Spring 2020

Investigations into Bolzano's Bounded Set Theorem

Dave Ruch
ruch@msudenver.edu

Investigations into Bolzano's Bounded Set Theorem

David Ruch*

February 21, 2023

The foundations of calculus were not yet on firm ground in the early 1800s. Mathematicians such as J. L. Lagrange (1736–1813) made efforts to put limits and derivatives on firmer logical ground, but were not entirely successful. It took even longer for mathematicians to fully develop the notion of completeness of the real numbers.

Bernard Bolzano (1781–1848) was one of the great success stories of the foundations of analysis. He was a theologian with interests in mathematics and a contemporary of Gauss and Cauchy, but was not well known in mathematical circles. Despite his mathematical isolation in Prague, Bolzano was able to read works by Lagrange and others, and published mathematical work of his own.

This project investigates a key result from his important pamphlet *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*[1] [Bolzano, 1817]. We will read excerpts from this paper related to Bolzano's main theorem on a property of bounded sets. The proof of that theorem, discussed in Section 3 of this project, gives some insight into the completeness of the real numbers. Bolzano's proof also inspired Karl Weierstrass decades later in his proof of what is now known as the Bolzano-Weierstrass Theorem.[2]

1 Bolzano's Property S for Series

Before proving the main results of his 1817 pamphlet, Bolzano developed some preliminaries. In Sections 1–9 of that work, he discussed series and their convergence. He introduced his own partial sum notation, quite different from the modern sigma notation:

$$A + Bx + Cx^2 + \cdots + Rx^n = \overset{n}{F}x. \tag{1}$$

He also talked about finite geometric series and their sums. In Section 6 of his 1817 paper, Bolzano created the sequence of partial sums[3]

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \cdots, \overset{n}{F}x, \cdots, \overset{n+k}{F}x, \cdots \tag{2}$$

*Department of Mathematical and Computer Sciences, Metropolitan State University of Denver, Denver, CO; emailruch@msudenver.edu.

[1]The title of Bolzano's pamphlet translates into English as *A Purely analytic Proof of the Theorem, that between any two Values that give opposite [sign] Results, there lies at least one real Root of the Equation.*

[2]According to Kline, Bolzano's proof method "was used by Weierstrass in the 1860s, with due credit to Bolzano, to prove what is now called the Bolzano-Weierstrass theorem"[Kline, 1972, p. 953].

[3]For the convenience of modern readers, we have made minor variable changes in the source material, such as replacing Bolzano's use of "r" by "n" or "k."

and discussed the:[4]

special property that the difference between its nth term $\overset{n}{F}x$ and any subsequent $\overset{n+k}{F}$, no matter how distant it is from the nth [term], remains smaller than any given quantity once n is taken to be large enough.

Throughout his 1817 paper, Bolzano always set $x = 1$ in $\overset{n}{F}x$, so for ease of reading we will use the modern subscript notation

$$F_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{i=0}^{n} a_i \,(1)^i \tag{3}$$

in place of the Bolzano's notation[5] $\overset{n}{F}x$ in (1) and (2) where $x = 1$.

Task 1 Consider the example series where $x = 1$ and $a_i = 1/2^i$ so

$$F_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$$

(a) Find F_2, F_3, F_4.

(b) Using your introductory calculus course knowledge, find the limit of the sequence $\{F_n\}_{n=1}^{\infty}$.

(c) Do you think this sequence $\{F_n\}_{n=1}^{\infty}$ has the "special property" Bolzano defined in the source excerpt above? Justify your answer.

Task 2 Bolzano discussed the example of the series where $x = 1$ and the sequence of partial sums (2) is

$$0.1, 0.11, 0.111, 0.1111, \ldots$$

and claimed that "the quantity which the terms approach as closely as desired" is $1/9$.

(a) For this example, what are the values of A, B, C, D in (1)? Find a general formula for a_n in (3) in terms of n for this example.

(b) Write $F_{n+k} - F_n$ using sigma notation.

(c) Using your introductory calculus course knowledge, explain how Bolzano got $1/9$ as the limit of this sequence.

(d) Do you think this sequence $\{F_n\}_{n=1}^{\infty}$ has the "special property" Bolzano defined in the source excerpt above? Justify your answer.

[4] All translations of Bolzano excerpts in this project were prepared by Michael P. Saclolo, St. Edward's University, 2023, with minor changes made by the project author for readability. The translations [Russ, 1980] and [Russ, 2004, pp. 251–278] were also consulted by the project author.

[5] See [Russ, 1980] for a discussion of Bolzano's notational choices.

We will see later in the project that Bolzano was especially interested in using geometric series to prove his major theorem on a property of certain bounded sets of real numbers. In Section 5 of his paper, he stated the well-known finite geometric series summation formula

$$ar^{n+1} + ar^{n+2} + \cdots + ar^{n+k} = ar^{n+1} \cdot \frac{1-r^k}{1-r} \tag{4}$$

whenever $r \neq 1$.

Task 3 Use formula (4) for the series in Task 2 with $x = 1$ to find:

(a) $F_{n+k} - F_n$ for $n = 4$, $r = 3$.

(b) the minimal n for which $|F_n - F_{n+k}| < 0.003$ holds for all $k \in \mathbb{N}$.

Part (b) of the previous task is an illustration of Bolzano's "special property,"with "given quantity" 0.003, defined in the source excerpt above. We will now formally name this attribute to be *Property S*, where the letter S is chosen to remind us of these special series.

Definition. A series with sequence of partial sums $F_1, F_2, F_3, \ldots$ has **Property S** if the difference between term F_n and every later term F_{n+k} stays smaller than any given quantity, provided n has first been taken large enough.

Task 4 Write Property S in modern notation using quantifiers $\forall$ and $\exists$.

Task 5 Use formula (4) to prove that the series in Task 2 has Property S.

Task 6 Use formula (4) to prove that the series

$$1 + 1/2 + 1/2^2 + 1/2^3 + \cdots$$

from Task 1 has Property S.

Task 7 Use formula (4) to prove that any geometric series with $|r| < 1$ has Property S. You may assume that $\lim_{n\to\infty} r^{n+1} = 0$.

If you have studied Cauchy sequences, you should try the next task.

Task 8 If a series with sequence of partial sums $F_1, F_2, F_3, \ldots$ has Property S, must the sequence be Cauchy? Conversely, if a sequence of partial sums $F_1, F_2, F_3, \ldots$ is a Cauchy sequence, must the series have Property S?

After some discussion of standard geometric series in Section 5 of his paper, Bolzano argued that:

Therefore every geometric progression whose ratio[6] is a proper fraction can be continued so far that the increase that can then be accrued through every further continuation must remain smaller than any given quantity. This must apply all the more to a series whose terms decrease even more than those of a decreasing geometric progression.

We shall see in Section 2 of this project that Bolzano was particularly interested in showing the convergence of series of the form

$$u + \sum_{j=0}^{\infty} D/2^{m_j} \tag{5}$$

where $u, D > 0$ and $\{m_j\}$ is a strictly increasing sequence of integers, $m_{j+1} > m_j > m_0 \geq 1$ for all j. For example, the following series has the form (5):

$$7 + \frac{3}{2^1} + \frac{3}{2^{1+3}} + \frac{3}{2^{1+3+5}} + \frac{3}{2^{1+3+5+7}} + \frac{3}{2^{1+3+5+7+9}} + \cdots . \tag{6}$$

Task 9 Identify u, D, m_0, m_1, m_2 for the example series (6) above.

Task 10 Prove that series of form (5) satisfy Property S.

In Section 7 of his paper, Bolzano attempted to prove that any series with Property S must converge to a unique real number. As you saw in Task 10, this would imply that any series with form (5) must converge to a unique real number. Perhaps this seems obvious to you: if the terms of a sequence $F_1, F_2, F_3, \ldots$ are getting closer together, surely they must have some real number limit L. However, mathematicians after Bolzano's time have shown that the existence of limit L depends on a deeper "completeness" assumption that the real number line has no "holes" in it. The details of this "completeness" assumption are quite complex, so for the purposes of this project we will distill Bolzano's discussion on this topic into the following axiom, which we will name Axiom C for "**C**ompleteness."

Axiom C. Any series of the following form has a unique real number sum (limit):

$$u + \sum_{j=0}^{\infty} \frac{D}{2^{m_j}},$$

where $u, D > 0$ and $\{m_j\}$ is a strictly increasing sequence of integers.

This completeness Axiom C will be essential for proving Bolzano's big result in the next section of this project. *We will assume the truth of Axiom C as a fact.*

[6] *Translation note:* Bolzano used the German word "Exponent" here, which is generally translated as "exponent." However, it is clear from the context of his discussion that he was referring to the ratio of a geometric series.

2 Bolzano's Bounded Set Theorem

We are now ready to examine the main theorem for this project.

The theory of sets had not been developed during Bolzano's era, so he did not use the same set theoretic language we might expect in a modern discussion of his ideas. As you read the next excerpt from Bolzano's pamphlet, think about how you could translate his ideas into set terminology.

§11

Prelude. In the studies of applied mathematics the case happens on occasion that one learns about a [nonnegative][7] variable quantity x, all of whose values less than a certain u have a particular property M, without at the same time learning the property no longer applies to those values greater than u. In such cases there can exist perhaps some u' that is $> u$, for which all values x below it have property M, in much the same way that applies to u. Indeed, this property can perhaps apply to all x without exception. If on the other hand one learns that M does not apply to all x at all, then from the combination of these two statements one will now be justified to conclude that there is a certain value U that is the largest of which it is true that all smaller x have property M. The following theorem demonstrates this.

§12

Theorem. If property M does not apply to all values of a [nonnegative] variable quantity x, but rather to all such that are less than a certain u, then there is always a quantity U that is the largest among them for which it can be asserted that all smaller x have property M.

Before reading Bolzano's proof of this theorem in his Section 12, let's look at some examples of this concept he was discussing.

Task 11 Let M be the property "$x^2 < 3$" applied to the set $\{x \in \mathbb{R} : x \geq 0\}$.

(a) Find rational numbers u, u' such that the property M applies to all smaller nonnegative values for this example and $u < u'$. (The values of u, u' are not unique.) What is the value of "the quantity U that is the largest among them for which it can be asserted that all smaller x have property M" for this example?

(b) Let S_M be the set of ω values for which all ω' values satisfying $0 \leq \omega' < \omega$ possess property M. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property M?

(c) Does U possess property M?

[7] Bolzano intended to discuss only $x \geq 0$ in this note and his Section 12 theorem statement. The term "nonnegative" has been included in this project for clarity.

Task 12 Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$, and let $\alpha \in \mathbb{R}$ be arbitrary. Let M be the property "$f(\alpha + \omega) \leq f(\alpha) + 2$" applied to the set $\{\omega \in \mathbb{R} : \omega \geq 0\}$.

(a) Find rational numbers u, u' such that the property M applies to all smaller nonnegative values for this example and $u < u'$. Are these values unique? What is the value of U for this example?

(b) Let S_{M} be the set of ω values for which all ω' values satisfying $0 \leq \omega' < \omega$ possess property M. Sketch S_{M} on a ω number line. Are the theorem hypotheses met for this property M?

We will refer to Bolzano's theorem from Section 12 as the *Bounded Set Theorem.*

Task 13 Rewrite Bolzano's Bounded Set Theorem using modern terminology and set notation.

Now we are ready to examine Bolzano's proof of his Bounded Set Theorem, which he broke into a number of parts. We begin with Part 1, and restate the theorem for ease of reference.

Theorem. If property M does not apply to all values of a [nonnegative] variable quantity x, but rather to all such that are less than a certain u, then there is always a quantity U that is the largest among them for which it can be asserted that all smaller x have property M.

Proof.

1. As property M is valid for all [nonnegative] x smaller than u, yet by no means valid for all values, there is surely some quantity $V = u + D$ (where D represents something positive), for which it can be asserted that M does not apply to all x that are $< V = u + D$. From there, if I raise the question of whether M in fact applies to all x that are $< u + \frac{D}{2^m}$ where I let the exponent m denote the values, in order, first 0, then 1, then 2, then 3, and so on, then I am sure that the first of my questions will have to be answered in the negative. For the question of whether M applies to all x that are $< u + \frac{D}{2^0}$ is the same as whether M applies to all [nonnegative] x that are $< u + D$, which, according to the assumption, is to be answered in the negative. It is only a matter of whether all the subsequent questions that arise, in which I set m to be progressively larger, can be answered in the negative as well. If this were to be the case, then it is clear that u itself is the largest among the [nonnegative] values for which the assertion is valid, that all x that are smaller than it possess property M. For if there would be yet a larger one, for example, $u + d$, to which the assertion applies, that is, all x that are $< u + d$ possess property M as well, it is obvious that when I take m large enough, $u + \frac{D}{2^m}$ becomes $=$ or $< u + d$ at some point. And, consequently, it must be that if M applies to all x that are $< u + d$, the same goes for those that are $< u + \frac{D}{2^m}$, so that this question should not have been answered in the negative, but rather in the affirmative. From there it is proved that in this case (where all the above-mentioned questions are answered in the negative), there is a certain quantity U (namely u itself) that is the largest among them for which the assertion, that all x below it possess property M, is valid.

Part 1 of Bolzano's proof includes several claims and their justifications. Let's investigate them carefully, beginning with the examples from Tasks 11 and 12.

Task 14 Consider Bolzano's argument using property M of Task 11 with $u = 3/2$ and $D = 8$.

(a) Find the numerical value of quantity V, and verify Bolzano's claim in the first sentence of Part 1 of his proof for this example.

(b) Find the integers $m = 0, 1, 2, \ldots$ for which property M holds for all $x < u + \frac{D}{2^m}$. A calculator will come in handy.

Task 15 Consider Bolzano's argument using property M of Task 12 with $u = 2/5$ and $D = 1$.

(a) Find the numerical value of quantity V, and verify Bolzano's claim in the first sentence of Part 1 of his proof for this example.

(b) Find the integers $m = 0, 1, 2, \ldots$ for which property M holds for all $x < u + \frac{D}{2^m}$.

In his proof, Bolzano raised the sequence of questions

> "of whether M in fact applies to all x that are $< u + \frac{D}{2^m}$ where I let the exponent m denote the values, in order, first 0, then 1, then 2, then 3, and so on ...."

Task 16 For which values of m will the answers to this question be "no" for:

(a) the example in Task 11 with $u = 3/2$ and $D = 8$.

(b) the example in Task 12 with $u = 2/5$ and $D = 1$.

Bolzano then made a crucial claim in the two sentences:

> "It is only a matter of whether all the subsequent questions that arise, in which I set m to be progressively larger, can be answered in the negative as well. If this were to be the case, then it is clear that u itself is the largest among the [nonnegative] values for which the assertion is valid, that all x that are smaller than it possess property M."

Let's call this **Claim U=u** for the case where "all the subsequent questions that arise ... can be answered in the negative as well."

Task 17 Rewrite this case and Claim $\mathsf{U} = u$ in your own words and modern notation as an if-then statement with appropriate quantifiers, without the question/answer format.

In his proof, Bolzano also stated that "it is obvious that when I take m large enough, $u + \frac{D}{2^m}$ becomes $=$ or $< u + d$ at some point." You might agree this is obvious, but this statement actually relies on an important property of the real numbers:

> **Archimedean Property**. For every positive real number p, there exists a natural number n for which $p > 1/n$.

Task 18 Verify Bolzano's claim when we set $d = 1/16$ and $D = 8$ by finding an appropriate integer m so that $u + \frac{D}{2^m} \leq u + d$. More generally, explain how to find such an m if d and D are arbitrary positive numbers.

Task 19 Rewrite Bolzano's proof of Claim $\mathsf{U} = u$ in your own words and modern notation. Note that this is a proof by contradiction.

Before reading the rest of Bolzano's proof, let's summarize his progress and try to anticipate where the proof goes next.

Task 20 Recall that Bolzano was trying to prove the existence of this special quantity U.

(a) Fill in the blank: In Part 1 of this proof, Bolzano showed that ______ for the case "where all the above-mentioned questions are answered in the negative."

(b) What is the alternative to the case "where all the above-mentioned questions are answered in the negative"? If U exists, how do you think u, d and U are related in this alternative case? Try to write your answer using an inequality involving a certain integer m from one of Bolzano's questions.

Now let's continue with Part 2 of Bolzano's proof.

2. If, on the other hand, the above question is answered once in the affirmative, and m is the particular value of the exponent for which it is first answered in the affirmative (where m can be 1, but, as we have seen, not 0), then now I know that property M applies to all x that are $< u + \frac{D}{2^m}$ but not to all that are $< u + \frac{D}{2^{m-1}}$. The difference of $u + \frac{D}{2^{m-1}}$ and $u + \frac{D}{2^m}$ is $= \frac{D}{2^m}$. Hence, if I proceed with this again as before with the difference D, i.e., if I pose the question of whether property M applies to all x that are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}},$$

where here the exponent n denotes first 0, then 1, then 2, and so on, I am once more assured that one will have to answer at least the first of these questions in the negative. For asking whether M applies to all x that are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+0}}$$

means (inherently) the same as asking whether M applies to all x that are $< u + \frac{D}{2^{m-1}}$, which was answered in the negative before. But suppose one is to answer all of my subsequent questions in the negative as well, as I make n larger bit by bit, then it would come to light, as before, that $u + \frac{D}{2^m}$ is the greatest of those values, or U, for which the assertion that all x under it possess property M is valid.

Let's examine some details of Part 2 of Bolzano's proof.

Task 21 Carefully re-read the first sentence of Part 2 of Bolzano's proof.

(a) In your own words, justify Bolzano's claim in this first sentence.

(b) Illustrate this claim with the property M of Task 11 using $u = 3/2$ and $D = 1/2$. What is the first integer m for which "the above question is answered in the affirmative"?

(c) Display a number line and mark on it the values $u, u + D, u + D/2^m, u + D/2^{m-1}$ and U from part (b) of this Task.

Task 22 Give a general proof of Bolzano's claim that

$$\text{“the difference between } u + \frac{D}{2^{m-1}} \text{ and } u + \frac{D}{2^m} \text{ is } = \frac{D}{2^m}.\text{”}$$

Task 23 Rewrite Bolzano's claim in the last sentence of Part 2 of his proof in your own words and modern notation as an if-then statement with appropriate quantifiers.

Before reading the rest of Bolzano's proof, let's summarize his progress and try to anticipate where the proof goes next.

Task 24 Bolzano raised a sequence of questions in Part 1 of his proof, and showed that U exists, with $\mathsf{U} = u$, when all of these questions are "answered in the negative."

(a) In your own words, summarize what Bolzano showed in Part 2 of his proof.

(b) What is the alternative to the case where "one is to answer all of my subsequent questions in the negative as well, as I make n larger bit by bit" in the last sentence of Part 2 of his proof? If U exists, how do you think u, D, m and U are related in this alternative case? Try to write your answer using an inequality involving a certain integer n from one of Bolzano's questions in Part 2 of his proof.

Task 25 At the beginning of Part 2 of his proof, Bolzano wrote "if the above question is answered once in the affirmative, and m is the particular value of the exponent for which it is first answered in the affirmative." If we are being very careful with assumptions, what property allows us to justify the claim that there must be a **first** value of the exponent?

Now let's continue with Part 3 of Bolzano's proof.

3. If one of these questions were to be answered in the affirmative, and this happens first for a particular value n, then I know that M applies to all x that are $< u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$, but not all that are $< u + \frac{D}{2^m} + \frac{D}{2^{m+n-1}}$. The difference of these quantities is $= \frac{D}{2^{m+n}}$; and I proceed again with it as I did before with $= \frac{D}{2^m}$, and so forth.

Task 26 Compare your results from Task 24 with Part 3 of Bolzano's proof.

Task 27 Consider the property M of Task 11.

(a) Find the value of n using $u = 3/2$, $D = 1/2$, and the value of m from Task 21.

(b) How close are $u + \frac{D}{2^m} + \frac{D}{2^{m+n-1}}$ and $u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$ to the true U value?

Before reading the rest of Bolzano's proof, let's reflect on his progress thus far. Can you see how Bolzano was building a procedure for closing in on and locating U? Let's now try a step in this procedure on an example.

Task 28 Carry out the next step in Bolzano's procedure for finding U with the example of property M from Task 11 using $u = 3/2$, $D = 1/2$, the value of m from Task 21, and the value of n from the previous task. That is, raise and answer Bolzano's sequence of questions about property M with a new exponent p. Do you arrive at U itself? If not, what interval must U lie in, assuming U exists?

Now we continue to Part 4 of Bolzano's proof.

4. If I continue in this way for as long as desired, one sees that the result that I ultimately obtain must be one of two cases.

(a) Either I find a value of the form

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

that emerges as the greatest for which the assertion that all x below it have property M is valid. This happens in the case when the questions of whether M applies to all x that are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r+s}}$$

for each value of s are answered in the negative.

(b) Or I find at least that M applies indeed to all x that are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

but not all that are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}}.$$

Here I am free to make the number of terms even larger by means of new questions.

Task 29 Again consider the Property M of Task 11 using $u = 3/2$, $D = 1/2$, and the values of m, n you found in the tasks above. Recall from Task 11 that the value of U is $\sqrt{3}$. If we continue to carry out Bolzano's procedure indefinitely for this example, Part 4 of Bolzano's proof claims the result will be that we "ultimately obtain must be one of two cases." Do you think we obtain Bolzano's case (a) or case (b) in this example? Justify your answer.

We are now ready for the final part of Bolzano's proof, where he used Axiom C (stated on page 4 of this project). As you read the next excerpt, see if you can spot it!

In this last part of his proof, Bolzano also used an ϵ and a δ, where he implicitly assumed these are *positive* quantities. As you read, think about how you would use quantifiers with ϵ, δ in a modern version of Bolzano's proof.

5. If the first case prevails, then the truth of the theorem is already established. In the second case, let us observe that the quantity

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

represents a series, whose number of terms I can increase at will, and which belongs to the class described in [*Axiom C*]. This is so because, depending on whether $m, n, \ldots, r$ are either all $= 1$ or at times even larger, the series either decreases just like or even more sharply than a geometric progression whose ratio is the proper fraction $\frac{1}{2}$. As a result, it has the property that ... there is a certain constant quantity to which it can come as close as desired when the number of its terms is sufficiently increased. Let this value be U.

Then I claim that property M applies to all x that are $< U$. Were it to be the case that M is not valid for some x that is $< U$, e.g. $U - \delta$, the quantity

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

would have to maintain a distance δ from U, because property M is supposed to hold for all x less than it [i.e., the stated quantity]. Then property M holds for each x that is

$$= u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}} - \omega,$$

however small ω may be. On the other hand M does not hold for $x = U - \delta$. So it must be that

$$\mathsf{U} - \delta > u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}} - \omega$$

or

$$\mathsf{U} - \left[u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}\right] > \delta - \omega$$

Consequently the difference between U and the series could not become as small as one desires, since $\delta - \omega$ cannot become as small as desired, in that δ does not change while ω is able to become smaller than any given quantity.

But even less so can M be valid for all x that are $< \mathsf{U} + \epsilon$. For since the value of the sum

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}}$$

can be brought as close to the value of the sum

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

as desired, in that the difference of the two is $\frac{D}{2^{m+n+\cdots+r}}$; and further, the value of the last sum can come as close as desired to the quantity U; therefore so can the value of the first sum and U come as close to one another as desired. Thus,

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}}$$

can surely become $< \mathsf{U} + \epsilon$. But now under the assumption, M is not valid for all x that are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}};$$

and even less so [is M valid] for all x that are $< \mathsf{U} + \epsilon$. Thus U is the largest value for which the claim is valid, that all x below it possess property M.

In this part of his proof, Bolzano finally established the existence of the sum U of his series for all cases, under the assumption of Axiom C. Let's analyze his argument.

Task 30 Write out the sum U of his series using sigma notation.

Hint: Recall formula (5) and let $m = m_0$. What is m_1?

Task 31 Identify precisely where Bolzano used Axiom C in his argument.

Having established the existence of the sum U of the series in the first paragraph of Part 5 of his proof, Bolzano then argued that "property M applies to all x that are $< U$."

Task 32 Make a diagram of the x-axis for this argument, with marks and labels for the key quantities.

Task 33 Rewrite Bolzano's argument that "property M applies to all x that are $< U$" using modern terminology and quantifiers for the proof.

Finally, Bolzano showed that the property M does not hold for all x which are less than $\mathsf{U} + \epsilon$.

Task 34 In your own words, with modern terminology and quantifiers, explain why this means that U is in fact the *greatest* value for which the property M holds for all lesser values of x.

Task 35 Make a diagram of the x-axis for Bolzano's argument that the property M does not hold for all x which are less than $\mathsf{U} + \epsilon$, with marks and labels for the key quantities.

Task 36 Rewrite Bolzano's argument that the property M does not hold for all x which are less than $\mathsf{U} + \epsilon$ using modern terminology and quantifiers for the proof.

Congratulations, you have now worked through a very complex argument to prove Bolzano's Bounded Set Theorem!

3 Applications of Bolzano's Bounded Set Theorem

After proving his Bounded Set Theorem, Bolzano went on to use it in his remarkable proof of the Intermediate Value Theorem, which you can explore in a related project based on Bolzano's pamphlet.[8] For this project, we will use Bolzano's Bounded Set Theorem to prove the *least upper bound property* and the *greatest lower bound property* of the real numbers. These are extremely useful and powerful properties of $\mathbb{R}$.

> **Greatest Lower Bound Property of** $\mathbb{R}$. Every nonempty set of real numbers that has a lower bound has a greatest lower bound in $\mathbb{R}$.
>
> **Least Upper Bound Property of** $\mathbb{R}$. Every nonempty set of real numbers that has an upper bound has a least upper bound in $\mathbb{R}$.

Here are some standard definitions for these terms.

> **Definition.** Let S be a nonempty subset of $\mathbb{R}$.
> The set S is **bounded below** if there exists a number $\ell \in \mathbb{R}$ for which $\ell \leq s$ for all $s \in S$. Each such number ℓ is called a **lower bound** of S. We say g is the **greatest lower bound** of S if (i) g is a lower bound of S, and (ii) $g \geq \ell$ for all lower bounds of S.

Task 37 Let S be a nonempty subset of $\mathbb{R}$ such that $s > 0$ for all $s \in S$. Use Bolzano's Bounded Set Theorem to prove that S has a greatest lower bound.

Task 38 Prove the Greatest Lower Bound Property of the real numbers.

Task 39 Define the terms *bounded above, upper bound,* and *Least Upper Bound Property* by analogy with the terms in the definition above.

[8] This related project, "Bolzano on Continuity and the Intermediate Value Theorem," is available at `https://digitalcommons.ursinus.edu/triumphs_analysis/9/`.

Task 40 Prove the Least Upper Bound Property of the real numbers.

Hint: Use the Greatest Lower Bound Property on the set $-S$.

Task 41 Use the Greatest Lower Bound or Least Upper Property of $\mathbb{R}$ to prove Bolzano's Bounded Set Theorem.

4 Conclusion

The completeness property of the real numbers is a challenging topic. Many textbooks start with the least upper bound property as an axiom. It is then used to prove the Archimedean Property and *Cauchy completeness* for real numbers. A sequence is called a *Cauchy sequence* if it satisfies a property analogous to Property S for series. The *Cauchy completeness* of real numbers means that every Cauchy sequence of real numbers converges to a real number, very much like Axiom C for series in this project. Some textbooks begin with Cauchy completeness and the Archimedean Property as axioms, and use them to prove the least upper bound property for $\mathbb{R}$, which is close to Bolzano's approach.

The term "Cauchy" refers to the mathematician Augustin-Louis Cauchy (1789–1857), a contemporary of Bolzano and a key player in building the theory of calculus. Cauchy defined a sequence version of Property S and assumed the Cauchy completeness of real numbers to give a proof of the Intermediate Value Theorem, probably a few years later than Bolzano did his work [Jahnke, 2003].

While Cauchy worked in Paris and was very influential, Bolzano's mathematics was largely ignored during his lifetime. Bolzano attempted to prove in his paper [Bolzano, 1817] that any series with Property S must converge to a unique real number, but without assuming the completeness of $\mathbb{R}$. While his proof could not succeed without this assumption, it is remarkable that Bolzano saw the need for justifying relationships between these concepts. He was well ahead of his time in this regard!

References

B. Bolzano. *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege (A Purely analytic Proof of the Theorem, that between any two Values that give opposite [sign] Results, there lies at least one real Root of the Equation).* Gottlieb Haase, Prague, 1817.

H. N. Jahnke, editor. *History of Analysis.* American Mathematical Society, Providence, RI, 2003.

M. Kline. *Mathematical Thought from Ancient to Modern Times.* Oxford University Press, New York, NY, 1972.

S. B. Russ. A Translation of Bolzano's Paper on the Intermediate Value Theorem. *Historia Mathematica*, 5:156–185, 1980.

S. B. Russ. *The Mathematical Works of Bernard Bolzano.* Oxford University Press, New York, NY, 2004.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed to be used in a course on real analysis or foundations of the real numbers. Specifically, its content goals are to:

1. Use fundamentals of infinite series to modernize Bolzano's discussion of infinite series convergence.
2. Analyze Bolzano's statement and proof of his Bounded Set Theorem.
3. Modernize Bolzano's proof of his Bounded Set Theorem.
4. Clarify and use the completeness property of the real numbers in the special "Axiom C" form assumed by Bolzano.
5. Explore the connections between the Least Upper Bound property of $\mathbb{R}$ and Bolzano's Bounded Set Theorem.

Student Prerequisites

The PSP assumes that students have seen infinite series in an introductory calculus course. Instructors may find helpful some just-in-time review of series convergence as the convergence of a sequence of partial sums. The project also assumes that students have done a rigorous study of quantifiers.

PSP Design and Task Commentary

This PSP is designed to take around two weeks of classroom time, with some reading and tasks done outside class.

Bolzano proved that certain bounded sets have a least upper bound, but he did not phrase his results as we would today. In his proof, Bolzano implicitly assumed that (1) a Cauchy sequence in $\mathbb{R}$ must converge, and that (2) the real numbers have the Archimedean Property. The project draws attention to these hidden assumptions, but does not assume that students have seen the concepts of Cauchy sequences, least upper bounds, or the Archimedean Property. Indeed, the project can be used to introduce these ideas. On the other hand, instructors will need to discuss the interplay between these ideas early in the project if their course has already used the existence of least upper bounds or the Nested Interval Property as completeness axioms for $\mathbb{R}$.

Task 1 is useful for getting students comfortable with the partial sum notation using a common geometric series. Moreover, this series will play an important role later in the PSP.

The examples in Tasks 11 and 12 will be used multiple times later in the project to illustrate Bolzano's proof and make sure students can follow his arguments with concrete examples.

LaTeXcode of this entire PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Suggestions for Classroom Implementation

This is roughly a one or two week project under the following methodology (basically David Pengelley's "A, B, C" method described on his website https://www.math.nmsu.edu/~davidp/):

1. Students do some advanced reading and light preparatory tasks before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the tasks or your grading load will get out of hand! Some instructor have students write questions or summaries based on the reading.

2. Class time is largely dedicated to students working in groups on the project - reading the material and working tasks. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the tasks they missed. This is usually a good incentive not to miss class!

3. Some tasks are assigned for students to do and write up outside of class. Careful grading of these tasks is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

Sample Implementation Schedule (based on a 50-minute class period)

Full implementation of the project can be accomplished in 6 class days, as outlined below.

Students read through the introductory material and do Tasks 1, 2 before the first class. After discussing their results at the beginning of Class 1, students work on and discuss Tasks 3, 4, 6, 9, 10. Tasks 5, 7, 10 can be assigned for homework.

As preparation for Class 2, students read about Axiom C and the first Bolzano excerpt with Bolzano's Bounded Set theorem, and do Task 11. After discussing their results at the beginning of Class 2, students work on and discuss Tasks 12, 13. Students then read the first part of Bolzano's proof and do Task 14.

As preparation for Class 3, students do Tasks 15–17. After discussing their results at the beginning of Class 3, students work on and discuss Tasks 19–20. Students then read the second part of Bolzano's proof and do Task 21. Task 22 and 25 can be assigned for homework.

As preparation for Class 4, students do Tasks 23 and 24. After discussing their results at the beginning of Class 4, students read the third part of Bolzano's proof and work on Tasks 26–28. Students then read the fourth part of Bolzano's proof and work on Task 29. Some of these tasks can be given as homework, as time permits.

As preparation for Class 5, students read the fifth part of Bolzano's proof and do Tasks 30, 31. After discussing their results at the beginning of Class 5, students work on Tasks 32–36. Some of these tasks can be given as homework, as time permits.

As preparation for Class 6, students read the beginning of Section 4 of the PSP and do Task 37. After discussing their results at the beginning of Class 6, students finish the project.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be completed in at most 2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from `https://digitalcommons.ursinus.edu/triumphs\_topology`; all other listed projects are available at `https://digitalcommons.ursinus.edu/triumphs\_analysis`.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Heine Barnett)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)
  Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)
  A second version of this project suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *Understanding Compactness: Early Work, Uniform Continuity to the Heine-Borel Theorem* (Naveen Somasunderam)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Mean Value Theorem*(David Ruch)
- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Heine Barnett)
- *Euler's Rediscovery of e** (David Ruch; sequence convergence, series & sequence expressions for e)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

Recommendations for Further Reading

The translations by Russ in [Russ, 1980] and [Russ, 2004] also include interesting background on Bolzano as well as commentary on some of the subtleties of Bolzano's work. The articles in [Jahnke, 2003] give some perspective on other works in analysis during Bolzano's era.

Acknowledgments

The development of this student project has been partially supported by the TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Program with funding from the National Science Foundation's Improving Undergraduate STEM Education Program under Grant Nos. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily represent the views of the National Science Foundation.

For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.