}$ and the finite set in Task 7 are not.

[18] Brief biographies of Borel and Lebesgue can be found at [O'Connor and Robertson, 2008] and [O'Connor and Robertson, 2004].

[19] The Riemann integral itself was developed by the celebrated German mathematician Bernhard Riemann (1826–-1866) as a generalization of Cauchy's basic "sum of areas of rectangles" definition, which we discussed earlier in this project. Despite having several useful properties, late 19th-century mathematicians found that Riemann's version of integration was not perfect. Lebesgue's version of integration — which remains the current standard used in mathematical research today — improved upon certain weaknesses of Riemann's version. For an introduction to the Lebesgue integral and how it relates to the Riemann integral, see the project "Henri Lebesgue and the Development of the Integral Concept" (author Janet Heine Barnett), available at `https://digitalcommons.ursinus.edu/triumphs_analysis/2/`.

As mentioned in the Introduction section of this project, the concept of compactness is another way that mathematicians have developed to characterize the "size" or "finiteness" of a set. Although we have not yet defined compactness to see how this might be done, Borel stated and proved a theorem which will let us do so in an 1895 article "Sur quelques points de la théorie des fonctions" ("On some points in the theory of functions") [Borel, 1895]. In fact, the idea he presented in his statement was generalized later on to give us the modern open cover definition of compactness. Although he stated it for a different reason himself, Borel's Theorem also gives us a proof of the Uniform Continuity Theorem, as we will see below.

Borel stated his theorem as follows:[20]

> Here is the theorem: If one has on a straight line an infinite number of partial intervals, such that any point on the line is interior to at least one of the intervals, one can effectively determine a LIMITED NUMBER of intervals chosen among the given intervals and having the same property (any point on the line is interior to at least one of them).

By a "straight line" Borel meant a segment of the real line $\mathbb{R}$; we will deduce exactly what type of a segment in Task 8. By the term "partial intervals" Borel meant "open intervals" of the form (a, b), the phrase "Limited Number" meant a finite number. You may want to read Borel's statement again with these clarifications in mind.

Task 8

(a) Let's show that Borel's statement does not hold for the entire real line $\mathbb{R}$ by a counterexample. Consider the set of open intervals of the form $(-n, n)$ for $n = 1, 2, 3, \ldots$. Is every point of $\mathbb{R}$ interior to at least one such interval? Can a *finite* number of such intervals have the same property i.e. every point of $\mathbb{R}$ is contained in one of the intervals from such a finite set?

(b) Consider the set of open intervals of the form $(1/n,\ 1 + 1/n)$. Is every point of the interval $(0,\ 1]$ interior to at least one of these intervals? Show that a finite number of such intervals cannot exhibit the property claimed by Borel.

(c) From your answers to the previous parts, can you conclude what Borel meant by a "straight line"?

Morever, Borel also assumed, in his own proof of this theorem, that an infinite number of partial intervals (i.e., open intervals) involved only a *countably infinite* collection of such intervals [Andre et al., 2013]. Lebesgue later noticed that Borel's theorem could be extended to the general case of any collection of open intervals, both countable *and* uncountable.[21] For this reason, we will look

[20] See p. 51 of [Borel, 1895].

[21] The concept of cardinality is yet another way to characterize the size of sets, based on a generalization of counting that was developed by Cantor as a result of his work in analysis. Cantor's ideas surprised the late-19th century mathematical world (and dismayed some of its members) with the discovery that not all infinite sets have the same cardinality, where the two basic distinctions of infinite cardinalities are defined as follows.

Definition 4.1. Given a set A, we say that

i. A is *countable* if it has the cardinality of the set of natural numbers.

ii. A is *uncountable* if it is neither finite or countable.

at Lebesgue's proof of Borel's Theorem, which he presented in his book *Leçons Sur l'Intégration et la recherche des fonctions primitives* (*Lessons on integration and research on primitive functions*) [Lebesgue, 1902].[22] We begin with Lebesgue's statement of the theorem, which he attributed to Borel.

If we have a family of intervals Δ such that any point of an interval $(a,\, b)$, including a and b, is inside at least one of the Δ, there exists a family formed by a finite number of intervals Δ with the same property [i.e., any point of $(a,\, b)$ is inside one of them].

Notice that Lebesgue has made it explicitly clear that when he wrote $(a,\, b)$, he intended to include the end points. That is, he was referring to the closed interval $[a,\, b]$. *We will stick to the modern interval notations below.*

Another way of stating the role of the family of intervals[23] here is that the interval $[a,\, b]$ is contained in the union of all the open intervals in the family. We say that the family *covers* the interval $[a,\, b]$. The claim of the theorem is that we can find a finite number of intervals in the given family that also cover $[a,\, b]$.

Consider for example, the interval $[0,\, 1]$ and let $\mathcal{G} = \{(-1/n,\, 1+1/n) \mid n = 1, 2, 3, 4, \ldots\}$. Then $\mathcal{G}$ is a cover of $[0,\, 1]$. Indeed, it is easily seen any *one* of the intervals from $\mathcal{G}$ is sufficient to cover $[0,\, 1]$. Lebesgue claimed that no matter how many intervals might be in the family $\mathcal{G}$ that covers $[0,\, 1]$, we can always find a finite number of intervals from $\mathcal{G}$ that also do the job.

On the other hand, this property will not hold for an interval that is either not closed or bounded. We considered some examples in Task 8. Here is another such example. Consider the open interval $(0,\, 1)$ and let $\mathcal{G} = \{(1/n,\, 1-1/n) \mid n = 1, 2, 3, 4, \ldots\}$. Convince yourself that $\mathcal{G}$ is a cover of $(0,\, 1)$. Clearly, no finite number of intervals from this family could cover $(0,\, 1)$.

Let's prove Borel's theorem in the next task by essentially following Lebesgue's argument.

Task 9 Let $\mathcal{G}$ be a family of open intervals that covers the interval $[a,\, b]$.

Letting $(\alpha,\, \beta)$ be one of the intervals Δ [from the given family] that contains a, the property to be proven is obvious for the interval $[a,\, x]$, if x is between α and β; by this, I mean that this interval[24] can be covered using a finite number of intervals Δ [from the given family], which I express by saying that the point x has been reached. We must prove that b is reached.

[22] See Chapter 7, p. 105 of [Lebesgue, 1902].
[23] Note that Lebesgue used 'Δ' to represent an arbitrary interval in the family.
[24] By "this interval," Lebesgue meant the interval $[a, x]$.

Following Lebesgue, we will say that a point x in $[a,\, b]$ has been *reached* if the subinterval $[a,\, x]$ can be covered by a finite number of intervals from the given family $\mathcal{G}$. Our ultimate goal is to show that b is a reached point.

(a) State why a is a reached point.

(b) If x is a reached point, state why all the points in between a and x are also reached points.

To prove that b is a reached point, Lebesgue set up a contradiction.

If x is reached, all points in $[a,\, x]$ are reached; if x is not reached, one of the points of $[x,\, b]$ is not reached. There is therefore, if b is not reached, a first point not reached, or a last point reached; let x_0 be this point.

As you likely suspected when you read this reference to a "first point not reached, or a last point reached," a modern version of Lebesgue's proof will use some form of completeness to fill in the details of this part of his argument. Let's do this before we continue.

Consider the set of all reached points within the bounded interval $[a, b]$:

$$A = \{x \in [a,\, b] \;\mid\; x \text{ is reached}\}.$$

(c) State why A is non-empty and bounded above.

(d) State why the point x_0 exists, and why all points $x < x_0$ are reached points. Hint: consider the supremum of A.

Note that at this juncture in the proof, we do not know if x_0 itself is or is not a reached point. We do, however, know that $a < x_0 \leq b$. (*Make sure you see why!)* Let's suppose that $x_0 < b$ (so that, as Lebesgue assumed above, b is not reached), and see how this leads to a contradiction.

[This point x_0] is within an interval Δ [from the given family], (α_1, β_1). Let x_1 be a point of (α_1, x_0) [and] x_2 a point of (x_0, β_1); x_1 is reached by hypothesis, [so that] the finite number of intervals Δ [from the given family] which serve to reach it, plus the interval (α_1, β_1), allow [us] to reach $x_2 > x_0$: x_0 is [then] neither the last point reached nor the first not reached; therefore b is reached.

(e) State why x_2 is not reached.

(f) Using the fact that x_1 is reached, show that x_2 can in fact be reached, arriving at a contradiction. Conclude that $x_0 = b$.

(g) Finally, complete the proof by showing that b is reached. Hint: we know x_1 is reached.

After completing the proof of Borel's Theorem that we have just worked through, Lebesgue added a footnote[25] in which he stated that the use of Borel's theorem gives a *nice* proof of uniform continuity of a continuous real-valued function on $[a, b]$.

We have deduced from the theorem, as stated in the text, a nice demonstration of the uniformity of continuity.

He then proceeded to give that proof in the same footnote. Let's work through his proof and compare it to Dirichlet's proof in Task 3.

Task 10 To begin, let's read Lebesgue's proof of the Uniform Continuity Theorem in its entirety.

Let $f(x)$ be a function continuous at all points of $[a, b]$, including a and b: each point of $[a, b]$ is, by definition [of continuity], inside an interval Δ in which the oscillation of $f(x)$ is less ϵ. Using a finite number of these [intervals], we can cover $[a, b]$; letting l be the length of the smallest interval used, in any interval of length l the oscillation of f is at most 2ϵ, because such an interval spans at most two [of the] intervals; the continuity is uniform.

Let's now verify the details of Lebesgue's very concise argument.

Letting $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, we wish to show that f is uniformly continuous. To this end, let $\epsilon > 0$.

(a) Use the continuity of f to define a set of values $\delta_x > 0$ and a collection (or family) of open intervals $I_{\delta_x}(x) = (x - \delta_x, x + \delta_x)$ "in which the oscillation of $f(x)$ is less ϵ;" that is, a collection of open intervals $I_{\delta_x}(x)$ for which f lies within the range $(f(x) - \epsilon/2,\ f(x) + \epsilon/2)$ for all input values from the interval $I_{\delta_x}(x)$.

(b) Justify why you can pick a finite subcover of $[a, b]$ from the collection of $I_{\delta_x}(x)$ intervals.

[25]See p. 105 of [Lebesgue, 1902].

Let this finite subcover be $\{I_{\delta_1}(x_1), \ldots, I_{\delta_n}(x_n)\}$, with radii $\delta_1, \ldots, \delta_n$. Now define

$$\delta = \min\{\delta_1, \ldots, \delta_n\}.$$

(c) Suppose we pick any two points x and y such that $|x - y| < \delta$. Argue why x and y must belong either to the same interval $I_{\delta_j}(x_j)$ or belong to two intersecting intervals $I_{\delta_j}(x_j)$ and $I_{\delta_k}(x_k)$. Hint: assume that x and y *neither* belong in the same interval *nor* belong to two intersecting intervals, then derive a contradiction.

(d) Show that given any interval $I_\delta(x)$ of radius δ centered at a point x, the range of f is within 2ϵ of the point $f(x)$. Explain why this allows us to now conclude that f is uniformly continuous on $[a, b]$.

(e) Compare and reflect on this proof versus Dirichlet's proof in Task 3. Why was Dirichlét's proof longer? In what ways are the proofs similar? In what ways are they different? What characteristics of the proof that Lebesgue gave do you think convinced him that using Borel's Theorem is a "nice" way to prove the Uniform Continuity Theorem?

Task 11 (Optional). This task takes us on a quick detour to see an example of how Lebesgue applied Borel's Theorem in connection with his primary interest: the development of a new theory of integration (i.e., Lebesgue integration) and the associated theory of measure that he needed to define that new integral.

As noted earlier, the goal of measure theory is to generalize the concepts of length, area, and volume in order to try to assign a number that measures the "size" of a set E, no matter how complicated that set may be. As a key part of Lebesgue's approach to doing this for sets of real numbers, he used open intervals to define two numbers called the *outer measure of* E (denoted $m_e(E)$) and the *inner measure of* E (denoted $m_i(E)$). A set is then said to be measurable if these two numbers are equal.

Open covers of a set are used to define its outer measure in the following way.

> Let $E \subseteq \mathbb{R}$ and let $\mathcal{G}$ be a *countable* cover of E by open intervals (a, b).
> *(The restriction to a countable cover is because we will be taking a sum.)*
> We take the length of an open interval (a, b) to be $l((a, b)) = b - a$.
>
> We then define the outer measure of E to be
>
> $$m_e(E) = \inf_{\mathcal{G}} \sum_{G \in \mathcal{G}} l(G).$$
>
> That is, for each specific countable cover $\mathcal{G}$ of E by open intervals, we take the sum of the lengths of all the open intervals in the cover $\mathcal{G}$. *(Note that this sum may be infinite for some covers.)*
> The outer measure of E is then defined as the infimum of the set of all such sums.

Now let $E = [a, b]$. Complete the following to show that $m_e(E) = b - a$.

(a) Argue why $m_e(E) \leq b - a$.

(b) Use Borel's Theorem to argue why $m_e(E) \geq b - a$.

5 The Open Cover Definition of Compactness

Note that open sets in $\mathbb{R}$ are the union of open intervals *(can you reason why?)*. Hence, we may take the family $\mathcal{G}$ that covers the interval $[a, b]$ in Lebesgue's statement of Borel's theorem to be a family of open sets in $\mathbb{R}$. The same proof given for that theorem in Task 9 will then still work for this more general setting.[26]

In modern terms, we say that a family of open sets $\mathcal{G} = \{G_\alpha \mid \alpha \in \Lambda\}$ (where Λ is an index set) is an open cover of a set A provided A is contained in the union of all the sets from $\mathcal{G}$; that is, $A \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha$. The family $\mathcal{G}$ is said to be a finite cover if it is finite. A sub-collection of sets from $\mathcal{G}$ is said to be a subcover of A if A is contained in the union of sets from this sub-collection.

Task 12 Restate Lebesgue's statement of Borel's theorem using open covers.

We also say that a subset A of $\mathbb{R}$ has the open cover property if every open cover of A has a finite subcover. Using this terminology, Borel's Theorem simply asserts that every closed bounded interval $[a, b]$ of $\mathbb{R}$ has the open cover property. It turns out that every closed subset of $[a, b]$ will also have the open cover property. This follows from the next task, in which you will prove the more general fact that if A is *any* set that satisfies the open cover property, then every closed subset of A also has the open cover property.

Task 13 Let $A \subseteq \mathbb{R}$ have the open cover property, and $K \subseteq A$ be closed in $\mathbb{R}$. Let $\mathcal{G}_\alpha$ be an open cover of K. We wish to find a finite subcover of K.

(a) Find an open cover $\mathcal{G}$ of A by extending the open cover $\mathcal{G}_\alpha$ of K. Hint: The set K is closed in $\mathbb{R}$ by assumption.

(b) There exists a finite subcover of A of the open cover in the previous step. Show that we can extract a finite subcover of K from this. Conclude that K has the open cover property.

Let's pause to think about these ideas in connection to the paragraph that started this project:

> When you hear the word "compact," you most likely think of something that is small and takes up little space, or perhaps a collection of things that are closely packed together. Usually in mathematics, the simplest case in examining certain properties of a set is when the set is finite. It is also the case in mathematics, that we can extend such properties of a finite set to more general sets also satisfying similar properties, but under certain constraints placed on the sets.

Sets with the open cover property may not be small in terms of their cardinality — indeed, every interval $[a, b]$ contains uncountably many points — but they do have the helpful property that no

[26] In fact, we can just use Borel's Theorem directly (without recreating the proof from Task 9) to get a quick proof for this more general case. If $\mathcal{G}$ is an open cover of $[a, b]$, then we can define a cover consisting of open intervals by setting

$$\mathcal{G}' = \{(\alpha, \beta) \mid (\alpha, \beta) \subseteq G \text{ for some } G \in \mathcal{G}\}.$$

Borel's Theorem allows us to choose a finite subcover of $[a, b]$ by intervals from $\mathcal{G}'$, which we can then use to obtain a corresponding finite subcover of open sets from $\mathcal{G}$.

matter how many opens sets we use to cover them, that open cover can be reduced to just a finite subcover. And, just as we expect a subset of a finite set to be finite, a closed subset of a set with the open cover property will itself have the open cover property. With these comments in mind, we offer the following definition of compactness.

Definition 5.1 (Open Cover Compactness)**.** A set $A \subseteq \mathbb{R}$ is compact if and only if A has the open cover property.

With this definition in hand, we can now state the modern version of Borel's theorem, or as it is more typically called today, the Heine-Borel Theorem.[27]

Theorem 5.1 (Heine-Borel)**.** *A subset A of $\mathbb{R}$ is closed and bounded if and only if it is compact.*

Note first of all that the Heine-Borel Theorem generalizes the original theorem given by Borel by asserting that *all* closed and bounded subsets of $\mathbb{R}$ — and not just intervals of the form $[a, b]$ — are compact (i.e., have the open cover property). Note also that the Heine-Borel Theorem asserts that the *converse* of this generalized version of Borel's Theorem holds; that is, a subset A of $\mathbb{R}$ which is compact (i.e., has the open cover property) must also be closed and bounded, as well as vice versa. In the next two tasks, we will prove both these extensions of Borel's original theorem.[28]

Task 14 Let's prove that Borel's original theorem generalizes to all closed and bounded sets, as stated in the Heine-Borel Theorem. Most of the work has already been done. From Borel's theorem, we know that any interval $[a, b]$ is compact (i.e., satisfies the open cover property). Extend Borel's theorem by showing any closed and bounded $K \subseteq \mathbb{R}$ is compact using Task 13.

[27] Borel's name is attached to this theorem for reasons that we have seen in this project; namely, the term "compact" simply means "every open cover has a finite subcover" and Borel was the first to publish a statement and proof of the key ideas asserted in today's Heine-Borel Theorem. The name of German mathematician Eduard Heine (1821–1881) came to be associated with this theorem as a result of a proof of the Uniform Continuity Theorem that Heine published in 1872, in which he used a method similar to that which Borel used to prove his version of Borel's Theorem. This situation is somewhat similar to what occurred in the case of Bolzano's name becoming inaccurately attached to the Bolzano-Weierstrass Theorem (as discussed in footnote 14); in this case, however a number of mathematicians (and especially Lebesgue) objected to assigning the name "Heine" to a theorem that Heine himself never stated or proved. A more complete history of the Heine-Borel theorem along with a discussion of its various versions and their proofs can be found in [Andre et al., 2013] and [Hildebrandt, 1926].

[28] The Heine-Borel theorem also holds in $\mathbb{R}^n$ under the Euclidean metric. The required extra step is to show that any closed and bounded hypercube in $\mathbb{R}^n$ satisfies the open cover property. You may refer to, for example, p. 39 of [Rudin, 1976].

Task 15 Now let's prove that the converse of the generalized version of Borel's original theorem also holds — that is, a subset A of $\mathbb{R}$ is compact only if A is closed and bounded — as stated in the Heine-Borel Theorem. We follow the proof presented in Rudin [1976].[29]

To begin, let $A \subseteq \mathbb{R}$ be compact.

(a) We first show that A is closed in $\mathbb{R}$ by showing its complement, A^c, is open. That is, given a point y in A^c, we will find an open interval around y that is contained in A^c.

  (i) For each x in A, pick $r = |x - y|/2$ and consider the interval $I_r(x) = (x - r, x + r)$. Justify the claim that there exist finitely many such intervals $I_{r_1}(x_1), \ldots, I_{r_n}(x_n)$ that cover A.

  (ii) Consider the intervals $I_{r_1}(y), \ldots, I_{r_n}(y)$, and take their intersection $W = I_{r_1}(y) \cap \cdots \cap I_{r_n}(y)$. Justify the claim that $W \cap A = \emptyset$. State why this completes the proof that A is closed.

(b) Now show that A is bounded. Hint: Use contradiction to produce an open cover with no finite subcover.

While the open cover definition of compactness encapsulated by Borel's Theorem continues to serve as a useful tool in analysis, later mathematicians developed other ways to capture this idea of "finiteness" in connection to their efforts to generalize the Uniform Continuity Theorem as well as the Extreme Value Theorem. Other projects in this series will examine these alternative notions of compactness, their connections to those two theorems, and the ways in which these ideas influenced, and were influenced by, the concept of a general topological space which evolved in the 20th century. For example, we will see how the steps outlined in Task 15 above can be generalized to any topological space satisfying a property known as the Hausdorff condition. Spaces with this property include, for example, Euclidean n-space $\mathbb{R}^n$, as well as other so-called metric spaces in which we are able to define a notion of distance. For those who have already studied the basic properties of metric spaces, we close this project with a task that foreshadows the ongoing trend of generalization that will be explored in future installments of the "Understanding Compactness through Primary Sources" series.

[29] See p. 37 of Rudin [1976].

Task 16 (Optional). This task requires familiarity with the basic theory of metric spaces. For such a space X with metric d, the definitions of open cover and compactness in $\mathbb{R}$ generalize in the natural way, with open intervals of the form $I_r(x) = (x - r, x + r)$ replaced by open balls centered at $x \in X$, defined as $B_r(x) = \{y \in X \mid d(x, y) < r\}$.

(a) Let (X, d) be a metric space and $A \subseteq X$ be compact. Generalize the steps in Task 15 to show that A is a closed subset of X, where a set is closed if it contains all its limit points.

(b) Let X be an infinite set with the discrete metric. That is,

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Show that X is not compact.

(c) Using the same argument as in Task 13, show that any closed subset of a compact metric space is compact.

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6 Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is ideally suited for a first or second semester course in undergraduate analysis. It could be covered for example in the final two weeks of a first semester course or at the beginning of a second semester course. It could also be used as a review for students entering graduate school. Many of the presented tasks and their solutions have been taken directly from primary historical sources. However, it should be noted that the ideas and techniques used are in no way obsolete. In fact, they are standard techniques students of mathematics are expected to master in order to demonstrate mathematical maturity.

The primary goal of the PSP is to understand the idea of open cover compactness in $\mathbb{R}$. In particular, students should be able to understand

a. The role of the completeness property of $\mathbb{R}$ and its connection to compactness.

b. That the open cover definition of compactness is a form of a "finiteness condition." That is, it leads to interesting properties that could be viewed as an extension of the properties exhibited by finite subsets of $\mathbb{R}$ such as

   i. Any compact (finite) subset of $\mathbb{R}$ is closed and bounded.
   ii. Any closed (finite) subset of a compact (finite) set is compact (finite).
   iii. Any continuous real valued function on a compact set achieves a maximum and minimum (so does any real valued function on a finite set).
   iv. Any continuous function on a compact set is uniformly continuous (as is any function on a finite set).

c. That the open cover property in $\mathbb{R}$ is equivalent to being closed and bounded (Heine-Borel Theorem).

Moreover, the PSP aims to expose students to using topological arguments based on open and closed sets.

Student Prerequisites

Students are expected to have had a beginning level exposure to real analysis, as covered by the introductory part of a first semester course. It is assumed that students have sufficient experience with using the $\epsilon - \delta$, $\epsilon - N$ definitions, and are comfortable with concepts such as sup and inf. It is also assumed that students have some prior familiarity with the concept of uniform continuity, and a brief review is given in Section 2. In Section 3, Task 7 requires an understanding of the induced metric on a subset A of $\mathbb{R}$. With this metric and $A = \{1, 2, 3, 4, 5\}$ as given in Task 7, every subset of A is open in A. The instructor should discuss this in class if required.

PSP Design and Task Commentary

The PSP develops the idea of open cover compactness by examining various proofs of the Uniform Continuity Theorem, and the Heine-Borel Theorem. In particular, some early work of Dirichlet on uniform continuity is presented in detail. Dirichlet's proof of the Uniform Continuity Theorem contains key ideas that sets the stage for Borel's definition of the open cover property. We then examine Lebesgue's discussion of Borel's open cover property, and his proof that any closed and bounded interval in $\mathbb{R}$ satisfies such a property. Lebesgue's short and elegant proof of the Uniform Continuity Theorem using the open cover property is presented and compared to Dirichlet's earlier proof. The PSP ends with the modern definition of open cover compactness and the proof of the Heine-Borel Theorem.

- Section 2 introduces the idea of uniform continuity. Tasks 1 and 2 discuss examples where uniform continuity is lost when we don't have a closed or bounded interval. For example, one could consider the function $f(x) = 1/x$ which is continuous but not bounded on (0, 1].

  The function $\sin\left(\dfrac{1}{1-x}\right)$ is an example of a bounded continuous function on $[0, 1)$ that is not uniformly continuous. This function oscillates between 1 and -1 with infinitely many zero crossings.

  Dirichlet gives the function $f(x) = \sin(x^2)$ as an example of a bounded continuous function that is not uniformly continuous. Students could be asked to modify this example of Dirichlet to produce other functions with a similar behavior. Any function with a sequence of sharper and sharper spikes would work. For example, consider

  $$f(x) = \begin{cases} 2n(x - n + \frac{1}{n}) & x \in [n - \frac{1}{n}, n - \frac{1}{2n}] \\ -2n(x - n) & x \in [n - \frac{1}{2n}, n] \\ 0 & \text{else ,} \end{cases}$$

  for $n = 1, 2, ...$

- In Section 3, we examine Dirichlet's proof of the Uniform Continuity Theorem. In Task 4, students are asked to reflect on the ideas presented in the proof. Here are some key ideas from Dirichlet's proof for students to reflect on

  (i) The real numbers R is an ordered field that exhibits the least upper bound property. In other words, we can take the *inf* or *sup* of a non-empty bounded set of real numbers.

  (ii) When given a set that is possibly infinite, we may be able to argue that the set is finite or else reduce the problem to a finite set. This could make our work easier.

  (iii) Sometimes, we may require multiple applications of the triangle inequality. Mathematicians fondly call these $\epsilon/3$ tricks, etc.

  (iv) When doing an ϵ-N or ϵ-δ argument, it's okay if the final inequality is not strictly less than ϵ as needed but is some multiple of it. We could go back and adjust our initial parameters.

- In Section 4, we discuss Borel's formulation of the open cover definition of compactness, and Lebesgue's proof of the open cover property of a closed and bounded interval and the Uniform Continuity Theorem. The key idea for students to realize here is that the open cover property allows us to pick a "finite number of open intervals," which reduces our arguments to be done over a finite set.

  For example, in Task 10 it is the use of Borel's theorem that allowed Lebesgue to pick a finite subcover and hence a δ such that the oscillation of f was within 2ϵ at any given point x.

Dirichlet on the other hand, had to *prove* that an appropriate finite subdivision of the interval $[a, b]$ was possible in order to make similar choices. Students are asked to reflect on this point.

Task 11 is given as an optional exercise. Here too, Lebesgue exploits the finiteness condition given by Borel's open cover property. Let $E = [a, b]$. Then any interval of the form $(a - \epsilon/2, b + \epsilon/2)$ is an open cover of E. Hence, $m^*(E) \le b - a + \epsilon$. On the other hand, let $\mathcal{G} = \{G_n\}$ be a countable open cover of E. By Borel's theorem we can pick a finite subcover, say $\{G_{n_1}, ..., G_{n_k}\} \subseteq \mathcal{G}$. Then note that

$$\sum_n l(G_n) \ge l(G_{n_1}) + ... + l(G_{n_k}).$$

Argue why for any such finite cover, $l(G_{n_1}) + ... + l(G_{n_k}) \ge b - a$. So,

$$\sum_n l(G_n) \ge b - a.$$

Taking the infimum, we get the desired result.

- In Section 5 we present the modern open cover definition of compactness and the Heine-Borel Theorem. It would be pertinent to point out to students here the essentially topological nature of both the statements considered and their proofs. Task 16 is optional, and generalizes the open cover property to metric spaces. Even if students have not had previous exposure to metric spaces, the instructor could define it and discuss how the ideas, theorems and proofs generalize in a natural way.

Suggestions for Classroom Implementation

The PSP would ideally require six 50-minute class room periods, i.e. two weeks of a regular semester course meeting three times a week. The class discussions could cover Sections 1 to 3 in the first week, and the later sections in the second week. The PSP is well suited for group work and class discussions.

Sample Implementation Schedule (based on a 50-minute class period)

The following schedule is based on six class periods.

- **Day 1:**
  Advance preparation: Read Sections 1 & 2 and complete Task 1.

  In-class: Working in small groups, compare answers to Task 1 and complete Task 2. A whole-class discussion on uniform continuity, based on the Tasks 1 and 2, could follow. Time permitting, students could also be asked to use the examples and ideas presented in these tasks to construct other examples of continuous functions that fail to be uniformly continuous, or to begin reading Section 3.

- **Day 2:**
  Advance preparation: Read the beginning of Section 3 and complete parts (a), (b) of Task 3.

  In-class: Working in small groups, compare answers to Task 3(a), (b) and continue working on the remaining parts of that Task.

- **Day 3:**
  Advance preparation: Complete any remaining parts of Task 3 and prepare notes for discussion of Tasks 4 and 6.

  In-class: Working in small groups, compare answers to any parts of Task 3 that were not completed on Day 2. By the end of this period, students should have completed Section 3 (with the exception of Tasks 5 & 7, which could be assigned as individual homework), and understand Dirichlet's proof of the Uniform Continuity Theorem. As Task 5 asks students to write a proof that can be used to assess whether they have achieved that understanding, it makes for a good homework problem. It would be worthwhile to spend some time reflecting on the central ideas of the proof, based on the prompts in Tasks 4 and 6. This could be done first in groups, and then having a general class discussion.

  Homework (need not be due on Day 4): Task 5, Task 7.

- **Day 4:**
  Advance preparation: Read the beginning of Section 4 (stopping just above Task 9) and complete Task 8 for class discussion.

  In-class: Working in small groups, compare answers to Task 8, then work on Task 9.

- **Day 5:**
  Advance preparation: Complete any remaining parts of Task 9 and also parts (a) and (b) of Task 10.

  In-class: Working in small groups, compare answers to any parts of Task 9 that were not completed on Day 4. By the end of this period, students should have completed Section 4 on the ideas of Borel and Lebesgue. Students should be asked to compare and contrast Lebesgue's proof of the Uniform Continuity Theorem versus Dirichlet's proof, as prompted in Task 10(e). This could be done first in groups, and then having a general class discussion.

  Optional Homework: Task 11.

- **Day 6:**
  Advance preparation: Read the beginning of Section 5 (stopping just above Task 14), but completing Tasks 12 and 13 for class discussion along the way.

  In-class: Working in small groups, compare answers to Tasks 12 and 13, then work on Tasks 14 & 15. Any remaining work on these two Tasks can be assigned as homework. Time permitting, the instructor could take an extra day to introduce the concept of metric spaces. Task 16 shows how the ideas presented previously generalize in a natural way, and is a precursor for things to come.

  Homework : Complete formal write up of Tasks 14 & 15.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be completed in at most 2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from https://digitalcommons.ursinus.edu/triumphs_topology; all other listed projects are available at https://digitalcommons.ursinus.edu/triumphs_analysis.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Heine Barnett)
- *Investigations into Bolzano's Bounded Set Theorem* (David Ruch)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)
  Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)
  A second version of this project suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real Analysis* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Mean Value Theorem*(David Ruch)
- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Heine Barnett)
- *Euler's Rediscovery of e** (David Ruch; sequence convergence, series & sequence expressions for e)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.