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# Investigations into Bolzano's Bounded Set Theorem

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## 1 Introduction

The foundations of calculus were not yet on firm ground in the early 1800s. Mathematicians such as J. L. Lagrange (1736-1813) made efforts to put limits and derivatives on firmer logical ground, but were not entirely successful. It took even longer for mathematicians to fully develop the notion of completeness of the real numbers.

Bernard Bolzano (1781-1848) was one of the great success stories of the foundations of analysis. He was a theologian with interests in mathematics and a contemporary of Gauss and Cauchy, but was not well known in mathematical circles. Despite his mathematical isolation in Prague, Bolzano was able to read works by Lagrange and others, and published mathematical work of his own.

This project investigates a key result from his important pamphlet *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*[1] [Bolzano, 1817]. We will read excerpts from this paper, as translated in [Russ, 2004], with very minor changes for readability by the project author.

Bolzano's proof of the main theorem on a property of bounded sets, discussed in Section 3 of this project, gives some insight into the completeness of the real numbers. Bolzano's proof also inspired Karl Weierstrass decades later in his proof of what is now known as the Bolzano-Weierstrass Theorem.[2]

## 2 Bolzano's Property S for series

Before proving the main results of his 1817 pamphlet, Bolzano developed some preliminaries. In Sections 1–9 of that work, he discussed series and their convergence. He introduced his own partial sum notation, quite different from the modern sigma notation:

$$A + Bx + Cx^2 + \cdots + Rx^n = \overset{n}{F}x. \tag{1}$$

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[1]The title of Bolzano's pahmplet translates into English as *A Purely Analytic Proof of the Theorem that between two values which give results of opposite sign there lies at least one real root of the equation.*

[2]According to Kline, Bolzano's proof method "was used by Weierstrass in the 1860s, with due credit to Bolzano, to prove what is now called the Bolzano-Weierstrass theorem"[Kline, 1972, p. 953].

He also talked about finite geometric series and their sums. In Section 6 of his 1817 paper, Bolzano created the new sequence of partial sums[3]

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \cdots, \overset{n}{F}x, \cdots, \overset{n+k}{F}x, \cdots \tag{2}$$

and discussed the:

special property that the difference between its $n$th term $\overset{n}{F}x$ and every later term $\overset{n+k}{F}x$ (no matter how far from that nth term) stays smaller than any given quantity, provided $n$ has first been taken large enough.

Throughout his 1817 paper, Bolzano always set $x = 1$ in $\overset{n}{F}x$, so for ease of reading we will use the modern subscript notation

$$F_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{i=0}^{n} a_i (1)^i \tag{3}$$

in place of the partial sums $\overset{n}{F}x$ in (1) and (2) where $x = 1$.[4]

**Task 1** Consider the example series where $x = 1$ and $a_i = 1/2^i$ so

$$F_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$$

(a) Find $F_2, F_3, F_4$.

(b) Using your introductory calculus course knowledge, find the limit of this sequence $\{F_n\}_{n=1}^{\infty}$.

(c) Do you think this sequence $\{F_n\}_{n=1}^{\infty}$ has the "special property" Bolzano defines in the source excerpt above? Justify your answer.

**Task 2** Bolzano discussed the example series where $x = 1$ and the sequence of partial sums (2) is

$$0.1, 0.11, 0.111, 0.1111, \ldots$$

and claimed "the quantity which the terms approach as closely as desired" is 1/9.

(a) For this example, what are the values of $A, B, C, D$ in (1)? Find a general formula for $a_n$ in (3) in terms of $n$ for this example.

(b) Write $F_{n+k} - F_n$ using sigma notation.

[3]For the convenience of modern readers, we have made minor variable changes in the source material, such as replacing Bolzano's use of "$r$" by "$n$" or "$k$."

[4]See [Russ, 1980] for a discussion of Bolzano's notational choices.

(c) Using your introductory calculus course knowledge, explain how Bolzano gets $1/9$ as the limit of this sequence.

(d) Do you think this sequence $\{F_n\}_{n=1}^{\infty}$ has the "special property" Bolzano defines in the source excerpt above? Justify your answer.

We will see later in the project that Bolzano was especially interested in using geometric series to prove his major theorem on a property of certain bounded sets of real numbers. In Section 5 of his paper, he stated the well-known finite geometric series summation formula

$$ar^{n+1} + ar^{n+2} + \cdots + ar^{n+k} = ar^{n+1} \cdot \frac{1 - r^k}{1 - r} \tag{4}$$

whenever $r \neq 1$.

**Task 3** Use formula (4) for the series in Task 2 with $x = 1$ to find:

(a) $F_{n+k} - F_n$ for $n = 4$, $r = 3$.

(b) the minimal $n$ for which $|F_n - F_{n+k}| < 0.003$ holds for all $k \in \mathbb{N}$.

Part (b) of the previous task is an illustration of Bolzano's "special property", with "given quantity" 0.003, defined in the source excerpt above. We will now formally name this attribute to be *Property S*, where the letter S is chosen to remind us of these special series.

**Definition.** A series with sequence of partial sums $F_1, F_2, F_3, \ldots$ has **Property S** if the difference between term $F_n$ and every later term $F_{n+k}$ stays smaller than any given quantity, provided $n$ has first been taken large enough.

**Task 4** Write Property S in modern notation using quantifiers $\forall$ and $\exists$.

**Task 5** Use formula (4) to prove that the series in Task 2 has Property S.

**Task 6** Use formula (4) to prove that the series

$$1 + 1/2 + 1/2^2 + 1/2^3 + \cdots$$

from Task 1 has Property S.

**Task 7** Use formula (4) to prove that any geometric series with $|r| < 1$ has Property S. You may assume that $\lim_{n\to\infty} r^{n+1} = 0$.

If you have studied Cauchy sequences, you should try the next task.

**Task 8** If a series with sequence of partial sums $F_1, F_2, F_3, \ldots$ has Property S, must the sequence be Cauchy? Conversely, if a sequence of partial sums $F_1, F_2, F_3, \ldots$ is a Cauchy sequence, must the series have Property S?

After some discussion of standard geometric series in Section 5 of his paper, Bolzano argued that:

Therefore every geometric progression whose ratio is a proper fraction can be continued so far that the increase caused by every further continuation must remain smaller than some given quantity. This must hold all the more for series whose terms decrease even more rapidly than those of a decreasing geometric progression.

We shall see in Section 2 of this project that Bolzano was particularly interested in showing the convergence of series of the form

$$u + \sum_{j=0}^{\infty} D/2^{m_j} \tag{5}$$

where $u, D > 0$ and $\{m_j\}$ is a strictly increasing sequence of integers, $m_{j+1} > m_j > m_0 \geq 1$ for all $j$. For example, the series

$$7 + \frac{3}{2^1} + \frac{3}{2^{1+3}} + \frac{3}{2^{1+3+5}} + \frac{3}{2^{1+3+5+7}} + \frac{3}{2^{1+3+5+7+9}} + \cdots \tag{6}$$

has the form (5).

**Task 9** Identify $u, D$, $m_0, m_1, m_2$ for the example series (6) above.

**Task 10** Prove that series of form (5) satisfy Property S.

In Section 7 of his paper, Bolzano attempted to prove that any series with Property S must converge to a unique real number. As you saw in Task 10, this would imply that any series with form (5) must converge to a unique real number. Perhaps this seems obvious to you: if the terms of a sequence $F_1, F_2, F_3, \ldots$ are getting closer together, surely they must have some real number limit $L$. However, mathematicians after Bolzano's time have shown that the existence of limit $L$ depends on a deeper "completeness" assumption that the real number line has no "holes" in it. The details of this "completeness" assumption are quite complex, so for the purposes of this project we will distill Bolzano's discussion on this topic into the following axiom, which we will name Axiom C for "<u>C</u>ompleteness". We will *assume the truth of Axiom C as a fact.*

**Axiom C.** Any series of the following form has a unique real number sum (limit):

$$u + \sum_{j=0}^{\infty} D/2^{m_j} \tag{7}$$

where $u, D > 0$ and $\{m_j\}$ is a strictly increasing sequence of integers, $m_{j+1} > m_j > m_0 \geq 1$ for all $j$.

This completeness Axiom C will be essential for proving Bolzano's big result in the next section of this project.

## 3 Bolzano's Bounded Set Theorem

We are now ready to examine the main theorem for this project, which we will refer as the " Bounded Set Theorem."

The theory of sets had not been developed during Bolzano's era, so he did not use the same set theoretic language we might expect in a modern discussion of his ideas. As you read the next excerpt from Bolzano's pamphlet, think about how you could translate his ideas into set terminology.

§11

*Preamble*. In investigations of applied mathematics it is often the case that we learn that a definite property M applies to all values of a [nonnegative[5]] variable quantity $x$ which are smaller than a certain $u$ without at the same time learning that this property M does not apply to values which are greater than $u$. In such cases there can still perhaps be some $u'$ that is $> u$ for which, in the same way as it holds for $u$, all values of $x$ lower than $u'$ possess property M. Indeed this property M may even belong to all values of $x$ without exception. But if this alone is known, that M does not belong to all $x$ in general, then by combining these two conditions we will now be justified in concluding: there is a certain quantity U which is the greatest of those for which it is true that all smaller values of $x$ possess property M. This is proved in the following theorem.

§12

Theorem. If a property M does not apply to all values of a [nonnegative] variable quantity $x$ but does apply to all values smaller than a certain $u$, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller $x$ possess the property M.

Before reading Bolzano's proof of this theorem in his Section 12, let's look at some examples of this concept he is discussing.

**Task 11** Let M be the property "$x^2 < 3$" applied to the set $\{x \in \mathbb{R} : x \geq 0\}$.

(a) Find rational numbers $u$, $u'$ for this example (these values are not unique). What is the value of U for this example?

(b) Let $S_{\mathsf{M}}$ be the set of $\omega$ values for which all $\omega'$ values satisfying $0 \leq \omega' < \omega$ possess property M. Sketch $S_{\mathsf{M}}$ on a $\omega$ number line. Are the theorem hypotheses met for this property M?

(c) Does U possess property M?

[5]Bolzano intended to discuss only $x \geq 0$ in this note and his Section 12 theorem statement. The term "nonegative" has been included in this project for clarity.

**Task 12** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$, and let $\alpha \in \mathbb{R}$ be arbitrary. Let M be the property "$f(\alpha + \omega) \le f(\alpha) + 2$" applied to the set $\{\omega \in \mathbb{R} : \omega \ge 0\}$.

(a) Find rational numbers $u$, $u'$ for this example. Are these values unique? What is the value of U for this example?

(b) Let $S_{\mathsf{M}}$ be the set of $\omega$ values for which all $\omega'$ values satisfying $0 \le \omega' < \omega$ possess property M. Sketch $S_{\mathsf{M}}$ on a $\omega$ number line. Are the theorem hypotheses met for this property M?

We will refer to Bolzano's theorem from Section 12 as the *Bounded Set Theorem.*

**Task 13** Rewrite Bolzano's Bounded Set Theorem using modern terminology and set notation.

Now we are ready to examine Bolzano's proof of his Bounded Set Theorem, which he broke into a number of parts. We begin with Part 1, and restate the theorem for ease of reference.

Theorem. If a property M does not apply to all values of a [nonnegative] variable quantity $x$ but does apply to all values smaller than a certain $u$, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller $x$ possess the property M.

Proof.

1. Because the property M holds for all [nonnegative] $x$ smaller than $u$ but nevertheless not for all $x$, there is certainly some quantity $V = u + D$ (where $D$ represents something positive) of which it can be asserted that M does not apply to all $x$ which are $< V = u + D$. If I then raise the question of whether M in fact applies to all $x$ which are $< u + \dfrac{D}{2^m}$, where the exponent $m$ is in turn first $0$, then $1$, then $2$, then $3$. etc., I am sure the first of my questions will have to be answered 'no'. For the question of whether M applies to all $x$ which are $< u + \dfrac{D}{2^0}$ is the same as that of whether M applies to all [nonnegative] $x$ which are $< u + D$, which is ruled out by assumption. What matters is whether all the succeeding questions, which arise as $m$ gradually gets larger, will also be ruled out. Should this be the case, it is evident that $u$ itself is the greatest value for which the assertion holds that all smaller [nonnegative] $x$ have property M. For if there were an even greater value, for example $u + d$, i.e. if the assertion held that also all $x$ which are $< u + d$ have the property M, then it is obvious that if I take $m$ large enough, $u + \dfrac{D}{2^m}$ will at some time be $=$ or $< u + d$. Consequently if M applies to all $x$ which are $< u + d$, it also applies to all $x$ which are $< u + \dfrac{D}{2^m}$. We would therefore not have said 'no' to this question but would have had to say 'yes'. Thus it is proved that in this case (when we say 'no' to all the above questions) there is a certain quantity U (namely $u$ itself) which is the greatest for which the assertion holds that all $x$ below it possess the property M.

Part 1 of Bolzano's proof includes several claims and their justifications. Let's investigate them carefully, beginning with the examples from Tasks 11 and 12.

**Task 14** Consider Bolzano's argument using property $\mathsf{M}$ of Task 11 with $u = 3/2$ and $D = 8$.

(a) Find the numerical value of quantity $V$, and verify Bolzano's claim in the first sentence of Part 1 of his proof for this example.

(b) Find the integers $m = 0, 1, 2, \ldots$ for which property $\mathsf{M}$ holds for all $x < u + \dfrac{D}{2^m}$. A calculator will come in handy.

**Task 15** Consider Bolzano's argument using property $\mathsf{M}$ of Task 12 with $u = 2/5$ and $D = 1$.

(a) Find the numerical value of quantity $V$, and verify Bolzano's claim in the first sentence of Part 1 of his proof for this example.

(b) Find the integers $m = 0, 1, 2, \ldots$ for which property $\mathsf{M}$ holds for all $x < u + \dfrac{D}{2^m}$.

In his proof, Bolzano raised a sequence of questions:

> "of whether M in fact applies to all $x$ which are $< u + \dfrac{D}{2^m}$, where the exponent $m$ is in turn first 0, then 1, then 2, then 3. etc."

**Task 16** For which values of $m$ will the answers to this question be "no" for:

(a) the example in Task 11 with $u = 3/2$ and $D = 8$.

(b) the example in Task 12 with $u = 2/5$ and $D = 1$.

Bolzano then made a crucial claim in the two sentences:

> "What matters is whether all the succeeding questions, which arise as $m$ gradually gets larger, will also be ruled out. Should this be the case, it is evident that $u$ itself is the greatest value for which the assertion holds that all smaller [nonnegative] $x$ have property M."

Let's call this Claim $\mathsf{U} = u$ for the case "when we say 'no' to all the above questions".

**Task 17** Rewrite this case and Claim $\mathsf{U} = u$ in your own words and modern notation as an if-then statement with appropriate quantifiers, without the question/answer format.

In his proof, Bolzano stated that:

> "it is obvious that if I take $m$ large enough, $u + \dfrac{D}{2^m}$ will at some time be $=$ or $< u + d$."

You might agree this is obvious, but this statement actually relies on an important property of the real numbers:

> **Archimedean Property**. For every positive real number $p$, there exists a natural number $n$ for which $p > 1/n$.

**Task 18** Verify Bolzano's claim when we set $d = 1/16$ and $D = 8$ by finding an appropriate integer $m$ so that $u + \frac{D}{2^m} \le u + d$. More generally, explain how to find such an $m$ if $d$ and $D$ are arbitrary positive numbers.

**Task 19** Rewrite Bolzano's proof of Claim $\mathsf{U} = u$ in your own words and modern notation. Note that this is a proof by contradiction.

Before reading the rest of Bolzano's proof, let's summarize his progress and try to anticipate where the proof goes next.

**Task 20** Recall that Bolzano was trying to prove the existence of this special quantity U.

(a) Fill in the blank: In Part 1 of this proof, Bolzano showed that ____ for the case "when we say 'no' to all the above questions".

(b) What is the alternative to the case "when we say 'no' to all the above questions"? If U exists, how do you think $u, d$ and U are related in this alternative case? Try to write your answer using an inequality involving a certain integer $m$ from one of Bolzano's questions.

Now let's continue with Part 2 of Bolzano's proof.

2. However, if one of the above questions is answered 'yes' and $m$ is the particular value of the exponent for which this happens first ($m$ can be $1$ but, as we have seen, not $0$), then I now know that the property M applies to all $x$ which are $< u + \frac{D}{2^m}$ but not to all $x$ which are $< u + \frac{D}{2^{m-1}}$. But the difference between $u + \frac{D}{2^{m-1}}$ and $u + \frac{D}{2^m}$ is $= \frac{D}{2^m}$. If I therefore deal with this as I did before with the difference $D$, i.e. if I raise the question of whether M applies to all $x$ which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$$

and here the exponent $n$ denotes first $0$, then $1$, then $2$, etc., then I am sure once again that at least the first of these questions will have to be answered 'no'. For to ask whether M applies to all $x$ which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+0}}$$

is just the same as asking whether M applies to all $x$ which are $< u + \frac{D}{2^{m-1}}$, which had previously been denied. But if all my succeeding questions are also to be answered negatively as I gradually make $n$ larger and larger, then it would appear, as before, that $u + \frac{D}{2^m}$ is that greatest value, or the U, for which the assertion holds that all $x$ below it possess the property M.

Let's examine some details of Part 2 of Bolzano's proof.

**Task 21** Carefully reread the first sentence of Part 2 of Bolzano's proof.

(a) In your own words, justify Bolzano's claim in this first sentence.

(b) Illustrate this claim with the property M of Task 11 using $u = 3/2$ and $D = 1/2$. What is the first integer $m$ for which "one of the above questions is answered 'yes'".

(c) Display a number line and mark on it the values $u, u + D, u + D/2^m, u + D/2^{m-1}$ and U from part (b) of this Task.

**Task 22** Give a general proof of Bolzano's claim that

$$\text{"the difference between } u + \frac{D}{2^{m-1}} \text{ and } u + \frac{D}{2^m} \text{ is } = \frac{D}{2^m}.\text{"}$$

**Task 23** Rewrite Bolzano's claim in the last sentence of Part 2 of his proof in your own words and modern notation as an if-then statement with appropriate quantifiers.

Before reading the rest of Bolzano's proof, let's summarize his progress and try to anticipate where the proof goes next.

**Task 24** Bolzano raised a sequence of questions in Part 1 of his proof, and showed that U exists, with $\mathsf{U} = u$, when all of these questions were "answered 'no'".

(a) In your own words, summarize what Bolzano showed in Part 2 of his proof.

(b) What is the alternative to the case "all my succeeding questions are also to be answered negatively as I gradually make $n$ larger and larger " in the last sentence of Part 2 of his proof? If U exists, how do you think $u, D, m$ and U are related in this alternative case? Try to write your answer using an inequality involving a certain integer $n$ from one of Bolzano's questions in Part 2 of his proof.

**Task 25** At the beginning of Part 2 of his proof, Bolzano stated that "if one of the above questions is answered 'yes' and $m$ is the particular value of the exponent for which this happens first."If we are being very careful with assumptions, what property allows us to justify the claim that there must be a **first** value of the exponent?

Now let's continue with Part 3 of Bolzano's proof.

3. However, if one of these questions is answered positively and this happens first for the particular value $n$, then I now know M applies to all $x$ which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$$

but not to all $x$ which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n-1}} .$$

The difference between these two quantities is $= \frac{D}{2^{m+n}}$ and I deal with this again as before with $\frac{D}{2^m}$, etc.

**Task 26** Compare your results from Task 24 with Part 3 of Bolzano's proof.

**Task 27** Consider the property M of Task 11.

(a) Find the value of $n$ using $u = 3/2$, $D = 1/2$, and the value of $m$ from Task 21.

(b) How close are $u + \frac{D}{2^m} + \frac{D}{2^{m+n-1}}$ and $u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$ to the true U value?

Before reading the rest of Bolzano's proof, let's reflect on his progress thus far. Can you see how Bolzano is building a procedure for closing in on and locating U? Let's now try a step in this procedure on an example.

**Task 28** Carry out the next step in Bolzano's procedure for finding U with the example of property M from Task 11 using $u = 3/2$, $D = 1/2$, the value of $m$ from Task 21, and the value of $n$ from the previous task. That is, raise and answer Bolzano's sequence of questions about property M with a new exponent $p$. Do you arrive at U itself? If not, what interval must U lie in, assuming U exists?

Now we continue to Part 4 of Bolzano's proof.

4. If I continue this way as long as I please it may be seen that the result that I finally obtain must be one of two things.

(a) Either I find a value of the form

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

which appears to be the greatest for which the assertion holds that all $x$ below it possess the property M. This happens in the case when the questions of whether M applies to all $x$ which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r+s}}$$

are answered with 'no' for every value of $s$.

(b) Or I at least find that M does indeed apply to all $x$ which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

but not to all $x$ which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}} .$$

Here I am always free to make the number of terms in these two quantities even greater through new questions.

**Task 29** Again consider the Property M of Task 11 using $u = 3/2$, $D = 1/2$, and the values of $m, n$ you found in the tasks above. Recall from Task 11 that the value of U is $\sqrt{3}$. If we continue to carry out Bolzano's procedure indefinitely for this example, Part 4 of Bolzano's proof claims the result we "finally obtain must be one of two things". Do you think we obtain Bolzano's case (a) or case (b)? Justify your answer.

We are now ready for the final part of Bolzano's proof, where he used Axiom C (stated on page 4 of this project). As you read the next excerpt, see if you can spot it!

In this last part of his proof, Bolzano also used an $\epsilon$ and a $\delta$, where he implicitly assumed these are *positive* quantities. As you read, think about how you would use quantifiers with $\epsilon, \delta$ in a modern version of Bolzano's proof.

5. Now if the first case occurs the truth of the theorem is already proved. In the second case we may remark that the quantity

$$u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}$$

represents a series whose number of terms I can increase arbitrarily and which belongs to the class described in [*Axiom C*]. This is because, depending on whether $m, n, ..., r$ are all $= 1$ or some of them are greater than $1$, the series decreases at the same rate or more rapidly than a geometric progression whose ratio is the proper fraction $1/2$. From this it follows that it has the property that there is a certain constant quantity to which it can come as close as we please if the number of its terms is increased sufficiently. Let this quantity be U; then I claim the property M holds for all $x$ which are $<$ U. For if it did not hold for some $x$ which is $<$ U, e.g. for U $-\,\delta$, then the quantity

$$u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}$$

must always keep at the distance $\delta$ from U because for all $x$ that are smaller than it, the property M is to hold. Since every $x$ that is

$$=u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}-\omega,$$

however small $\omega$ is, possesses the property M, while on the other hand, M is not to apply to $x=$ U$-\delta$, it must therefore be that

$$\mathsf{U}-\delta>u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}-\omega$$

or

$$\mathsf{U}-\left[u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}\right]>\delta-\omega$$

Hence the difference between U and the series cannot become as small as we please, since $\delta-\omega$ cannot become as small as we please because $\delta$ does not change, while $\omega$ can become smaller than any given quantity. But just as little can M hold for all $x$ which are $<$ U $+\,\epsilon$. For the value of the series

$$u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r-1}}$$

can be brought as close to the value of the series

$$u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}$$

as we please because the difference between the two is only $\frac{D}{2^{m+n+\cdots+r}}$. Further, the value of the latter series can be brought as close as we please to the quantity U. Therefore the value of the first series can also come as close to U as we please. So

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}}$$

can certainly become $< \mathsf{U} + \epsilon$. But now by assumption M does not hold for all $x$ which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}};$$

so much less therefore [does M hold] for all $x$ which are $< \mathsf{U} + \epsilon$. Therefore U is the greatest value for which the assertion holds that all $x$ below it possess the property M.

In this part of his proof, Bolzano finally established the existence of the sum U of his series for all cases, under the assumption of Axiom C. Let's analyze his argument.

**Task 30** Write out the sum U of his series using sigma notation.

Hint: Recall formula (5) and let $m = m_0$. What is $m_1$?

**Task 31** Identify precisely where Bolzano uses Axiom C in his argument.

In Part 5 of his proof, having established the existence of the sum U of the series, Bolzano then argued that "the property M holds for all $x$ which are $< \mathsf{U}$."

**Task 32** Make a diagram of the $x$-axis for this argument, with marks and labels for the key quantities.

**Task 33** Rewrite Bolzano's argument that "the property M holds for all $x$ which are $< \mathsf{U}$" using modern terminology and quantifiers for the proof.

Finally, Bolzano showed that the property M does not hold for all $x$ which are less than $\mathsf{U} + \epsilon$.

**Task 34** In your own words, with modern terminology and quantifiers, explain why this means that U is in fact the *greatest* value for which the property M holds for all lesser values of $x$.

**Task 35** Make a diagram of the $x$-axis for Bolzano's argument that the property M does not hold for all $x$ which are less than $\mathsf{U} + \epsilon$, with marks and labels for the key quantities.

**Task 36** Rewrite Bolzano's argument that the property M does not hold for all $x$ which are less than $\mathsf{U} + \epsilon$ using modern terminology and quantifiers for the proof.

Congratulations, you have now worked through a very complex argument to prove Bolzano's Bounded Set Theorem!

## 4 Applications of Bolzano's Bounded Set Theorem

After proving his Bounded Set Theorem, Bolzano went on to use it in his remarkable proof of the Intermediate Value Theorem, which you can explore in a related PSP.[6] For this project, we will use Bolzano's Bounded Set Theorem to prove the *least upper bound property* and the *greatest lower bound property* of the real numbers. These are extremely useful and powerful properties of $\mathbb{R}$.

**Greatest Lower Bound Property of** $\mathbb{R}$. Every nonempty set of real numbers that has a lower bound also has a greatest lower bound in $\mathbb{R}$.

**Least Upper Bound Property of** $\mathbb{R}$. Every nonempty set of real numbers that has an upper bound also has a least upper bound in $\mathbb{R}$.

Here are some standard definitions for these terms.

**Definition.** Let $S$ be a nonempty subset of $\mathbb{R}$. The set $S$ is **bounded below** if there exists a number $\ell \in \mathbb{R}$ for which $\ell \leq s$ for all $s \in S$. Each such number $\ell$ is called a **lower bound** of $S$. We say $g$ is the **greatest lower bound** of $S$ if (i) $g$ is a lower bound of $S$, and (ii) $g \geq \ell$ for all lower bounds of $S$.

**Task 37** Let $S$ be a nonempty subset of $\mathbb{R}$ such that $s > 0$ for all $s \in S$. Use Bolzano's Bounded Set Theorem to prove that $S$ has a greatest lower bound.

**Task 38** Prove the Greatest Lower Bound Property of the real numbers.

**Task 39** Define the terms *bounded above, upper bound,* and *Least Upper Bound Property* by analogy with the terms in the definition above.

**Task 40** Prove the Least Upper Bound Property of the real numbers.

Hint: Use the Greatest Lower Bound Property on the set $-S$.

**Task 41** Use the Greatest Lower Bound or Least Upper Property of $\mathbb{R}$ to prove Bolzano's Bounded Set Theorem.

## 5 Conclusion

The completeness property of the real numbers is a challenging topic. Many textbooks start with the least upper bound property as an axiom. It is then used to prove the Archimedean Property and *Cauchy completeness* for real numbers. A sequence is called a *Cauchy sequence* if it satisfies a property analogous to Property S for series. The *Cauchy completeness* of real numbers means that every Cauchy sequence of real numbers converges to a real number, very much like Axiom C for series in this project. Some textbooks begin with Cauchy completeness and the Archimedean Property as axioms, and use them to prove the least upper bound property for $\mathbb{R}$, which is close to Bolzano's approach.

[6] *Bolzano's Definition of Continuity, his Bounded Set Theorem, and an Application to Continuous Functions* `https://digitalcommons.ursinus.edu/triumphs_analysis/13`.

The term "Cauchy" refers to the mathematician A. L. Cauchy (1789-1857), a contemporary of Bolzano and a key player in building the theory of calculus. Cauchy defined a sequence version of Property S and assumed the Cauchy completeness of real numbers to give a proof of the intermediate value theorem, probably a few years later than Bolzano did his work [Jahnke, 2003].

While Cauchy worked in Paris and was very influential, Bolzano's mathematics was largely ignored during his lifetime. Bolzano attempted to prove in his paper [Bolzano, 1817] that any series with Property S must converge to a unique real number, but without assuming the completeness of $\mathbb{R}$. While his proof could not succeed without this assumption, it is remarkable that Bolzano saw the need for justifying relationships between these concepts. He was well ahead of his time in this regard!

## References

B. Bolzano. *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege (A Purely Analytic Proof of the Theorem that between two values which give results of opposite sign there lies at least one real root of the equation).* Prague, 1817. English translations of the complete pamphlet appear in [Russ, 1980] and [Russ, 2004, pp. 251–278]; see references below.

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# Notes to Instructors

## PSP Content: Topics and Goals

This PSP is designed to be used in a course on real analysis or foundations of the real numbers. Specifically, its content goals are to:

1. Use fundamentals of infinite series to modernize Bolzano's discussion of infinite series convergence.

2. Analyze Bolzano's statement and proof of his Bounded Set Theorem.

3. Modernize Bolzano's proof of his Bounded Set Theorem.

4. Clarify and use the completeness property of the real numbers in the special "Axiom C" form assumed by Bolzano.

5. Explore the connections between the Least Upper Bound property of $\mathbb{R}$ and Bolzano's Bounded Set Theorem.

## Student Prerequisites

The PSP assumes that students have seen infinite series in an introductory calculus course. Instructors may find helpful some just-in-time review of series convergence as the convergence of a sequence of partial sums. The project also assumes that students have done a rigorous study of quantifiers.

## PSP Design, and Task Commentary

This PSP is designed to take around two weeks of classroom time, with some reading and tasks done outside class.

Bolzano proved that certain bounded sets have a least upper bound, but he does not phrase his results as we would today. In his proof, Bolzano implicitly assumed that (1) a Cauchy sequence in $\mathbb{R}$ must converge, and that (2) the real numbers have the Archimedean Property. The project draws attention to these hidden assumptions, but does not assume that students have seen the concepts of Cauchy sequences, least upper bounds, or the Archimedean Property. Indeed, the project can be used to introduce these ideas. On the other hand, instructors will need to discuss the interplay between these ideas early in the project if their course has already used the existence of least upper bounds or the Nested Interval Property as completeness axioms for $\mathbb{R}$.

Task 1 is useful for getting students comfortable with the partial sum notation using a common geometric series. Moreover, this series will play an important role later in the PSP.

The examples in Tasks 11 and 12 will be used multiple times later in the project to illustrate Bolzano's proof and make sure students can follow his arguments with concrete examples.

LaTeXcode of this entire PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

## Suggestions for Classroom Implementation

This is roughly a one or two week project under the following methodology (basically David Pengelley's "A, B, C" method described on his website https://www.math.nmsu.edu/~davidp/ ):

1. Students do some advanced reading and light preparatory tasks before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the tasks or your grading load will get out of hand! Some instructor have students write questions or summaries based on the reading.

2. Class time is largely dedicated to students working in groups on the project - reading the material and working tasks. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the tasks they missed. This is usually a good incentive not to miss class!

3. Some tasks are assigned for students to do and write up outside of class. Careful grading of these tasks is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

## Sample Implementation Schedule (based on a 50-minute class period)

Students read through the introductory material and do Tasks 1, 2 before the first class. After discussing their results at the beginning of Class 1, students work on and discuss Tasks 3, 4, 6, 9, 10. Tasks 5, 7, 10 can be assigned for homework.

As preparation for Class 2, students read about Axiom C and the first Bolzano excerpt with Bolzano's Bounded Set theorem, and do Task 11. After discussing their results at the beginning of Class 2, students work on and discuss Tasks 12, 13. Students then read the first part of Bolzano's proof and do Task 14.

As preparation for Class 3, students do Tasks 15–17. After discussing their results at the beginning of Class 3, students work on and discuss Tasks 19–20. Students then read the second part of Bolzano's proof and do Task 21. Task 22 and 25 can be assigned for homework.

As preparation for Class 4, students do Tasks 23 and 24. After discussing their results at the beginning of Class 4, students read the third part of Bolzano's proof and work on Tasks 26–28. Students then read the fourth part of Bolzano's proof and work on Task 29. Some of these tasks can be given as homework, as time permits.

As preparation for Class 5, students read the fifth part of Bolzano's proof and do Tasks 30, 31. After discussing their results at the beginning of Class 5, students work on Tasks 32–36. Some of these tasks can be given as homework, as time permits.

As preparation for Class 6, students read the beginning of Section 4 of the PSP and do Task 37. After discussing their results at the beginning of Class 6, students finish the project.

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students.

## Connections to other Primary Source Projects

Other projects for real analysis written by the author of this PSP (Dave Ruch) are listed below. "Mini-PSPs," designed to be completed in 1–2 class periods, are designated with an asterisk (*).

- *Bolzano's Definition of Continuity, his Bounded Set Theorem, and an Application to Continuous Functions*
  `https://digitalcommons.ursinus.edu/triumphs_analysis/13`
- *An Introduction to a Rigorous Definition of Derivative*
  `https://digitalcommons.ursinus.edu/triumphs_analysis/7`
- *The Mean Value Theorem*
  `https://digitalcommons.ursinus.edu/triumphs_analysis/5/`
- *The Definite Integrals of Cauchy and Riemann*
  `https://digitalcommons.ursinus.edu/triumphs_analysis/11/`
- *Investigations Into d'Alembert's Definition of Limit** (sequences)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/13/`
- *Euler's Rediscovery of e**
  `https://digitalcommons.ursinus.edu/triumphs_analysis/3/`
- *Abel and Cauchy on a Rigorous Approach to Infinite Series*
  `https://digitalcommons.ursinus.edu/triumphs_analysis/4/`

Additional PSPs that are suitable for use in introductory real analysis courses include the following; the PSP author name for each is listed parenthetically.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/1/`
- *Topology from Analysis** (Nick Scoville)
  Also suitable for use in a course on topology.
  `https://digitalcommons.ursinus.edu/triumphs_topology/1/`
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real Analysis* (Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/10/`
- *The Cantor set before Cantor** (Nick Scoville)
  Also suitable for use in a course on topology.
  `https://digitalcommons.ursinus.edu/triumphs_topology/2/`
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/2/`

### Recommendations for Further Reading

The translations by Russ in [Russ, 1980] and [Russ, 2004] also include interesting background on Bolzano as well as commentary on some of the subtleties of Bolzano's work. The articles in [Jahnke, 2003] give some perspective on other works in analysis during Bolzano's era.

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