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Investigations Into d'Alembert's Definition of Limit (Real Analysis Version)

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1 Introduction

The modern definition of a limit evolved over many decades. One of the earliest attempts at a precise definition is credited to Jean-Baptiste le Rond d'Alembert (1717–1783), a French mathematician, philosopher and physicist.[1] Among his many accomplishments, d'Alembert was a co-editor of the *Encyclopédie*, an important general encyclopedia published in France between 1751 and 1772. This work is regarded as a significant achievement of the Enlightenment movement in Europe.

D'Alembert argued in two 1754 articles of the *Encyclopédie* that the theory of limits should be put on a firm foundation. As a philosopher, d'Alembert was disturbed by critics who pointed out logical problems with limits and the foundations of calculus. He recognized the significant challenges of these criticisms, writing in [d'Alembert, 1754b] that

> This metaphysics [of calculus], of which so much has been written, is even more important, and perhaps as difficult to develop as these same rules of the calculus.

In this project we will investigate d'Alembert's limit definition and study the similarities and differences with our modern definition.

2 D'Alembert's Limit Definition

By 1754 mathematical techniques using calculus were quite advanced. D'Alembert won a 1747 prize for his work in partial differential equations, but became embroiled in arguments with Leonhard Euler (1707–1783) and others over methodology and foundational issues. These squabbles contributed to his interest in clearing up the foundations of limits and convergence.

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[1]Early chapters of d'Alembert's biography read like something out of *Masterpiece Theater*. He was born out of wedlock and left as an infant at the church Saint Jean le Rond in Paris. His mother, Claudine Guérin de Tencin, was a runaway nun who established a well-known Paris *salon*, a carefully orchestrated social gathering that brought together important writers, philosophers, scientists, artists and aristocrats for the purpose of intellectual and political discussions. Tencin never acknowledged d'Alembert as her son, and his father, Louis-Camus Destouches, found another woman to raise young Jean. Destouches died in 1726, but left funds for Jean's education. D'Alembert did well in school and became active as an adult in the philosophy, literature, science and mathematics of his day, standing "at the very heart of the Enlightenment with interests and activities that touched on every one of its aspects" [Hankins, 1990].

Here is d'Alembert's limit definition from the *Encyclopédie* [d'Alembert, 1754a]:

Limit. (Mathematics) One says that a magnitude is the *limit* of another magnitude, when the second may approach the first more closely than by a given quantity, as small as one wishes, moreover without the magnitude approaching, being allowed ever to surpass the magnitude that it approaches; so that the difference between a quantity and its *limit* is absolutely unassignable.

For example, suppose we have two polygons, one inscribed and the other circumscribed about a circle; it is clear that one may increase the sides as much as one wishes, and in that case each polygon will approach ever more closely the circumference of the circle; the perimeter of the inscribed polygon will increase, and that of the circumscribed polygon will decrease; but the perimeter or edge of the first will never surpass the length of the circumference, and that of the second will never be smaller than that same circumference; the circumference of the circle is therefore the limit of the increase of the first polygon and the decrease of the second.

...

Strictly speaking, the *limit* never coincides, or is never equal to the quantity of which it is the *limit*; but the latter approaches it ever more closely, and may differ from it by as little as one wishes. The circle, for example, is the *limit* of the inscribed and circumscribed polygons; for strictly it never coincides with them, though they may approach it indefinitely.

Note that this definition is lacking in precise, modern mathematical notation. Also observe that the polygon/circle example is for the limit of a *sequence*. Here is a standard first-year calculus book definition of limit for a sequence:

First-Year Calculus Definition. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

Let's examine some examples.

Task 1 Draw a diagram for a circle of radius 1 and an inscribed regular polygon with $n = 8$ sides. Use some basic trigonometry to find the exact length of the polygon's perimeter. How close is it to the circle's circumference?

Task 2 Use modern subscript notation for an appropriate sequence to rewrite d'Alembert's "inscribed polygon $\to$ circle" limit example. Assume for simplicity that the inscribed polygons are regular with n sides centered at the circle's center. These polygons have perimeter formula

$$perimeter = 2n \cdot radius \cdot \sin(\pi/n)\,.$$

As a bonus, derive the perimeter formula, and use Calculus to confirm this limit.[2]

[2]Notice this example gives a way to approximate π. There are many other ways to estimate π without trigonometry, including Archimedes' method.

Task 3 Consider d'Alembert's "inscribed polygon $\rightarrow$ circle" limit example and his definition. For 'given quantity' 0.1 and a circle of radius 1, how many sides for the inscribed polygon are needed to guarantee the 'second may approach the first more closely than' given quantity 0.1? Technology will be helpful! How many sides are needed for given quantity 0.01?

Task 4 Consider the sequence $\{a_n\}$ with $a_n = \dfrac{n}{2n+1}$.

(a) Find the limit of this sequence by any means.

(b) For 'given quantity' 0.01, suppose we want a_n and its limit to 'differ by as little as' 0.01. What is "sufficiently large" for n to guarantee that a_n and its limit differ by 0.01 or less?

We have seen that this 'given quantity' is a measure of an allowable difference or tolerance between a sequence term a_n and the limit itself. We next generalize this example a bit, replacing 'given quantity' 0.01 by a generic tolerance value ϵ.

Task 5 For sequence $\left\{\dfrac{n}{2n+1}\right\}$, let ϵ be an arbitrary small positive number. Suppose we want a_n and its limit to differ by less than ϵ. In terms of ϵ, what is "sufficiently large" for n?

Task 6 Look closely at d'Alembert's phrase 'Strictly speaking, the *limit* never coincides, or is never equal to the quantity of which it is the limit' and notice that it does not appear in the First-Year Calculus definition. Find a simple convergent sequence that violates this requirement of d'Alembert's limit definition.

Task 7 Consider d'Alembert's phrase 'without the magnitude approaching, being allowed ever to surpass the magnitude that it approaches' and notice that it does not appear in the First-Year Calculus definition. Find a simple convergent sequence that violates this requirement of d'Alembert's limit definition.

Task 8 Use modern notation to help rewrite d'Alembert's limit definition for sequences with the quantifiers "for all" and "there exists" and inequalities. The First-Year Calculus Definition and a graph of the sequence $\{a_n\}$ should be helpful in getting started. You should introduce a variable ϵ to bound the distance between the quantities, and another variable M to measure n being "sufficiently large." Be sure to include d'Alembert's requirements that sequence terms can neither surpass nor coincide with the limit in your answer.

As we have seen, d'Alembert's 1754 limit definition doesn't fully apply to some types of sequences studied by today's mathematicians. It is interesting to note that during d'Alembert's era there was some debate regarding whether or not a quantity could ever reach or surpass its limit.[3] Based on your work with d'Alembert's definition of limit, what do you think was d'Alembert's opinion on these questions?

During the 1800s mathematicians reached a consensus that limits could be attained, and a convergent sequence could indeed oscillate about its limit. We see the First-Year Calculus definition allows for these possibilities; however, it is too vague for actually constructing complex proofs. We can remedy this problem by clarifying the logic and converting some verbal descriptions into algebraic inequalities.

[3]For more on these issues in the evolution of the limit concept, see J. Grabiner's fascinating book [Grabiner, 2010].

Task 9 Use the quantifiers "for all" and "there exists" and inequalities to rewrite the First-Year Calculus limit definition for sequences, without the extra requirements that d'Alembert imposed in his definition. Then comment on the differences between this definition and your definition from Task 8.

Task 10 Use your definition from Task 9 to prove that sequence $\left\{\dfrac{n}{2n+1}\right\}$ converges.

Task 11 Suppose that a sequence $\{c_n\}$ converges to limit 1. Use your definition from Task 9 to prove that there exists a natural number M for which $0.9 < c_n < 1.1$ whenever $n \geq M$.

3 Limit Properties

D'Alembert also made two assertions about limit properties in his article [d'Alembert, 1754a], and gave a proof of one property using his limit definition.

> [Claim] 1st. If two values are the *limit* of the same quantity, these two quantities are equal to each other.
>
> [Claim] 2nd. Suppose A $\times$ B is the product of two magnitudes A, B. Let us suppose that C is the *limit* of the quantity A, and D the *limit* of the quantity B; I say that C $\times$ D, the product of the *limits*, will necessarily be the *limit* of A $\times$ B, the product of the two quantities A, B.

One quality of a good modern definition is that it should be useful in constructing proofs of a concept's properties. Let's investigate d'Alembert's first uniqueness claim and his proof.

Task 12 Write d'Alembert's Claim 1st for sequences in modern notation.

Here is d'Alembert's proof of uniqueness from [d'Alembert, 1754b].

> Given Z and X, limits of the same quantity Y, I say that X = Z, because if there is some difference between them, such as V, it would be X = Z ± V. By hypothesis, the quantity Y can approach X as closely as we desire. That is to say that the difference between Y and X can be as small as wished. Therefore, since Z differs from X by the quantity V, it follows that Ycannot approach Z any closer than the quantity V, and consequently, that Z is not the limit of Y , which is contrary to the hypothesis.

Task 13 Rewrite this uniqueness proof using your modern definition from Task 9.

The proof of d'Alembert's Claim 2nd is harder and d'Alembert did not give one in his article.

Task 14 Write d'Alembert's Claim 2nd for sequences in modern notation.

The next task investigates a proof for a special case of the second claim on the product of sequences to give you some appreciation of the challenges. It may give you ideas for writing a general proof!

Task 15 Suppose you know a sequence $\{a_n\}$ is within 0.01 of its limit $C = 5$ if n is larger than the integer $N_1 = 47$. Also suppose you know a sequence $\{b_n\}$ is within 0.01 of its limit $D = 3$ if n is larger than the integer $N_2 = 92$. Determine how far you must go with sequence $\{a_n b_n\}$ to get close to the product of limits $CD = 15$. How little difference between $a_n b_n$ and CD can you **guarantee** if you go out far enough?

Task 16 Use ideas from the previous task to prove d'Alembert's Claim 2nd.

4 Conclusion

Historians have noted that definitions of limit were given verbally by mathematicians of the 1600s and 1700s. However, to make these ideas useful in rigorous proofs, it is important to translate the verbal limit definition into one with clear logic and algebraic language, as you accomplished in Task 9. The mathematician Augustin-Louis Cauchy (1789–1867) is usually credited with being the first to do this, using ϵ and precise inequalities in some of his proofs. Even so, his definition of limit was verbal and similar to d'Alembert's, except that for Cauchy limits could be attained and surpassed, as in the modern definition. The modern limit definition we see today finally matured in the work of Karl Weierstrass (1789–1867) and his students.

How influential was d'Alembert's limit definition? This is hard to say, since d'Alembert only used his definition to carry out one proof. Certainly his advocacy for a precise limit definition may have influenced mathematicians such as Cauchy, and can thus be considered a worthy contribution to the evolution of the rigorous limit definition we use today.

References

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Jean le Rond d'Alembert. Calcul différentiel. In *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, volume 9, pages 985–988. Paris, 1754b. Translation by Gregory Bringman (2003), in *The Encyclopedia of Diderot & d'Alembert Collaborative Translation Project*, Ann Arbor: Michigan Publishing, University of Michigan Library. `http://hdl.handle.net/2027/spo.did2222.0001.091`.

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Notes to Instructors

This mini-Primary Source Project (mini-PSP) is designed to investigate the definition of limit for sequences, beginning with d'Alembert's definition and a modern Introductory Calculus text definition. Similarities and differences are explored.

Two versions of this project are available, for very different audiences.

- One version is aimed at Real Analysis students. **This is the version you are currently reading.** D'Alembert's definition is completely verbal, and the Section 2 tasks lead students through some examples and a translation of this definition to one with modern notation and quantifiers. Students are asked to find examples illustrating the difference between the modern and d'Alembert definitions. Section 3 investigates two limit properties stated by d'Alembert, including modern proofs of the properties. Some historical remarks are given in a concluding section.

- A shorter version of this mini-PSP is aimed at Calculus 2 students studying sequences. Section 3 on limit properties and some of the more technical tasks in Section 2 are omitted from this version.

PSP Content: Topics and Goals

1. Develop a modern limit definition with quantifiers for sequences based on d'Alembert's definition and an Introductory Calculus text definition.

2. Analyze subtleties of the limit definitions: whether sequence terms can surpass or coincide with the limit.

3. Develop facility with the modern limit definition by using it to prove a given sequence converges.

4. Students also analyze the uniqueness property of limits and explore the limit of a *product* of convergent sequences. A final task asks for a general proof of the limit of a *product* of convergent sequences.

Student Prerequisites

This version of the project is written for a course in Real Analysis with the assumption that students have become somewhat comfortable with quantifiers, but no other background is assumed. (The author has used this mini-PSP on Day 1 of the course.)

PSP Design, and Task Commentary

This mini-PSP is designed to take two days of classroom time where students work through tasks in small groups. Some reading and tasks are done before and after class. If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

The PSP is designed to be used largely in place of a textbook section introducing the definition of limit for sequences. The differences between the d'Alembert and modern definition can help students realize subtleties and the precision of the modern definition.

Instructors may want to use Task 5 to discuss the Archimedean property of the real numbers, which most students take for granted.

Task 8 may be difficult for students. Encouraging students to draw a plot and labels for ϵ and M should help. Leading questions to help them realize that the definition needs to start with "for all $\epsilon > 0$" may also be helpful. Including d'Alembert's requirements that sequence terms can't "surpass" or coincide with the limit is challenging but pedagogically useful.

Note that d'Alembert's uniqueness proof, using contradiction, is different than most traditional Analysis book proofs. Students may need help with the inequalities for Task 13. Task 11 should be helpful for converting to modern epsilon terminology the fact that sequence Y is assumed to converge to limit X.

Students may need a hint for Task 15 on the product of limits, something like $a_n b_n - 5 \cdot 3 = (a_n b_n - a_n \cdot 3) + (a_n \cdot 3 - 5 \cdot 3)$ or a verbal or visual version of this identity. The proof of the more general result in the last task is very similar, but gives students practice writing proofs with epsilons.

Suggestions for Classroom Implementation

Advanced reading of the project and some task work before each class is ideal but not necessary. See the sample schedule below for ideas.

LaTeX code of this entire mini-PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The mini-PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

Students read through the d'Alembert excerpt and do Tasks 1–3 before the first class. After a class discussion of these tasks, students work through Tasks 4–8 in groups. Task 8 is critical, so a class discussion after the group work is advisable to make sure everyone understands this task. Tasks 9–11 are assigned for homework, but Tasks 9 and 10 need to discussed at the beginning of the second class. Then students read d'Alembert's limit properties and his uniqueness proof, and work through Tasks 12 and 13 in groups. For time purposes, you may want to give a polished version of the proof in Task 13 as homework. Student groups then work on Tasks 14 and 15. If you have only two class periods, you may want to have them finish Task 15 for homework and save Task 16 for a later class discussion. A third class day might be preferable if time permits.

Connections to other Primary Source Projects

Other projects for real analysis written by the author of this PSP (Dave Ruch) are listed below. "Mini-PSPs," designed to be completed in 1–2 class periods, are designated with an asterisk (*).

- *Bolzano's Definition of Continuity, his Bounded set Theorem, and an Application to Continuous Functions*, `https://digitalcommons.ursinus.edu/triumphs_analysis/9/`
- *An Introduction to a Rigorous Definition of Derivative* `https://digitalcommons.ursinus.edu/triumphs_analysis/7`
- *The Mean Value Theorem* `https://digitalcommons.ursinus.edu/triumphs_analysis/5/`

- *The Definite Integrals of Cauchy and Riemann*
  `https://digitalcommons.ursinus.edu/triumphs_analysis/11/`
- *Euler's Rediscovery of e**
  `https://digitalcommons.ursinus.edu/triumphs_analysis/3/`
- *Abel and Cauchy on a Rigorous Approach to Infinite Series*
  `https://digitalcommons.ursinus.edu/triumphs_analysis/4/`

Additional PSPs that are suitable for use in introductory real analysis courses include the following; the PSP author name for each is listed parenthetically.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/1/`
- *Topology from Analysis** (Nick Scoville)
  Also suitable for use in a course on topology.
  `https://digitalcommons.ursinus.edu/triumphs_topology/1/`
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real Analysis* (Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/10/`
- *The Cantor set before Cantor** (Nick Scoville)
  Also suitable for use in a course on topology.
  `https://digitalcommons.ursinus.edu/triumphs_topology/2/`
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Barnett)
  `https://digitalcommons.ursinus.edu/triumphs_analysis/2/`

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