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The Definite Integrals of Cauchy and Riemann

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1 Introduction

Rigorous attempts to define the definite integral began in earnest in the early 1800s. A major motivation at the time was the search for functions that could be expressed as Fourier series as follows:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kw) + b_k \sin(kw)\right) \tag{1}$$

where the coefficients are:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt \quad , \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt.$$

Expanding a function as an infinite series like this may remind you of Taylor series $\sum c_k x^k$ from introductory calculus, except that the Fourier coefficients a_k, b_k are defined by definite integrals and sine and cosine functions are used instead of powers of x.

Joseph Fourier (1768-1830) argued in 1807 that this series expansion was valid for *any* function f, and he used the expansion in his study of heat conduction. This ambitious claim was met with considerable skepticism among mathematicians, but it certainly motivated much research into the convergence of these infinite series. One of the pioneers in these developments was Augustin-Louis Cauchy (1789–1857). In particular, Cauchy published a study of the definite integral for continuous functions in his 1823 *Calcul Infinitésimal*[1] [Cauchy, 1823], from which we will read in Section 2 of this project.

Both Fourier and Cauchy attempted to prove that the Fourier series would converge to $f(x)$ under suitable conditions. Unfortunately, both proof attempts had flaws. Gustav Lejeune Dirichlet (1805–1859) read their work and, after pointing out an error in Cauchy's proof, set out to give a rigorous proof in his 1829 paper "Sur la convergence des séries" trigonométriques"[2] [Dirichlet, 1829].

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[1]The full title of Cauchy's textbook is *Résumé des leçons données à L'Ecole Royale Polytechnique sur le calcul infinitésimal*, or *Summary of lessons given on the infinitesimal calculus at the Royal Polytechnic School*.

[2]The full title of this paper was "Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données," or "On the convergence of trigonometric functions representing arbitrary functions between given limits."

Specifically, Dirichlet gave a proof of Fourier series convergence that is valid for a piecewise continuous function f with finitely many jump discontinuities[3] and a finite number of extrema. He then discussed the possibility of extending his proof for a function ϕ with infinitely many extrema (in a bounded interval), but he didn't hold much hope for functions with infinitely many discontinuities. To indicate why, he gave an example that quickly became famous in mathematical circles of his day. The next passage[4] is from Dirichlet's discussion of the Fourier series for a function with discontinuities. Dirichlet argued that he could extend his proof for a function ϕ with infinitely many extrema, assuming that:

> ... the function $\phi(x)$ be such that, if one denotes by a and b two arbitrary quantities lying between $-\pi$ and π, one can always find between a and b two quantities r and s so close that the function is continuous in the interval from r to s. One easily appreciates the necessity of this restriction by considering that the different terms of the series [(1)] are definite integrals and by going back to the fundamental notion of integrals. One will then see that the integral of a function only means something when the function satisfies the condition previously stated. One will have an example of a function that does not satisfy this condition, if one supposes that $\phi(x)$ is equal to a definite constant c when the variable x has a rational value, and is equal to a different constant d when this variable is irrational. The function thus defined has finite and definite values for each value of x, however, one cannot substitute it in the series because the different integrals which enter this series [(1)] lose all meaning in this case.

For the rest of project, we'll refer to this example function as "Dirichlet's function ϕ", and we will label a key property Dirichlet described:

Dirichlet Condition. For any two values a and b where $-\pi < a < b < \pi$, we can find two values r and s, with $a < r < s < b$, such that the function is continuous in the interval from r to s.

Task 1 Consider the example function $\phi(x)$ Dirichlet gave in the excerpt. Dirichlet claimed this function does not satisfy the Dirichlet Condition.

(a) Show that Dirichlet's function ϕ is not continuous at any rational x.

(b) Prove ϕ is not continuous at any irrational x.

(c) Use parts (a) and (b) to verify Dirichlet's claim that ϕ does not satisfy the Dirichlet Condition.

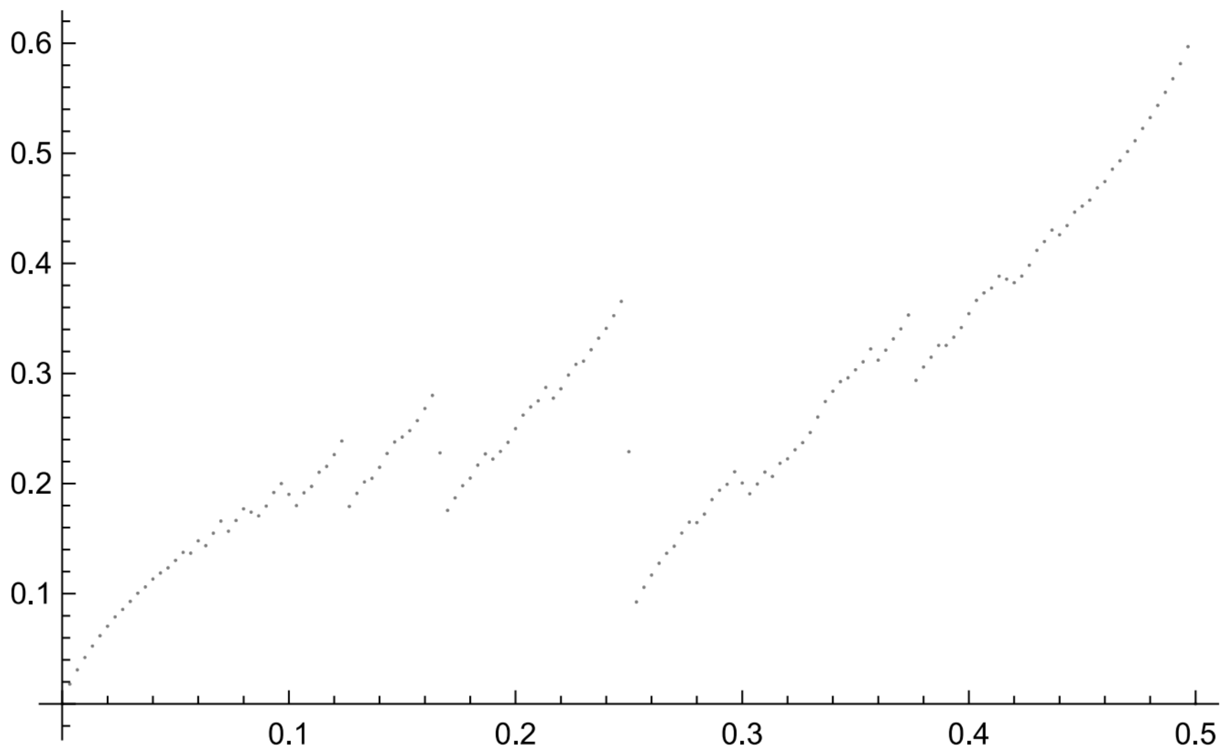
It is important to remember that in 1829 the only definition of the definite integral was the one given by Cauchy, and that definition was only for continuous functions. Thus we can see why Dirichlet

[3] At each point of discontinuity, the one-sided limits exist and are finite.

[4] This passage from [Dirichlet, 1829] was translated by Jeremy Gray in [Gray, 2015], with minor changes by the project author.

felt "One easily appreciates the necessity of . . . going back to the fundamental notion of integrals." In particular, notice that the Fourier series coefficients a_k and b_k in (1) are *defined* in terms of integrals. So, if the function $f(x)$ is strange enough that these coefficient integrals a_k, b_k don't make sense, then expanding $f(x)$ as a Fourier series won't be possible.

While the study of Fourier series raged on for the next couple decades, it wasn't until 1854 that Bernard Riemann developed a more general concept of the definite integral that could be applied to functions with infinite many discontinuities, in the paper "Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe" ("On the representability of a function by a trigonometric series")[5] [Riemann, 1854]. Amazingly, he also constructed an integrable function with infinitely many discontinuities that does not satisfy Dirichlet's Condition above — see this function's graph below. We will read about Riemann's work in Section 3 and the conclusion of this project.



[5] Riemann completed this work in partial fulfillment of the requirements for the extra post-doctoral qualification (or *habilitation*) that allowed one to become a lecturer in German universities. Its eventual posthumous publication was due to the efforts of Richard Dedekind (1831–1916), who arranged for it to appear in *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*(*Treatises of the Göttingen Royal Society of Sciences*), vol. 13, 1868.

2 Cauchy's Definite Integral

Most mathematicians before Cauchy's time preferred to think of integration as the inverse of differentiation: to evaluate $\int_a^b f(x)\,dx$ you found an antiderivative F of f and evaluated $F(b) - F(a)$. However, there was plenty of 18th century mathematics evaluating difficult integrals approximately using sums. Cauchy used many of their ideas in creating his new definition of the definite integral.

Cauchy was a professor at the École Polytechnique in Paris during the 1820s when he wrote two texts on the calculus. He developed his theory of the definite integral for continuous functions in his 1823 *Calcul Infinitésimal* [Cauchy, 1823]. In his foreword to this work, Cauchy stated that:

> My main goal has been to reconcile the rigor, which I have made a law in my *Analysis Course*, with the simplicity which results from the direct consideration of infinitely small quantities.

That is, he wanted to validate important and well-known calculus results using what he considered a more rigorous approach, without recourse to the use of infinitesimals.

We will read his development over the course of several excerpts in this section of the project, beginning with the following.

Cauchy Excerpt A

Definite Integrals.

Suppose that, the function $y = f(x)$, being continuous with respect to the variable x between two finite limits $x = x_0$, $x = X$, we denote by $x_1, x_2, \ldots, x_{n-1}$ new values of x interposed between these limits, which always go on on increasing or decreasing from the first limit up to the second. We can use these values to divide the difference $X - x_0$ into elements

$$x_1 - x_0, \quad x_2 - x_1, \quad x_3 - x_2, \quad \ldots, \quad X - x_{n-1}, \tag{2}$$

which will always be the same sign. This granted, consider that we multiply each element by the value of $f(x)$ corresponding to the origin of this same element, namely, the element $x_1 - x_0$ by $f(x_0)$, the element $x_2 - x_1$ by $f(x_1)$, ... , finally, the element $X - x_{n-1}$ by $f(x_{n-1})$; and, let ...

$$S = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (X - x_{n-1}) f(x_{n-1}) \tag{3}$$

be the sum of the products thus obtained. The quantity S will obviously depend upon: 1° the number of elements n into which we will have divided the difference $X - x_0$; 2° the values of these same elements, and by consequence, on the mode of division adapted. Now, it is important to remark that, if the numerical values of the elements become very small and the number n very considerable, the mode of division will no longer have a perceptible influence on the value of S.

The next task will help ensure that you understand Cauchy's notation and concepts.

Task 2 Consider the example $f(x) = x^2 - 2$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 3/2$, $X = 2$, $n = 3$.

(a) Use your calculus knowledge to find $\int_{x_0}^{X} f(x)\,dx$.

(b) Find the elements $x_1 - x_0, x_2 - x_1, x_3 - x_2$ for this example. Then calculate the sum S. How close is S to $\int_{x_0}^{X} f(x)\,dx$?

(c) Make and label a diagram that graphically represents what is going on with Cauchy's construction of S in (3) for this example.

We will find it convenient to give a name to the set $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_{n-1}, X\}$. We will call $\mathcal{P}$ a **partition** of the interval $[a, b]$ and require the x_k values to be distinct. Cauchy's "mode of division" is equivalent to choosing a partition for the interval. Also, rather than continuing to use the letter S for different things, a handy modern notation is to include the partition in the notation. We will use the modern notation $S(f, \mathcal{P})$ for Cauchy's sum S to indicate the dependence of S on both f and $\mathcal{P}$.

Observe that Cauchy made a bold claim at the very end of the excerpt that we will call Claim M for "mode of division."

Claim M.
The mode of division will no longer have a perceptible influence on the value of S.

Task 3 What two requirements did Cauchy place on the mode of division to justify this Claim M?

Task 4 Write Cauchy's Claim M with modern terminology and quantifiers.

You may have noticed in the last task that the maximum element value will be important, and so we will give it a modern name. Define $\text{mesh}(\mathcal{P})$, the **mesh** of a partition $\mathcal{P}$, to be its maximum element value. For example, $\text{mesh}(\mathcal{P}) = 1$ for the partition $\mathcal{P}$ in Task 2.

In order to prove his claim, Cauchy partitioned each subinterval (x_{k-1}, x_k) and considered the corresponding sum $S(f, \mathcal{P}')$ for the new partition $\mathcal{P}'$ of $[x_0, X]$. From inside the first subinterval $[x_0, x_1]$ he chose m points $\left\{x_j^1\right\}_{j=1}^{m}$ with

$$x_0 < x_1^1 < x_2^1 < \cdots < x_m^1 < x_1$$

and considered the sum

$$S_1 = \left(x_1^1 - x_0\right) f(x_0) + \left(x_2^1 - x_1^1\right) f\left(x_1^1\right) + \left(x_3^1 - x_2^1\right) f\left(x_2^1\right) + \cdots + \left(x_1 - x_m^1\right) f\left(x_m^1\right). \quad (4)$$

Cauchy then used some very clever algebra and the Intermediate Value Theorem (IVT) with the continuity of f to show that

$$S_1 = f(c_1)(x_1 - x_0) \quad (5)$$

for some c_1 between x_0 and x_1. He carried out this process for each subinterval and then added up the sums to show that

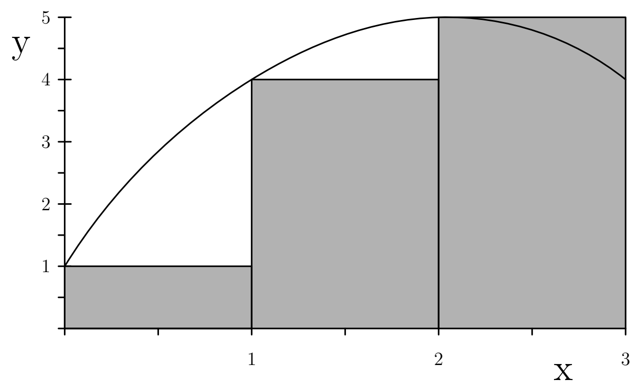
$$S\left(f, \mathcal{P}'\right) = f(c_1)(x_1 - x_0) + f(c_2)(x_2 - x_1) + \cdots + f(c_n)(X - x_{n-1})$$

for $c_k \in (x_{k-1}, x_k)$ chosen according to the IVT.

Task 5 Consider a continuous function $f(x)$ on the interval $[0,7]$, where part of the graph is given below. Let $x_0 = 0, x_1 = 3, m = 2$, where we partition the first subinterval $[x_0, x_1]$ as shown in the diagram below.

(a) Use the diagram and a rectangle area argument to estimate the value of c_1 for (5) with this example. Explain from the graph why we can be sure such a c_1 exists, even without knowing a formula for $f(x)$.

(b) Now assume $f(x) = 5 - (x-2)^2$. Calculate S_1 from the formula (4). Then find c_1 to one decimal place using algebra. Label c_1 on the diagram and compare with your estimate of c_1 in part (a).



Next we will read Cauchy's description of this process of partitioning each subinterval and deriving a new formula for the sum S.

Cauchy Excerpt B

To pass from the mode of division that we have just considered, to another in which the numerical values of the elements of $X - x_0$ are even smaller, it will suffice to partition each of the expressions in (2) into new elements. Then, we should replace, in the second member of equation (3), the product $(x_1 - x_0)\, f(x_0)$ by a sum of similar products, for which we can substitute an expression of the form

$$(x_1 - x_0) f\left[x_0 + \theta_0 (x_1 - x_0)\right], \qquad (6)$$

θ_0 being a number less than unity. ...

By the same reasoning, we should substitute for the product $(x_2 - x_1)\, f(x_1)$, a sum of terms which can be presented under the form

$$(x_2 - x_1) f\left[x_1 + \theta_1 (x_2 - x_1)\right],$$

θ_1 again denoting a number less than unity.

By continuing in this manner, we will finally conclude that, in the new mode of division, the value of S will be of the form

$$\begin{aligned} S &= (x_1 - x_0) f\left[x_0 + \theta_0 (x_1 - x_0)\right] \\ &\quad + (x_2 - x_1) f\left[x_1 + \theta_1 (x_2 - x_1)\right] + \cdots \\ &\quad + (X - x_{n-1}) f\left[x_{n-1} + \theta_{n-1} (X - x_{n-1})\right]. \end{aligned} \tag{7}$$

∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞

In modern terminology, we define a **refinement of partition** to describe what Cauchy called the "new mode of division" in which we partition each of the expressions in (2) into new elements. If we let $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_{n-1}, X\}$ be the original partition and let $\mathcal{P}'$ be the refinement, $\mathcal{P}'$ will include X and all the x_k plus some additional values between x_0 and X. For example, a refinement of the partition $\mathcal{P} = \{0, 1/2, 3/2, 2\}$ in Task 2 is $\mathcal{P}' = \{0, 1/3, 1/2, 7/8, 1, 3/2, 2\}$.

If we let $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_{n-1}, X\}$ be Cauchy's original partition and let $\mathcal{P}'$ be a refinement, then in modern terminology the sum in (3) is $S(f, \mathcal{P})$ and the sum in (7) is $S(f, \mathcal{P}')$.

Task 6 Consider the example in Task 2.

(a) Use your results from Task 2 to give $S(f, \mathcal{P})$ where $\mathcal{P} = \{0, 1/2, 3/2, 2\}$ is the original partition of interval $[0, 2]$.

(b) Find a refinement $\mathcal{P}''$ of $\mathcal{P}$ that is **not** a refinement of $\mathcal{P}' = \{0, 1/3, 1/2, 7/8, 1, 3/2, 2\}$.

2.1 Comparing $S(f, \mathcal{P})$ and $S(f, \mathcal{P}')$ for refinement $\mathcal{P}'$.

Let's reflect briefly on what Cauchy cleverly created with his expression (7) for the sum $S(f, \mathcal{P}')$ with *refined* partition $\mathcal{P}'$. He had

$$S(f, \mathcal{P}) = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (X - x_{n-1}) f(x_{n-1})$$

and

$$S\left(f, \mathcal{P}'\right) = (x_1 - x_0) f\left[x_0 + \theta_0 (x_1 - x_0)\right] + \cdots + (X - x_{n-1}) f\left[x_{n-1} + \theta_{n-1} (X - x_{n-1})\right],$$

which are both expressions in terms of the original partition $\mathcal{P}$ values $x_0, x_1, \ldots, x_{n-1}, X$. Then he could work more easily with the difference $S(f, \mathcal{P}) - S(f, \mathcal{P}')$, which is allegedly tiny, in his proof of Claim M. Let's see how he did it.

Cauchy Excerpt C

∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞

If in this last equation [(7)] we let

$$\begin{aligned} f\left[x_0 + \theta_0 (x_1 - x_0)\right] &= f(x_0) \pm \epsilon_0, \\ f\left[x_1 + \theta_1 (x_2 - x_1)\right] &= f(x_1) \pm \epsilon_1, \\ \cdots &\quad \cdots \\ f\left[x_{n-1} + \theta_{n-1} (X - x_{n-1})\right] &= f(x_{n-1}) \pm \epsilon_{n-1} \end{aligned} \tag{8}$$

we will derive

$$S = (x_1 - x_0)[f(x_0) \pm \epsilon_0] + (x_2 - x_1)[f(x_1) \pm \epsilon_1] + \cdots + (X - x_{n-1})[f(x_{n-1}) \pm \epsilon_{n-1}] \; ; \tag{9}$$

then, by developing products,

$$S = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (X - x_{n-1}) f(x_{n-1}) \pm \epsilon_0 (x_1 - x_0) \pm \epsilon_1 (x_2 - x_1) \pm \cdots \pm \epsilon_{n-1}(X - x_{n-1}) \, . \tag{10}$$

Add that, if the elements $x_1 - x_0, x_2 - x_1, \ldots, X - x_{n-1}$ have very small numerical values, each of the quantities $\pm\epsilon_0, \pm\epsilon_1, \ldots, \pm\epsilon_{n-1}$ will differ very little from zero; and as a result, it will be the same for the sum

$$\pm\epsilon_0 (x_1 - x_0) \pm \epsilon_1 (x_2 - x_1) \pm \cdots \pm \epsilon_{n-1}(X - x_{n-1})$$

...

This granted, it follows from equations (3) and (10), when compared to each other, that we will not significantly alter the calculated value of S for a mode of division in which the elements of the difference $X - x_0$ have very small numerical values, if we pass to a second mode in which each of these elements are found subdivided into several others.

Notice that Cauchy had not yet compared the sums $S(f, \mathcal{P}), S(f, \mathcal{Q})$ for two *arbitrary* partitions $\mathcal{P}$, $\mathcal{Q}$ with small mesh. At this point he was working only with refinements. Let's rewrite what Cauchy proved as a lemma, using modern terminology with quantifiers.

Lemma 1 *Suppose f is continuous on $[a, b]$. For any $\epsilon > 0$, we can find $d > 0$ such that if mesh$(\mathcal{P}) < d$ and $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then $|S(f, \mathcal{P}) - S(f, \mathcal{P}')| < \epsilon$.*

Task 7 A key to Cauchy's proof is his claim that "each of the quantities $\pm\epsilon_0, \pm\epsilon_1, \ldots, \pm\epsilon_{n-1}$ will differ very little from zero."What property of f allowed him to say this?

Task 8 Use Cauchy's ideas to give a modern proof of Lemma 1.

2.2 Comparing $S(f, \mathcal{P}_1)$ and $S(f, \mathcal{P}_2)$, and defining the definite integral

Cauchy next considered two "modes of division of the difference $X - x_0$, in each of which the elements of this difference have very small numerical values." That is, he wanted to compare the sums $S(f, \mathcal{P}_1), S(f, \mathcal{P}_2)$ for two arbitrary partitions $\mathcal{P}_1$,$\mathcal{P}_2$ with small mesh.

Cauchy Excerpt D

We can compare these two modes to a third, chosen so that each element, whether from the first or second mode, is found formed by the union of the various elements of the third. For this condition to be fulfilled, it will suffice that all the values of x interposed in the first two modes between the limits x_0, X are employed in the third, and we will prove that we alter the value of S very little by passing from the first or from the second mode to the third, and by consequence, in passing from the first to the second. Therefore, when the elements of the difference $X - x_0$ become infinitely small, the mode of division will no longer have a perceptible influence on the value of S; and, if we decrease indefinitely the numerical values of these elements, by increasing their number, the value of S will eventually be substantially constant, or in other words, it will finally attain a certain limit which will depend uniquely on the form of the function $f(x)$, and the extreme values x_0, X attributed to the variable x. This limit is what we call a definite integral.

Task 9 Explain what Cauchy meant by "it will suffice that all the values of x interposed in the first two modes between the limits x_0, X are employed in the third." Illustrate your explanation with an example where $\mathcal{P}_1 = \{1, 2, 3.5, 5\}$ and $\mathcal{P}_2 = \{1, 1.7, 2.9, 4.7, 4.8, 5\}$.

Task 10 Suppose we are given a continuous function g on $[a, b]$ and $\epsilon = 0.1$. Further, suppose we find the value d from Lemma 1 for $\epsilon/2$, and two partitions $\mathcal{P}_1$,$\mathcal{P}_2$ each with mesh less than d. Use Cauchy's reasoning and Lemma 1 to prove that

$$|S(g, \mathcal{P}_1) - S(g, \mathcal{P}_2)| \leq 0.1$$

Now we just need to generalize the previous task to finally give a modern equivalent to Cauchy's Claim M, that "when the elements of the difference $X - x_0$ become infinitely small, the mode of division will no longer have a perceptible influence on the value of S."

Task 11 State and prove a modern version of Claim M that generalizes Task 10.

After justifying Claim M, Cauchy then went on to define the definite integral $\int_a^b f$ as a limit, but he was not terribly precise about this limit. His basic idea was to choose any sequence of sums $S(f, \mathcal{P}_n)$ with $\lim_{n\to\infty} \operatorname{mesh}(\mathcal{P}_n) = 0$. Then your theorem from Task 11 can be used to show the sequence $\{S(f, \mathcal{P}_n)\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore has a limit, which we define to be the definite integral $\int_a^b f$. Riemann used a similar approach to define his definite integral, and the formal details of this development can be explored in the supplementary Section 5.

Many of Cauchy's ideas will work for finding integrals of functions with discontinuities, but he used continuity in a couple crucial spots.

Task 12 Reflect on Cauchy's development of the definite integral for continuous functions. Where did he use continuity? Which ideas would make sense even for functions with discontinuities?

To illustrate the problems with integrating functions with lots of discontinuities, we now look at Dirichlet's function ϕ and the theorem you proved in Task 11. We will label this as "Theorem M" as a reminder that it is a modern restatement of Cauchy's Claim M.

Theorem M. Suppose g is continuous on $[a,b]$. For any $\epsilon > 0$, we can find $d > 0$ such that if $\mathcal{P}_1$,$\mathcal{P}_2$ are partitions with mesh $(\mathcal{P}_1)$, mesh $(\mathcal{P}_2) < d$, then $|S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)| < \epsilon$.

Task 13 Prove that the conclusion of Theorem M is not valid for Dirichlet's function ϕ.

While we won't prove it here, the condition in Theorem M turns out to be necessary and sufficient for a function f to be integrable. We will see similar ideas developed—with some twists—by Riemann in the next section.

3 Riemann's Definite Integral

Cauchy's 1823 development of the definite integral for continuous functions was not extended to non-continuous functions for another three decades. While Dirichlet and others continued to research the problem of Fourier series convergence, no one looked hard at the definite integral itself until 1854, when Dirichlet's student Bernard Riemann took up the issue.

Riemann (1826–1866) was born near Hanover, Germany and studied mathematics at the University of Göttingen and Berlin University with strong influence by C. Gauss and Dirichlet. Despite his early death from tuberculosis, Riemann made major contributions in geometry, number theory and complex analysis, in addition to his work with Fourier series and the definite integral that bears his name.

Remember from the project introduction that Dirichlet was hoping to extend his Fourier series convergence proof to the case where the function $f(x)$ being expanded as a Fourier series has infinitely many but isolated discontinuities and infinitely many extrema. This clearly motivated his student Riemann to develop and use a more general definition of the definite integral. In particular, Riemann wanted an integral definition that could be used to find the Fourier coefficients a_k, b_k for functions with infinitely many discontinuities.

All excerpts in this section are from Riemann's 1854 paper [Riemann, 1854].

Riemann Excerpt A

Vagueness still prevails in some fundamental points concerning the definite integral. Hence I provide some preliminaries about the concept of a definite integral and the scope of its validity.

Hence first: What is one to understand by $\int_a^b f(x)\,dx$?

In order to establish this, we take a succession of values $x_1, x_2, \ldots, x_{n-1}$ between a and b arranged in succession, and denote, for brevity, $x_1 - a$ by δ_1, $x_2 - x_1$ by $\delta_2, \ldots$, $b - x_{n-1}$ by δ_n, and a positive number less than 1 by ϵ. Then the value of the sum

$$S = \delta_1 f(a + \epsilon_1\delta_1) + \delta_2 f(x_1 + \epsilon_2\delta_2) + \delta_3 f(x_2 + \epsilon_3\delta_3) + \cdots + \delta_n f(x_{n-1} + \epsilon_n\delta_n) \quad (11)$$

depends on the selection of the intervals δ and the numbers ϵ. If this now has the property, that however the δ's and ϵ's are selected, S approaches a fixed limit A when the δ's become infinitely small together, this limiting value is called $\int_a^b f(x)\,dx$.

If we do not have this property, then $\int_a^b f(x)\,dx$ is undefined. ... if the function $f(x)$ becomes infinitely large ... then clearly the sum S, no matter what degree of smallness one may prescribe for δ, can reach an arbitrarily given value. Thus it has no limiting value, and by the above $\int_a^b f(x)\,dx$ would have no meaning.

Observe that Riemann frequently wrote ϵ or δ where he clearly meant a set of ϵ_k or δ_k values.

From now on, we will say that if $\int_a^b f(x)\,dx$ exists according to Riemann's definition in Excerpt A, then f is **Riemann integrable** on $[a,b]$, and we will write $\int_a^b f$ for the definite integral.

Task 14 Consider the example with $f(x) = 2x^3 - 9x^2 + 12x + 1$, $a = 0, b = 3$, partition $\mathcal{P} = \{0, 1, 3\}$, and $\epsilon_1 = 1/2$, $\epsilon_2 = 3/4$. Make and label a diagram that graphically represents what is going on with Riemann's construction of S.

Task 15 Riemann had read Cauchy's work on the definite integral before writing his own paper. Compare and contrast Riemann's definition of the sum S in (11) with Cauchy's definition of sum S in (3) and Cauchy's reworked formulation of S in (7).

We've seen that in order to calculate the sum S for Riemann, we need to keep track of the ϵ_k values as well as the partition values x_k. For ease of notation, we will name the $x_{k-1} + \epsilon_k\delta_k$ values **tags** $t_k = x_{k-1} + \epsilon_k\delta_k$ and call the combined set of x_k and t_k values a **tagged partition**, writing $\dot{\mathcal{P}} = \{x_k, t_k\}_{k=1}^n$ for the tagged partition (with $x_0 = a$, $x_n = b$). Then we can write $S\left(f, \dot{\mathcal{P}}\right)$ for the sum S in (11) and call $S\left(f, \dot{\mathcal{P}}\right)$ a **Riemann sum**.

Task 16 What are the tags for the example in Task 14?

Task 17 Give a general inequality that relates the tags t_k and partition values x_k in Riemann's definition of $\int_a^b f$.

Task 18 Using appropriate quantifiers and modern notation for tagged partitions and mesh, rewrite Riemann's definition in Excerpt A for the existence of $\int_a^b f$.

After his definition of $\int_a^b f$, Riemann discussed the case where "the function $f(x)$ becomes infinitely large."We can rephrase his claim for this case with the following Theorem B, where the letter "B" suggests bounded:

Theorem B. If $f(x)$ is not bounded on $[a,b]$ then f is not Riemann integrable on $[a,b]$.

You will use Riemann's ideas in the next task to give a modern proof of Theorem B.

Task 19 To prove Theorem B, assume for the sake of contradiction that f is unbounded but integrable with $A = \int_a^b f$. Since f is integrable, using $\epsilon = 1$ we can find $\delta > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a,b]$ with mesh $\left(\dot{\mathcal{P}}\right) < \delta$ we have

$$\left|S\left(f;\dot{\mathcal{P}}\right) - A\right| < 1. \tag{12}$$

(a) Let $\mathcal{P}$ be a partition $\{x_k\}_{k=1}^n$ of $[a,b]$ with mesh $(\mathcal{P}) < \delta$. Explain why f must be unbounded on at least one subinterval of $[a,b]$, say $[x_{j-1}, x_j]$.

Now we will choose tags $\{t_k\}_{k=1}^n$ for $\mathcal{P}$ to get a contradiction to (12). Choose $t_k = x_k$ except for $[x_{j-1}, x_j]$ where f is unbounded. Then choose t_j so that

$$|f(t_j)| > \frac{1}{x_j - x_{j-1}}\left(|A| + 1 + \left|\sum_{k\neq j} f(t_k)(x_k - x_{k-1})\right|\right)$$

(b) Use part (a) and (12) to obtain a contradiction. The following triangle inequality may be helpful:

$$\left|\sum_{k=1}^n f(t_k)(x_k - x_{k-1})\right| \geq |f(t_j)(x_j - x_{j-1})| - \left|\sum_{k\neq j} f(t_k)(x_k - x_{k-1})\right|$$

The following exercises are not needed for the flow of Riemann's discussion, but will sharpen your skills in working with Riemann sums and Riemann's definition of the definite integral.

Task 20 Rewrite Theorem B in its equivalent contrapositive form.

Task 21 Use Riemann's definition to prove the following: Suppose g is Riemann integrable on $[a,b]$ and $c \in R$. Then cg is Riemann integrable on $[a,b]$.

Task 22 Use Riemann's definition to prove the following: Suppose f, g are Riemann integrable on $[a,b]$. Then $f+g$ is Riemann integrable on $[a,b]$.

Task 23 Is Dirichlet's function ϕ Riemann integrable on $[0,1]$? Prove your assertion.

Task 24 Define function h by $h(x) = 3$ for x in $[0,1]$ and $h(x) = 4$ on $(1,2]$. Is h Riemann integrable on $[0,2]$? Prove your assertion.

Task 25 Prove that changing the value of $f(x)$ at a finite number of points in $[a,b]$ will not change whether f is integrable, and will not change the value of $\int_a^b f$ when it exists.

Task 26 Use Riemann's definition to prove the following: Suppose $f(x) \geq 0$ on $[a,b]$ and f is Riemann integrable on $[a,b]$. Then $\int_a^b f \geq 0$.

After Riemann gave his new definition of the definite integral, he developed an alternate condition for the existence of $\int_a^b f$. Recall from Excerpt A that the Riemann sum (11) is

$$S = \delta_1 f(a + \epsilon_1\delta_1) + \delta_2 f(x_1 + \epsilon_2\delta_2) + \delta_3 f(x_2 + \epsilon_3\delta_3) + \cdots + \delta_n f(x_{n-1} + \epsilon_n\delta_n).$$

Riemann Excerpt B

Let us examine now, secondly, the range of validity of the concept, or the questions: In which cases can a function be integrated, and in which cases can it not?

We suppose that the sum S converges if the δ's together become infinitely small. We denote by D_1 the greatest fluctuation of the function between a and x_1, that is, the difference of its greatest and smallest values in this interval, by D_2 the greatest fluctuation between x_1 and $x_2, \ldots$, by D_n that between x_{n-1} and b. Then

$$\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n \tag{13}$$

must become infinitely small when the δ's do.

Task 27 Consider the example from Task 14 with $f(x) = 2x^3 - 9x^2 + 12x + 1$, $a = 0, b = 3$, and partition $\mathcal{P} = \{0, 1, 3\}$. Calculate D_1, D_2 and the "fluctuation" (13) for this partition $\mathcal{P}$. Are the tags relevant for the value of (13)?

Task 28 Write a brief "big picture" summary of this Excerpt B.

Task 29 Since f is not assumed to be continuous in general, we must actually define the D_k a bit differently than Riemann did.

(a) Explain why we need to adjust Riemann's D_k definition.

(b) Write a definition of the D_k using modern terms and set notation.

Note the expression in (13) appears frequently in Riemann's discussion, and roughly measures the total fluctuation of f across the entire partition $\mathcal{P}$. We will name this expression $\operatorname{Fluc}(f, \mathcal{P})$, a function of f and $\mathcal{P}$:

$$\operatorname{Fluc}(f, \mathcal{P}) = \delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n \tag{14}$$

We saw in Task 29 that the tags are not relevant for $\operatorname{Fluc}(f, \mathcal{P})$.

Task 30 Use quantifiers and $\operatorname{Fluc}(f, \mathcal{P})$ to rewrite Riemann's claim that the fluctuation (13) "must become infinitely small when the δ's do" for integrable f.

Task 31 Consider the example from Task 14 with $f(x) = 2x^3 - 9x^2 + 12x + 1$, $a = 0, b = 3$. For fixed $\epsilon = 0.1$, find a $d > 0$ such that for any partition $\mathcal{P}$ with $\operatorname{mesh}(\mathcal{P}) < d$, you can guarantee that $\operatorname{Fluc}(f, \mathcal{P}) < \epsilon$.

Task 32 Give a modern proof of Riemann's claim that the sum (13) "must become infinitely small when the δ's do" for integrable f, using Tasks 29, 30 and 18.

Observe that what Riemann was stating here is an indirect condition for integrability that doesn't involve $\int_a^b f$ itself: if f is integrable, then for any $\epsilon > 0$ we can find $d > 0$ so that for any partition $\mathcal{P}$ with mesh$(\mathcal{P}) < d$ we are guaranteed that the total fluctuation of f across $\mathcal{P}$ is less than ϵ. It turns out this condition is necessary and sufficient, which we record as Theorem R (for Riemann).

Theorem R A function f is Riemann integrable on $[a, b]$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ with mesh$(\mathcal{P}) < \delta$ then

$$\text{Fluc}(f, \mathcal{P}) < \epsilon. \tag{15}$$

You have shown the necessity of this condition (15) for integrability. The proof of sufficiency is rather technical. The basic idea is much the same as we outlined in Cauchy Section 2.2. We construct a sequence of partitions with mesh approaching zero and Riemann sums that converge, and prove, using Theorem R, that the limit of these Riemann sums is $\int_a^b f$. The details are given in the supplementary Section 5.

This characterization of integrability is very powerful. In the next two tasks you will use it to give fairly easy proofs that all continuous and monotone functions are integrable.

Task 33 Use Theorem R to prove that if f is continuous on $[a, b]$, then $\int_a^b f$ exists.

Task 34 Use Theorem R to prove that if f is monotone on $[a, b]$, then $\int_a^b f$ exists.

Later in his paper, Riemann constructed an integrable function with *infinitely many* discontinuities that does not satisfy Dirichlet's Condition on page 2. Amazing! This function's graph is displayed at the end of the project introduction and is discussed more in its conclusion. Well after Riemann's work, the mathematician Carl Thomae (1840–1921) devised another function with infinitely many discontinuities that is easier to show is integrable with the tools we've developed so far.

Thomae's Function. Define $T(x) : [0, 1] \to \mathbb{R}$ by $T(x) = 0$ for irrational x, $T(0) = 1$, and $T(m/n) = 1/n$ for rational $x = m/n$ where m/n is in reduced form.

Task 35 Show that Thomae's T function is continuous at all irrationals and discontinuous at all rationals.

Task 36 Use Theorem R to prove that T is integrable.

4 Conclusion

Riemann's definite integral raised new questions about the nature of $\int_a^b f$ as well as answering some old ones. On the one hand, he showed that you could integrate a function that has an infinite number of discontinuities densely packed into a bounded interval. This was mind-boggling for many mathematicians of his era! His necessary and sufficient conditions give new insight into how much a function can fluctuate at discontinuities and still remain integrable.

On the other hand, new questions about rules for handling integrals and infinite series occur naturally from his work. Riemann built his strange function, graphed on page 3, by starting with a piecewise linear function $E(x)$ that has a jump discontinuity at every odd multiple of 1/2, and defining a new function as the infinite sum

$$f(x) = \sum_{n=1}^{\infty} \frac{E(nx)}{n^2}.$$

Riemann used a fluctuation argument to indirectly show this $f(x)$ is integrable, without trying to actually calculate the integral's value. This raises the question: can you evaluate his function $f(x)$ legitimately by interchanging the integration and infinite sum? That is, can you integrate term by term:

$$\sum_{n=1}^{\infty} \int_a^b \frac{E(nx)}{n^2}\, dx \overset{???}{=} \int_a^b \sum_{n=1}^{\infty} \frac{E(nx)}{n^2}\, dx.$$

In general, this question of interchanging integration and infinite sums does not have an easy answer, and mathematicians in the 1800s had examples where term by term integration works fine, and other examples where it does not. The mathematician Henri Lebesgue (1875–1941) became convinced that an entirely new type integral was needed, and developed his own theory of integration, largely developed in his 1902 thesis. The Lebesgue integral has become very important in many fields of mathematics and statistics, and is frequently studied in graduate school.[6]

A final word of comfort to remember with all these new developments for strange functions and different integration theories: for any *continuous* function $f(x)$, the value of $\int_a^b f$ is the same for the integrals of Cauchy, Riemann and Lebesgue.

5 Appendix: Supplementary Tasks on the $\text{Fluc}(f, \mathcal{P})$ Sufficiency Condition

We saw in Section 2 that Cauchy defined the definite integral $\int_a^b f$ for continuous f in a rather imprecise way as a limit of sums $S(f, \mathcal{P})$. He also showed that if two partitions $\mathcal{P}_1, \mathcal{P}_2$ had sufficiently small mesh, then we could make the difference $S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)$ arbitrarily small.

Riemann also gave a condition for integrability in Theorem R using $\text{Fluc}(f, \mathcal{P})$ instead of $S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)$ and in Section 3 we proved the necessity but not the sufficiency of that condition. In the exercises below, you will prove the sufficiency. That is:

> If for all $\epsilon > 0$ there exists $d > 0$ such that
>
> $$\text{if } \mathcal{P} \text{ is a partition with } \text{mesh}(\mathcal{P}) < d, \text{ then } \text{Fluc}(f, \mathcal{P}) < \epsilon \tag{16}$$
>
> holds, then f is Riemann integrable on $[a, b]$.

To carry out this proof, a "Fluctuation Refinement Lemma" will be useful:

[6] The project "Henri Lebesgue and the Development of the Integral Concept" [Barnett, 2016] examines other shortcomings of the Riemann integral and explores the basic idea behind Lebesgue's theory of integration.

Fluctuation Refinement Lemma. Suppose f is bounded on $[a, b]$ and that partition $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Then

1. $\text{Fluc}(f, \mathcal{P}') \leq \text{Fluc}(f, \mathcal{P})$
2. $\left|S\left(f, \dot{\mathcal{P}}'\right) - S\left(f, \dot{\mathcal{P}}\right)\right| \leq \text{Fluc}(f, \mathcal{P})$ for any tags of $\mathcal{P}'$ and $\mathcal{P}$.

A complete proof by induction on the number of additional points in refinement is appropriate here. For ease of notation, the following task is for the case where $\mathcal{P}'$ adds just one point to $\mathcal{P}$ between a and x_1.

Task 37 Prove this lemma for the case $\mathcal{P} = \{a, x_1, x_2, \ldots, x_{n-1}, x_n\}$ and $\mathcal{P}' = \{a, x', x_1, x_2, \ldots, x_{n-1}, x_n\}$.

We don't yet have a candidate for $\int_a^b f$, so we will construct one using a Cauchy sequence of Riemann sums. To do this, first note that by condition (16) we can construct, for each $n \in \mathbb{N}$, a $d_n > 0$ so that:

1. $d_n \leq d_{n-1}$, and
2. for any partition $\mathcal{P}$ with $\text{mesh}(\mathcal{P}) < d_n$ we have $\text{Fluc}(f, \mathcal{P}) < 1/n$.

Next define a sequence of tagged partitions $\left\{\dot{\mathcal{P}}_n\right\}$ by

1. $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, and
2. $\text{mesh}\left(\dot{\mathcal{P}}_{n+1}\right) \leq \text{mesh}\left(\dot{\mathcal{P}}_n\right) < d_n$.

We will see that any tags will do for the $\dot{\mathcal{P}}_n$.

Task 38 Prove that $\left\{S\left(f, \dot{\mathcal{P}}_n\right)\right\}$ is a Cauchy sequence in $\mathbb{R}$.

Now let A denote $\lim_{n\to\infty} S\left(f, \dot{\mathcal{P}}_n\right)$, the limit for this Cauchy sequence of real numbers. This is our candidate for the integral of f! We will show this using the properties of $\text{Fluc}(f, \mathcal{P})$.

If $\dot{Q}$ is an arbitrary tagged partition with small mesh, we need to show its Riemann sum $S\left(f, \dot{Q}\right)$ is close to A. To do this, we will show $S\left(f, \dot{Q}\right)$ is close to some $S\left(f, \dot{\mathcal{P}}_K\right)$ where K is large enough to guarantee that $\left|S\left(f, \dot{\mathcal{P}}_K\right) - A\right|$ is tiny. The following tasks will be useful.

Task 39 Let $\dot{Q}$ be a tagged partition. For $K \in \mathbb{N}$ and any tags of partition $\mathcal{P}_K$, choose $\dot{\mathcal{P}}^*$ to be a refinement of both $\dot{Q}$ and $\dot{\mathcal{P}}_K$ with any tags. Then show that

$$\left|S\left(f, \dot{Q}\right) - A\right| \leq \left|S\left(f, \dot{Q}\right) - S\left(f, \dot{\mathcal{P}}^*\right)\right| + \left|S\left(f, \dot{\mathcal{P}}^*\right) - S\left(f, \dot{\mathcal{P}}_K\right)\right| + \left|S\left(f, \dot{\mathcal{P}}_K\right) - A\right|.$$

Task 40 Fix $\epsilon > 0$. Choose $K > 1/3\epsilon$. Choose d appropriately and use the Fluctuation Refinement Lemma and above tasks to show that

$$\left|S\left(f, \dot{Q}\right) - A\right| < \epsilon.$$

Task 41 Use the tasks above to prove that if f satisfies (16) then f is integrable on $[a, b]$.

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Notes to Instructor

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed to introduce the definite integral with some historical background for a course in Real Analysis. Specifically, its content goals are to:

1. Learn the basics of Cauchy's definite integral, including the concepts of partition, mesh, refinements and (Riemann) sums.
2. Learn the basics of Riemann's definite integral definition.
3. Develop elementary properties of the Riemann integral.
4. Learn about Riemann's "fluctuation" condition for integrability.
5. Apply the definition and properties of the Riemannn integral to develop more results on this integral.

Student Prerequisites

The project is written with the assumption that students have studied continuous functions and are well-versed in using quantifiers.

PSP Design and Task Commentary

The introductory, motivational discussion of Fourier series might merit some elaboration by the instructor, depending on the goals of the course and student interests. The project then moves to Cauchy's definite integral. Cauchy gives a more detailed development than Riemann, even if some aspects are specific to continuous functions. The first few Cauchy excerpts ease students into the ideas and notation of partitions, mesh, and refinements. It may interest students to see that Riemann's definition for a Riemann sum is nearly identical to Cauchy's reworked formulation (7).

A shorter version of the PSP could condense the Cauchy material into a short overview lecture, and focus on the Riemann material.

Additional comments about the design of and tasks in each section of the project follow.

- **Comments on Section 1**

  This material is included mostly to motivate the need for a rigorous treatment of the definite integral, especially for integrands with discontinuities. Analysis students have most likely encountered Dirichlet's function ϕ while studying continuity, but may not be aware of Dirichlet's motivations in creating this nowhere-continuous function.

- **Comments on Section 2**

  Cauchy's development of the integral with "new modes of division" (partition refinements) is quite useful for developing techniques for working with the Riemann integral, especially since Riemann did not spend much time developing properties of the integral.

  Task 2 is pretty basic, but may merit some extra time if students are rusty with Riemann sums from their introductory calculus courses. The function $f(x) = x^2 - 2$ is deliberately chosen to change sign in the interval $(0, 2)$ so that students are reminded of the "signed area" concept for integrals.

Between Cauchy Excerpts B and C, the project development includes the statement that "Cauchy then used some very clever algebra and the Intermediate value Theorem (IVT) with the continuity of f to show that $S_1 = f(c_1)(x_1 - x_0)$." This development from Cauchy's text is rather long and has been omitted for time purposes. It is probably too difficult for most students without considerable time and scaffolding, but might be of interest for a challenging bonus activity.

Cauchy's argument in Excerpt C uses the uniform continuity of integrand f; this will be needed again in a Section 3 task showing that continuous functions are Riemann integrable.

- **Comments on Section 3 and the Conclusion**

  Since Riemann did not spend much time developing properties of the integral, some tasks on elementary properties are inserted between Excerpts A and B. While these are not essential for reading the rest of Riemann's work, instructors may sample the set for classroom examples or homework problems.

  A detailed exploration of Riemann's "fluctuation" expression (13) is important. He explicitly used this fluctuation in a necessary condition for a function being integrable. He didn't show the sufficiency (which is difficult and left to the supplementary tasks in the optional Appendix section), but he did use it later in his paper. Some modern authors develop *oscillation* expressions very much like Riemann's fluctuation expression. It is interesting to note Riemann used the maximum of various expressions where a modern treatment requires a supremum.

  Riemann's work through Excerpt B, summarized in Theorem R, can be used to develop a great number of integration properties, some of which are given in the tasks. Thomae's function is given as a relatively simple example of an integrable function despite being discontinuous on the rationals.

Suggestions for Classroom Implementation

This is roughly a one or two week project under the following methodology (basically David Pengelley's "A, B, C" method described on his website.[7]:

1. Students do some advanced reading and light preparatory tasks before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the tasks or your grading load will get out of hand! Some instructor have students write questions or summaries based on the reading.

2. Class time is largely dedicated to students working in groups on the project—reading the material and working tasks. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some

[7] `https://web.nmsu.edu/~davidp/`

instructors collect the work. If a student misses class, I have them write up solutions to the tasks they missed. This is usually a good incentive not to miss class!

3. Some tasks are assigned for students to do and write up outside of class. Careful grading of these tasks is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

Sample Implementation Schedule (based on a 50-minute class period)

Full implementation of the project can be accomplished in 5–6 class days, as outlined below.

- Students read through the introductory material and do Task 1 before the first class. After discussing their results at the beginning of Class 1, students read Cauchy Excerpt A and work on and discuss Tasks 2–4.
- As preparation for Class 2, students read the material before Task 5 and do Task 5. After discussing their results at the beginning of Class 2, students read Cauchy Excerpt B, do Task 6, read Cauchy Excerpt C, and work on Tasks 7, 8.
- As preparation for Class 3, students read Cauchy Excerpt D and do Task 9. After discussing their results at the beginning of Class 3, students work on and discuss Tasks 10 and 12. Students then read Riemann Excerpt A and do Tasks 14–18. Tasks 11 and 13 can be assigned for homework.
- As preparation for Class 4, students do Task 19 and read Riemann Excerpt B. After discussing their results at the beginning of Class 4, students work on Tasks 27–31. Some subset of Tasks 20–26, 32 can be given as homework, as time permits.
- As preparation for Class 5, students do Task 33. After discussing their results at the beginning of Class 5, students do Tasks 35 and 36 on Thomae's function. Task 34 can be given as homework, as time permits.
- The Appendix material on $\mathrm{Fluc}\,(f, \mathcal{P})$ can be omitted, or another day can be used to work through this material.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be completed in at most 2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from `https://digitalcommons.ursinus.edu/triumphs\_topology`; all other listed projects are available at `https://digitalcommons.ursinus.edu/triumphs\_analysis`.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Heine Barnett)

- *Investigations into Bolzano's Bounded Set Theorem* (David Ruch)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)
  Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)
  A second version of this project suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Mean Value Theorem*(David Ruch)
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Heine Barnett)
- *Euler's Rediscovery of e** (David Ruch; sequence convergence, series & sequence expressions for e)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

Recommendations for Further Reading

The articles in [Jahnke, 2003] give some perspective on other 19th century works in analysis.

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