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# Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis

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Reflecting on the development of analysis during the nineteenth century, Henri Poincaré (1854–1912) lamented in [Poincaré, 1904, p. 263] that:[1]

Logic sometimes begets monsters. The last half-century saw the emergence of a crowd of bizarre functions, which seem to strive to be as different as possible from those honest [honnêtes] functions that serve a purpose. No more continuity, or continuity without differentiability, etc. ... In the old days, when a new function was invented, it was for a practical purpose; nowadays, they are invented for the very purpose of finding fault in the reasoning of our fathers, and nothing more will come out of it.

In this project, you will meet some of the strange functions that Poincaré so roundly condemned, together with certain mathematical concepts that grew out of efforts to tame these "monsters" in the latter part of the nineteenth century.

We begin in the next section with a brief overview of the background and motivations of one of the foremost "monster makers" of the nineteenth century, Gaston Darboux (1843-1917). In Section 2, we then examine a certain family of "monster functions" created by Darboux. Following this, Section 3 explores a function property — uniform differentiability — that Darboux created as a means to identify a new juncture in the hierarchy of function families. Section 4 focuses on two other function properties — continuity and the Intermediate Value Property — that were already well-known prior to Darboux's work. In fact, prior to Darboux's proof of an important theorem that now bears his name, these two function properties were often considered to be interchangeable. In Section 4, we will read and analyze Darboux's original proof of 'Darboux's Theorem' and see how that theorem implies that these two function properties are, in fact, quite distinct. In the closing section of the project, we then return to the question of what role (if any) these strange new beasts served, other than to merely find fault in our 'father's reasoning.'

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[1]All translations of excerpts from the works of Poincaré, Darboux, Hoüel and Borel in this project were prepared by the project author.

## 1 Gaston Darboux: Student, Teacher and Editor *par excellence*

We begin with some background information on Darboux and the setting in which he worked. Born on 14 August 1842, Darboux attended Lycée first in Nimes, and later in Montpellier. In 1861, he was admitted to both of the two most important Paris universities for the study of mathematics, the École Polytechnique and the École Normale Supérieur; he chose to attend the École Normale Supérieur. While a student there, he published his first paper on orthogonal surfaces; his 1866 doctoral thesis (under Michel Chasles) was on this same topic. From 1866–1867, Darboux taught at the Collège de France before spending five years at the Lycée Louis le Grand (1867–1872) and another four at the École Normale Supèrieur (1872–1881). He then moved to the Sorbonne where he taught for the remainder of his life. While at the Sorbonne, Darboux demonstrated his excellence as both a teacher and an organizer. For the last 17 years of his life (1900–1917), his talents as an organizer were also put to use in his capacity as the Secrétaire Perpétuel de l'Académie des Sciences.

Darboux excelled as an organizer and promoter of mathematical research as well. Of particular relevance to the story being told in this project was his role as a founding editor of the *Bulletin des Sciences*, sometimes referred to as "Darboux's *Bulletin*" in recognition of his role as its co-founder in 1870. The *Bulletin* published lists of titles of research papers from journals from outside of France, as well as summaries of the contents of the more important works and, when possible, complete translations of those papers. In this way, the *Bulletin* sought to provide the French mathematical community with access to cutting-edge research being conducted elsewhere that, for a variety of issues related to finance and infrastructure, was difficult to obtain inside France at the time.

Darboux was especially concerned that, without proper exposure to new research methodologies and standards then evolving outside of France, the research training of future generations of French mathematicians would be compromised. Echoes of this concern are heard in a letter that Darboux wrote to his collaborator on the *Bulletin*, the French mathematician Jules Hoüel (1823–1886).

> ... we need to mend our [system of] higher education. I think you agree with me that the Germans get the better of us there, as elsewhere. If this continues, I believe the Italians will surpass us before too long. So let us try, with our *Bulletin*, to wake the holy fire and the French understanding that there are many things in the world that they do not suspect, and that even if we are still the *Grrrand* [sic] nation, no one abroad perceives this.
>
> Darboux, as quoted in [Gispert, 1987, p. 160]

Although Hoüel agreed with Darboux's general concerns, we will soon see that the two men did not share a common understanding of all things mathematical. Their professional situations within the French mathematical community were also quite different. Senior to Darboux by twenty years, Hoüel had received his initial mathematical training at the École Normale Supérieur (entering in 1843), and his doctorate (in celestial mechanics) from the Sorbonne (in 1855). He then returned to his home town of Thaon for four years, pursuing mathematical research on his own despite an offer for a post at the Paris Observatory. In 1859, Hoüel accepted the Chair of pure mathematics in Bordeaux, located about 360 miles southwest of Paris, and remained in that position for the remainder of his life.

Despite being geographically removed from the intellectual center of France in Paris, Hoüel had already gained a reputation for excellence as a translator prior to joining Darboux as co-founder of the *Bulletin*. An early proponent of non-Euclidean geometry — he expressed doubts about the parallel postulate even before learning about the work of Nikolai Ivanovich Lobachevski (1792–1856) and János Bolyai (1802–1860) — Hoüel produced French translations of key papers by both these men, as well as other important works in non-Euclidean geometry by Eugenio Beltrami (1835–1900), Hermann von Helmholtz (1821–1894), and Bernhard Riemann (1826–1866). After the *Bulletin* was founded in 1870, Hoüel contributed numerous French translations to the new journal. Notable among these was his translation of Riemann's 1854 "Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe" ("On the representability of a function by a trigonometric series"). First published in German in 1868,[2] it was not until Hoüel's translation appeared in the *Bulletin* in 1873 that the contents of this important work, including Riemann's treatment of the integral, became generally known in France.

The year 1873 also marked the beginning of an exchange between the *Bulletin*'s founding co-editors in which several "monster functions" made their debut as Darboux sought to convince Hoüel of the need for increased rigor in the latter's own approach to analysis. In the next section, we meet one of these monsters through an excerpt from this correspondence.[3]

## 2 Monsters in the Darboux-Hoüel Correspondence

The impetus for the ten-year debate concerning rigor in analysis in the Darboux-Hoüel correspondence was Hoüel's request for feedback on preliminary drafts of his intended textbook on differential calculus, eventually published as *Cours de Calcul infinitésimal* in 1878. Throughout this debate, Darboux offered various counterexamples in a (vain) attempt to convince Hoüel of the need for greater care in certain of his (Hoüel's) proofs. The following excerpt from a letter written by Darboux on 24 January 1875 reveals one such example.

Go on then and explain to me a little, I beg you, why it is that when one uses the rule for composition functions, the derivative of $y = x^2 \sin \frac{1}{x}$ is found to be $-\cos\frac{1}{x} + 2x\sin\frac{1}{x}$, which is indeterminate for $x = 0$ even though the true value is $\lim \frac{y}{x} = 0$, . . . .

Darboux, as quoted in [Gispert, 1987, p. 101]

[2]Riemann submitted this work to the University of Göttingen in 1854, in partial fulfillment of the requirements for the extra post-doctoral qualification (or *habilitation*) that allowed one to become a lecturer in German universities. Its eventual posthumous publication was due to the efforts of Richard Dedekind (1831–1916), who arranged for it to appear in *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*(*Treatises of the Göttingen Royal Society of Sciences*), vol. 13, 1868. Although its main focus was the problem of characterizing those functions that are sums of everywhere-convergent trigonometric series, Riemann's habilitation thesis became quickly famous among mathematicians for its introduction of what is now called the Riemann integral. Riemann himself devoted only a small portion (5–6 pages) of his thesis to the question of how to define the integral. Instead, it was Darboux who provided the rigorous reformulation of the Riemann integral that is studied in most undergraduate level analysis courses, in the article on discontinuous functions from which we will read in Section 4 of this project [Darboux, 1875]. For some background on how and why the definition of integration evolved during the nineteenth and early twentieth centuries, see the projects *The Definite Integrals of Cauchy and Riemann* (by David Ruch) and *Henri Lebesgue and the Development of the Integral Concept* (by Janet Heine Barnett), both available at `https://digitalcommons.ursinus.edu/triumphs_analysis/`.

[3]This correspondence was not published until the 1980s, when it was extensively studied by historian of mathematics Hélène Gispert. Unedited excerpts from the letter collection appear in her articles [Gispert, 1983, 1987, 1990].

**Task 1**

(a) What is the name that is usually used in a current US calculus or analysis textbook for what Darboux called 'the rule for composition functions'? Use this rule to verify Darboux's claim about the derivative of $y = x^2 \sin \dfrac{1}{x}$ for $x \neq 0$. Why is this derivative function indeterminate for $x = 0$?

(b) Notice that the function $y = x^2 \sin \dfrac{1}{x}$ given by Darboux is undefined at $x = 0$. What did Darboux say in the preceding excerpt that gives us reason to believe that he was implicitly assuming that $y$ is continuous at $x = 0$?

(c) In order to make the assumption that $y$ is continuous at $x = 0$ explicit, we can stipulate a value for $y(0)$ and define $y$ as the piecewise function

$$y = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ A & \text{if } x = 0 \end{cases},$$

where $A$ is a well-chosen real number. What value must be assigned to $A$ in order to ensure that $y$ is continuous at 0. Justify your response.

*For the rest of this task, use this value of $A$ in the definition of the function $y$.*

(d) Now verify Darboux's claim that the 'true value of' of $y'(0)$ is 0 by computing $\lim\limits_{x \to 0} \dfrac{y}{x} = 0$. Then describe how this limit relates to the standard Calculus textbook definition(s) for the derivative at a specific point in order to explain why $\lim\limits_{x \to 0} \dfrac{y}{x} = 0$ gives us the value of $y'(0)$ Note also that Darboux himself did not specify that $x \to 0$ in his letter. What did he write that tells us that this is what he meant?

(e) Use the results from parts (a) and (d) to complete the following piecewise definition of the derivative function:

$$y' = \begin{cases} \underline{\hspace{3cm}} & \text{if } x \neq 0 \\ \underline{\hspace{3cm}} & \text{if } x = 0 \end{cases}$$

What function property does $y'$ fail to satisfy? Explain.

In the next section of this project, we will examine an excerpt from Darboux's letters in which he addressed what he felt was the underlying problem with Hoüel's overall approach to differentiable functions. Let's first examine some 'family relatives' of the particular monster $y = x^2 \sin \dfrac{1}{x}$ that Darboux attempted to use to show Hoüel that a problem with his (Hoüel's) approach did exist.

**Task 2** Note that Task 1 was based on the function $f_\alpha$ for $\alpha = 2$, where

$$f_\alpha(x) = \begin{cases} x^\alpha \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

In this task, you will examine properties of the function $f_\alpha$ for $\alpha = 3$.

To this end, define $f_3 : \mathbb{R} \to \mathbb{R}$ by $f_3(x) = \begin{cases} x^3 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Sketch a rough graph of this function. (You can use your calculator, if you like.)

(b) Use 'the rule for composition functions' (aka, the chain rule) to determine $f_3'(x)$ for $x \neq 0$.

(c) Use the definition of derivative to compute $f_3'(0)$.

(d) Use your answers from parts (b) and (c) to complete the following piecewise definition of the derivative function:

$$f_3'(x) = \begin{cases} \underline{\hspace{8cm}} & \text{if } x \neq 0 \\ \underline{\hspace{2cm}} & \text{if } x = 0 \end{cases}$$

(e) Show that **the derivative function** $f_3'$ is continuous at $x = 0$.
Is $f_3'$ also differentiable at $x = 0$? Explain why or why not.

**Task 3** This task extends our exploration of the function family

$$f_\alpha(x) = \begin{cases} x^\alpha \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

to include other values of $\alpha \in \mathbb{R}^+$.

Complete the following summary table by providing the requested information about the continuity and differentiability about the function $f_\alpha$ and its derivative $f_\alpha'$ for the indicated four values of $\alpha$. Justify each of your answers with an appropriate computation, or by citing a reference to your earlier work.

| $\alpha$ | $f_\alpha$ | $f_\alpha$ cont at 0? | $f_\alpha$ diff at 0? | $f_\alpha'$ cont at 0? | $f_\alpha'$ diff at 0? |
|---|---|---|---|---|---|
| 0 | $f_0(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | | | | |
| 1 | $f_1(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | | | | |
| 2 | $f_2(x) = \begin{cases} x^2\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | | | | |
| 3 | $f_3(x) = \begin{cases} x^3\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | | | | |

**Task 4** This task continues the exploration of the function family $f_\alpha : \mathbb{R} \to \mathbb{R}$, where $\alpha \in \mathbb{R}^+$ and

$$f_\alpha(x) = \begin{cases} x^\alpha \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

For each of the following, remember that $\alpha \in \mathbb{R}^+$ (but not necessarily a natural number). Justify each of your responses with an appropriate proof or limit calculation.

(a) Determine the values of $\alpha$ for which $f_\alpha$ is discontinuous at 0.

(b) Determine the values of $\alpha$ for which $f_\alpha$ is differentiable at 0, but $f'_\alpha$ is discontinuous at 0.

(c) Determine the values of $\alpha$ for which $f_\alpha$ is differentiable at 0, and $f'_\alpha$ is continuous but not differentiable at 0.

(d) Determine the values of $\alpha$ for which $f_\alpha$ is twice-differentiable at 0.

## 3 Defining Differentiability

In a second letter, written on 31 January 1875, Darboux expressed the following frustration with Hoüel's response to his (Darboux's) discussion of the function $y = x^2 \sin \dfrac{1}{x}$:

You have not addressed the nature of my objection. ... I have told you that, according to (your) rule of composite functions, we obtain

$$\frac{dy}{dx} = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

an expression that is indeterminate for $x = 0$, even though, according to first principals, the derivative is perfectly determined, it is zero. For your methods to be sound, you will need to explain very clearly what part of your reasoning is deficient in this particular case. Without that your proofs are not proof.

Darboux, as quoted in [Gispert, 1987, p. 102]

In this same letter, Darboux also returned to a second specific concern that he had about the proofs that Hoüel had provided for certain theorems involving derivatives. In fact, Darboux raised this other concern as early as January 18, 1875, when he wrote:[4]

[4]Notice that Darboux did not use absolute values in this excerpt, even though he was assuming that the quantity $\epsilon$ was positive and had no reason to expect that $\frac{f(x+h)-f(x)}{h} - f'(x)$ would be positive for all $h$. This practice was typical of nineteenth century analysts, who understood when absolute values were implied from the context. In keeping with today's customary practice, you should include absolute values as appropriate in inequalities and equations in the project tasks.

Here is what I reproach in your reasoning which no one would now find rigorous. When we have

$$\frac{f(x+h)-f(x)}{h} - f'(x) = \epsilon,$$

$\epsilon$ is a function of two variables $x$ and $h$ that approaches zero when, $x$ remaining fixed, $h$ approaches zero. But if $x$ and $h$ [both] vary as they do in your proof, or worse yet, if to each new subdivision of the intervals $x_1 - x_0$ there arise new quantities $\epsilon$, then I find it altogether unclear and your proof has nothing but the appearance of rigor.

Darboux, as quoted in [Gispert, 1983, p. 99]

The following quote from Hoüel, taken from a letter written 19 January 1875, is typical of Hoüel's replies to Darboux on this second issue.

Yes, I admit as a fact of experience (without looking to prove it in general, which might be difficult) that in the functions that I treat, one can always find $h$ satisfying the inequality $\frac{f(x+h)-f(x)}{h} - f'(x) < \epsilon$, no matter what the value of $x$, and I avow to you that I am ignorant of what the word derivative would mean if it is not this. ... I believe this hypothesis is identical with that of the existence of a derivative.

Hoüel, as quoted in [Gispert, 1987, pp. 56–57]

**Task 5** (a) Do you agree with Hoüel's assertion that 'this hypothesis is identical with that of the existence of a derivative'? Include a complete statement of what he meant by 'this hypothesis' in your response. How does what Darboux said in the excerpt at the top of this page seem to be different from what Hoüel is saying here?

(b) To make the connection between the definition of 'derivative at a point' and Darboux's concerns more clear, let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$, and begin with the standard definition of differentiability of $f$ at $x_0$; that is,

Given $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$, $f$ is differentiable at $x_0$ iff there is a real number $f'(x_0)$ for which $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$.

Set $x = x_0 + h$ and re-write this definition in terms of a limit that involves $x_0$ and the variable $h$, with $h \to 0$.

(c) Use symbolic notation to write an $\epsilon - \delta$ definition for the limit that you found in part (b). How does this relate to what Hoüel and Darboux were saying in the last two excerpts?

Darboux and Hoüel exchanged several other letters about this issue in early 1875. The next (more extensive) excerpt tells us how Darboux tried to explain his concern in his final letter to Hoüel about this topic, written on February 2.

As for the question of the derivative, this time you change the question. It is clear that for a value $x_0$ of $x$, that saying

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0)$$

is the same thing as saying:

> one can find $h$ such that
>
> $$\frac{f(x_0+h)-f(x_0)}{h}-f'(x_0)<\epsilon,$$
>
> for this value of $h$ and for all values that are smaller [than this $h$].

But there is an abyss between this proposition and the following:

> Being given a function $f(x)$ for which the derivative exists for all values of $x$ between $a$ and $b$, to every quantity $\epsilon$, one can find a corresponding quantity $h$ such that[5]
>
> $$\frac{f(x+h)-f(x)}{h}-f'(x)<\epsilon,$$
>
> for all values of $x$ between $a$ and $b$.

Because it is certainly true that for each value $x_1$ of $x$ between $a$ and $b$, there will be a quantity $h_1$ such that

$$\frac{f(x_1+h_1)-f(x_1)}{h_1}-f'(x_1)<\epsilon,$$

but there is nothing to imply that, as one allows $x_1$ to vary between $a$ and $b$, this quantity $h_1$ remains above a certain minimum.

Darboux, as quoted in [Gispert, 1983, pp. 103–104]

**Task 6** (a) The first 'proposition' that Darboux stated in the preceding excerpt is today written symbolically as follows (using the variable '$x$' where Darboux wrote '$x_0$'):

$$(\forall x\in[a,b])\,(\forall\epsilon>0)\,\,(\exists\delta>0)\,(\forall h\in\mathbb{R})\left(0<|h|<\delta\Rightarrow\left|\frac{f(x+h)-f(x)}{h}-f'(x)\right|<\epsilon\right)$$

Describe how this compares to your answer to part (c) of Task 5.

(b) Now write a symbolic version of the second proposition that Darboux stated in the preceding excerpt. You will need to pay special attention to Darboux's statement in order to place the quantifier '$\forall x$' at the correct place in the symbolic sentence.

[5] Translator's Note: A slight modification was made in the statement of this inequality in order to align it with how Darboux described it in his earlier letters.

**Task 6 - continued**

(c) Explain why Darboux was concerned about introducing a different quantity $h_1$ for 'each value $x_1$ of $x$ between $a$ and $b$.' How does this show that there is an 'abyss' between the two statements described symbolically in parts (a) and (b) above?

(d) Today, we say that:

- A function that satisfies the property in part (a) is 'differentiable on the interval $[a, b]$.'
- A function that satisfies the property in part (b) is '**uniformly** differentiable on the interval $[a, b]$.'

Look back at the excerpt taken from Hoüel's letter of 19 January 1875. (Just above Task 5.) Which of these two definitions do you think Hoüel was describing there? Explain why you think this by referencing what Hoüel himself actually said. Do you think Darboux was right to be worried about what Hoüel's assertions in this excerpt? Why or why not?

**Task 7** This task further explores the concept of uniform differentiability.

(a) Here is how the definition of uniform differentiability is generally stated today:

Given $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ with $f$ differentiable on $A$.
We say that $f$ is **uniformly differentiable** on $A$ iff

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in A)\left(|x - y| < \delta \Rightarrow \left|\frac{f(x) - f(y)}{x - y} - f'(x)\right| < \epsilon\right)$$

Explain how to obtain this definition (given in terms of the variables $x, y$) as a translation of the definition that you wrote in Task 6(b) (given in terms of the variables $x, h$).

*(If you get stuck, try a substitution similar to the one you used in Task 5(b).)*

(b) Use the definition from part (a) of this task to prove that the function $f(x) = x^2$ is uniformly differentiable on $\mathbb{R}$. What can you say about function properties that the derivative function $f'$ has in this case?

(c) Determine if the function $g(x) = x^3$ is uniformly differentiable on $\mathbb{R}$, and justify your response using the definition from part (a) of this task. How does this function $g$ (and its derivative function $g'$) differ from the function $f$ (and its derivative function $f'$) in part (b)?

(d) Use the definition from part (a) of this task to prove the following:

If $f$ is uniformly differentiable on $A$, then the derivative $f'$ is continuous on $A$.

Is the converse of this theorem is also true? If so, provide a proof. If not, provide a counterexample.

(e) Recall that continuous functions on compact sets are necessarily uniformly continuous. Does an analogous theorem hold for uniform differentiability? If so, provide a proof. If not, provide a counterexample.

## 4 Monsters in Darboux's Published Works in Analysis

Many of the monsters presented in Darboux's private letters to Hoüel remained hidden away from public sight until the publication of that correspondence by Hélène Gispert between 1983 and 1990. But other of his monster creations appeared in Darboux's three published works in analysis [Darboux, 1872, 1875, 1879]. In this section, we consider the contents of only the most influential of the three, his 1875 publication *Mémoire sur les fonctions discontinues (Memoire on discontinuous functions)*. Darboux described the goal of this work as follows [Darboux, 1875, p. 58].

At the risk of being too long, I have set out to be rigorous, perhaps without full success. Many points which would justly be considered obvious or would be granted in the applications of science to usual functions have to undergo rigorous criticism when it comes to expounding the propositions pertaining to the most general functions.

Among the "most general functions" that Darboux's *Mémoire* added to the existing menagerie of "monsters" were specimens of each of the following:

- A continuous, nowhere differentiable function[6]
- A continuous function that is neither increasing nor decreasing on any interval
- A discontinuous function that satisfies the Intermediate Value Property

Darboux's proof that this last example possesses the Intermediate Value Property followed from a theorem that now bears his name. We will read that proof in its entirely later in this section. First, let's pause to explore the notion that a continuous function can be neither increasing nor decreasing on any interval. Because the construction of such a monster requires techniques that go beyond the scope of the project, we will content ourselves with a slightly less bizarre example.

**Task 8** This task looks at an example of a differentiable function with a positive derivative at a point for which there is no interval containing that point on which $f$ is increasing.

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} \frac{1}{2}x + x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(a) Show that $g$ is differentiable on $\mathbb{R}$ with $g'(0) > 0$.

(b) Show that there is no open interval containing 0 on which $g$ is increasing. Why is this not a contradiction?

(c) Is there an open interval containing 0 on which $g$ is decreasing? Explain why or why not.

[6]Karl Weierstrass (1815–1897) is rightfully credited with being the first to define this type of function; although Weierstrass himself never published his example, his work became known through publications by some of this students. Darboux developed his example of such a function independently of the work of Weierstrass. Darboux did, however, have a strong grasp of recent developments in German analysis. He especially admired Riemann's concept of the integral, as the latter described it in a brief (5-6 page) discussion in his important 1853 *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe*. In fact, a primary goal of Darboux's Mémoire was to provide a rigorous reformulation of the Riemann integral.

We now turn to Darboux's proof of the theorem that currently bears his name, first stated and proven in [Darboux, 1875, pp. 109–110]. Here's what Darboux had to say by way of an introduction to this proof:

> ... we will show that there are discontinuous functions that satisfy [enjoy] a property that had sometimes been regarded as the distinctive characteristic of continuous functions, that of not being able to vary from one value to another without passing through all of the intermediate values.

Recall that the property that Darboux described as 'not being able to vary from one value to another without passing through all of the intermediate values' is today called the *Intermediate Value Property.* Here's a modern definition of this property; notice how both the domain $[a, b]$ and the function $j$ play a role in this definition:

**Definition**: The function $j$ has the **Intermediate Value Property** on the interval $[a, b]$ if and only if given any $u, v \in [a, b]$ with $u < v$ and any $L \in \mathbb{R}$ that lies between the values $j(u)$ and $j(v)$, there exists $c \in (u, v)$ such that $j(c) = L$.

As Darboux noted, the class of functions satisfying the Intermediate Value Property was often considered by (earlier) mathematicians to be identical to the class of continuous functions. Of course, a function that is continuous must also have the Intermediate Value Property — this is precisely what the Intermediate Value Theorem tells us. In the proof from Darboux that we are about to read, we will learn how to construct counterexamples to show that the *converse* of the Intermediate Value Theorem is false: a *discontinuous* function *can* satisfy the Intermediate Value Property!

In fact, Darboux actually showed that all members of a certain class of discontinuous functions are guaranteed to satisfy the Intermediate Value Property. Let's start by reading and making sense of Darboux's statement of this claim.

> Let $F(x)$ be a function for which the derivative exists at every value of $x$, but is discontinuous. Suppose that, for $x = x_0, x = x_1$, the derivative takes the values
>
> $$F'(x_0) = A, \quad F'(x_1) = B.$$
>
> I say that, if $x$ varies from $x_0$ to $x_1$, $F'(x)$ will pass at least once through all the values intermediate between $A$ and $B$.

**Task 9** This task examines Darboux's statement of the theorem to be proven.

(a) Explain why we could re-state it in somewhat more concise terms as follows:

If $F$ is differentiable on the interval $[A, B]$, then the derivative function $F'$
has the Intermediate Value Property on the interval $[A, B]$

In particular, explain how we can be sure, based on what Darboux said and other things that we know, that Darboux was talking about the **derivative** function $F'$ having the Intermediate Value Property (rather than the function $F$ itself).

(b) Now give an even more concise statement of the theorem that Darboux intended to prove by filling in the blanks below:

Every ____________________ function has the ____________________ Property.

(c) Use the idea described in this theorem to identify a specific function that is discontinuous on $\mathbb{R}$, but nevertheless satisfies the Intermediate Value Property on $\mathbb{R}$.

*(Hint: You worked with a family of such "monsters" earlier in this project!)*

Let's now return to Darboux's proof of his claim, which appears in its entirety below. Begin by reading through the proof at least twice, taking note of any questions or concerns you have about it. Task 10 then includes several exercises that should address most of those questions and concerns.

Let $F(x)$ be a function for which the derivative exists at every value of $x$, but is discontinuous. Suppose that, for $x = x_0, x = x_1$, the derivative takes the values

$$F'(x_0) = A, \quad F'(x_1) = B.$$

I say that, if $x$ varies from $x_0$ to $x_1$, $F'(x)$ will pass at least once through all the values intermediate between $A$ and $B$.

Indeed, let $M$ be one of these values,

$$A > M > B,$$

and form the function

$$F(x) - Mx.$$

This continuous function will have, for $x = x_0$, a positive derivative [value] $A - M$, and, for $x = x_1$, a negative derivative [value] $B - M$.

It will begin therefore by being increasing as $x$ varies from $x_0$, to $x_1$, but will finish by being decreasing at $x = x_1$.

Thus it will have a maximum that will be attained at a certain value

$$x_0 + \theta(x_1 - x_0),$$

and for which its derivative will be zero; one thus will have

$$F'(x_0 + \theta(x_1 - x_0)) - M = 0.$$

Hence, every number $M$ intermediate between $A$ an $B$ is a value of the derivative.

Now that you've read Darboux's proof at least twice, explore its details by completing the next two tasks. Then go back to your list of questions about the proof to be sure that they are resolved.

**Task 10** This task examines the details of the proof in the previous excerpt.

(a) Re-read the first paragraph and the beginning of the second paragraph of Darboux's proof. Explain how Darboux's assumptions (about $A$, $B$, $x_0$, $x_1$ and $M$) are setting up the assumptions needed to prove that the DERIVATIVE function $F'$ has the Intermediate Value Property. Also state what Darboux needed to prove in order to establish that the conclusion of the Intermediate Value Property also holds for the DERIVATIVE function $F'$.

(b) In the second paragraph of his proof, Darboux defined a new function $G(x) = F(x) - Mx$. Justify the following claims (made by Darboux at various points in his proof) about this function.

| RECALL DARBOUX'S ASSUMPTIONS: |
| --- |
| $F$ is differentiable with $F'(x_0) = A$ , $F'(x_1) = B$ , $A > M > B$ |

(i) The function $G(x)$ is differentiable.

(ii) The function $G(x)$ is continuous.

(iii) For $x = x_0$, the derivative of $G(x)$ has the positive value $A - M$.

(iv) For $x = x_1$, the derivative of $G(x)$ has the negative value $B - M$.

(v) If $c \in [x_0, x_1]$ with $G'(c) = 0$, then $F'(c) = M$.

(c) Based on the facts about the function $G$ summarized in part (b) of this task, Darboux asserted that:

> It [the function $G(x) = F(x) - Mx$] will begin therefore by being increasing as $x$ varies from $x_0$, to $x_1$, but will finish by being decreasing at $x = x_1$.

(i) Explain what you think Darboux was trying to say at this part of his proof.

(ii) Now look back at the function $g$ defined in Task 8 on the interval $[0, \frac{2}{3\pi}]$.
Recall that $g'(0) > 0$. Also verify that $g'(\frac{2}{3\pi}) < 0$. Based on the result of Task 8, do you find it convincing to say that this function $g$ "will begin to therefore appear increasing" as $x$ varies from $x_0 = 0$ to $x_1 = \frac{2}{3\pi}$? In what way (if any) does it make sense to say this function $g$ will "eventually appear decreasing at $x_1 = \frac{2}{3\pi}$"? (Notice that $x_1$ is just a single point!)

(iii) In order to make this part of Darboux's proof more rigorous, prove the following:

> *Lemma I:* Let $a, b \in \mathbb{R}$ with $a < b$. Assume that $g$ is differentiable on $[a, b]$ and satisfies $g'(a) > 0 > g'(b)$. Then there exists $x, y \in (a, b)$ such that $g(a) < g(x)$ and $g(y) > g(b)$.

(iv) Now use Lemma I together with theorems about continuous and differentiable functions from a Calculus or an analysis textbook to carefully prove the following:

> *Lemma II:* Let $a, b \in \mathbb{R}$ with $a < b$. Assume that $g$ is differentiable on $[a, b]$ and satisfies $g'(a) > 0 > g'(b)$. Then there exists $c \in (a, b)$ such that $g'(c) = 0$.

(d) Explain how Lemma II relates to the following claim made by Darboux in his proof:

$$\text{There exists a value } \theta \text{ such that } F'(x_0 + \theta(x_1 - x_0)) - M = 0. \qquad (\star)$$

In particular, identify restrictions on the value of $\theta$ for which the number $x_0 + \theta(x_1 - x_0)$ lies on the interval $(x_0, x_1)$. Then explain why the last line of Darboux's proof follows from statement $(\star)$.

**Task 11** This task looks back at the assumptions that Darboux made in this proof.

(a) Notice that Darboux began (in his first paragraph) by assuming that $F'$ was discontinuous. Did he use this hypothesis in the proof? If so, where and how? If not, why do you think he stipulated this assumption? Would his theorem be weaker or stronger with this condition as one of the assumptions?

(b) Suppose that we knew $F$ were uniformly differentiable on $[A, B]$, rather than just differentiable on $[A, B]$. Use a result from Task 7 to write a very quick and simple proof that $F'$ has the Intermediate Value Property on $[A, B]$ in this case.

## 5 The Monster Debate Revisited

Throughout the course of the debate between Darboux and Hoüel about how to properly approach definitions and proofs in analysis, one can hear Hoüel's increasing exasperation with Darboux's examples in his description of them as "drôlatiques" (humorous), "bizarres" (bizarre), "dereglés" (disorderly), "saugrenues" (absurd), and "gênantes" (obstructive). Darboux too became increasingly vexed by Hoüel's apparent inability to understand the underlying purpose of these examples.

Let's also recall what Poincaré had to say about these functions, where we will now read the complete quote [Poincaré, 1904, p. 263]:

> Logic sometimes begets monsters. The last half-century saw the emergence of a crowd of bizarre functions, which seem to strive to be as different as possible from those honest [honnêtes] functions that serve a purpose. No more continuity, or continuity without differentiability, etc. *What's more, from the logical point of view, it is these strange functions which are the most general, [while] those which arise without being looked for appear only as a particular case. They are left with but a small corner.* In the old days, when a new function was invented, it was for a practical purpose; nowadays, they are invented for the very purpose of finding fault in the reasoning of our fathers, and nothing more will come out of it.

Notice the italicized portion of this quote in particular — there are far more monsters in the mathematical world than you might expect!

Darboux's student Émile Borel (1871–1956) proposed two further reasons why these "refined subtleties with no practical use" should not be ignored [Borel, 1912, p. 14]:

> [O]n the one hand, until now, no one could draw a clear line between straightforward and bizarre functions; when studying the first, you can never be certain you will not come across the others; thus they need to be known, if only to be able to rule them out. On the other hand, one cannot decide, from the outset, to ignore the wealth of works by outstanding mathematicians; these works have to be studied before they can be criticized.

**Task 12** Look back at the mathematical work you completed in this project, along with the closing quotations included in this section. Then write a brief essay in response to the following.

- Summarize the ways in which Darboux used "monster functions" in his work in analysis.
- What was Darboux trying to accomplish with these examples? Did he succeed?
- What were the general consequences for the study of analysis that came out of the "monsters" created by Darboux and other nineteenth century mathematicians?

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# Notes to Instructors

## PSP Content: Topics and Goals

The Primary Source Project (PSP) *Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis* is designed for use in an introductory undergraduate course in analysis. It is intended to replace the standard modern textbook treatment of much of the content related to differentiability in such a course. The relationship between fundamental function properties (e.g., continuity, differentiability, Intermediate Value Property) is one of its central themes. This includes the result now known as *Darboux's Theorem* — that every derivative function possesses the Intermediate Value Property — which is developed through a guided reading of Darboux's original proof. Uniform differentiability is also considered, although that particular section could be omitted if this topic is not part of the course curriculum. Other standard topics related to differentiability on which this PSP touches are listed in the Student Prerequisites section below. Neither the Mean Value Theorem[7] nor the Extreme Value Theorem, two other standard analysis topics related to differentiability, is considered in any way in this PSP.

The function family $f(x) = x^\alpha \sin \frac{1}{x}$, where $\alpha \in \mathbb{R}^+$, plays a starring role within this PSP, as they did within Darboux's long-standing debate with Hoüel concerning rigor in analysis. These same functions also appear in the treatment of differentiability found in most modern undergraduate analysis textbooks. Missing from these modern treatments is a consideration of the historical context in which these examples were first considered. Why were these examples developed in the first place? What mathematical intuitions were refined and in what ways by studying them? Were they even accepted as legitimate examples of functions and, if not, why not?

Because most students enter an analysis course with a general understanding of the calculus (and the concept of continuity in particular) that differs little from the views of nineteenth century mathematicians like Hoüel, sharing Darboux's explanations and motivations for considering such functions with students serves two other important goals of this PSP. First, exposure to this historical context helps students develop the more rigorous and critical view of the basic ideas of calculus that an introductory analysis course seeks to achieve. A second closely-related companion goal is to help students develop an understanding of the language, techniques and theorems of elementary analysis that developed as mathematicians adopted such a critical perspective in the nineteenth century.

## Student Prerequisites

This PSP assumes that students have studied the basic material related to continuity and limits in an introductory analysis course. This includes especially the $\epsilon - \delta$ definition of limits, the Intermediate Value Property and the Intermediate Value Theorem. If Section 4 (on uniform differentiability) is completed, then students should also be familiar with the concept of uniform continuity; that section also assumes basic familiarity with the notation of symbolic logic.

A basic Calculus I level understanding of derivatives is also assumed, most notably prior exposure to the definition of the derivative as the limit of a difference quotient. However, there is no assumption that students have already encountered this definition in their analysis course. Instead, this PSP can be used as the students' first encounter with the definition of derivative within their introductory

[7] The PSP *The Mean Value Theorem* (author David Ruch) develops that theorem through excerpts from the efforts of Augustin-Louis Cauchy (1789–1857) to rigorously prove it for a function with a continuous derivative, and the very different approach developed some forty years later by the mathematicians Joseph Serret (1819–1885) and Pierre Ossian Bonnet (1819–1892). That project is available at `http://digitalcommons.ursinus.edu/triumphs\_analysis/5/`.

analysis course. The project also assumes (without proof) a few standard differentiability theorems. In particular, the sum/product/quotient/chain rules and the fact that differentiable functions are necessarily continuous are extensively used in Section 2. The Interior Extremum Theorem (also called Fermat's Theorem) is also needed for one part of one task in Section 4. All of these results will, however, be familiar to students from their prior calculus course work.

## PSP Design and Task Commentary

A sample implementation schedule for this particular PSP is included below. The following description of the mathematical content of each section should assist instructors in determining how best to adapt that recommended schedule to their own course goals and students' needs.

- Introductory Comments
  This (untitled) section sets the stage for the study of "monster functions" with a (partial) quote from Poincaré in which he lamented the path taken by analysis in the late nineteenth century. An outline of the project's contents is also provided.
- Section 1: Gaston Darboux: Student, Teacher and Editor *par excellence*
  This short section provides some biographical information about both Darboux and Hoüel, and describes the historical context of the debate between them concerning rigor in analysis that is explored in Sections 2 and 3 of the PSP.
- Section 2: Monsters in the Darboux-Hoüel Correspondence
  A primary objective of this section is to re-introduce students to the definition of differentiability as the limit of a difference quotient, but with an emphasis on the derivative as a function in its own right, which may or may not also be continuous, or differentiable, etc. The function family $f_\alpha(x) = x^\alpha \sin \frac{1}{x}$, where $\alpha \in \mathbb{R}^+$, is explored in some detail as a concrete example that illustrates how the properties of differentiability and continuity interact with each other. **Tasks 1, 2, 3 and 4 in this section are core tasks of the PSP.**
- Section 3: Defining Differentiability
  Its title not withstanding, this section is really about the distinction between *differentiability* and *uniform differentiability*, and the slippery nature of quantifiers. Depending on the instructor's course goals, this section can be omitted completely or in part. The following commentary on the tasks in this section are offered to help instructors decide how much of this section they wish to implement, and how.
  - Task 5 and Task 6 are directly tied to the historical exchange between Darboux and Hoüel in which uniform differentiability emerged as a concept distinct from that of differentiability. Completion of these tasks, along with Task 7(a), is recommended to promote student understanding of this new concept, and to expand/consolidate their understanding of issues related to quantifier placement more generally.
  - Task 5(a), Task 6(c) and Task 6(d) pertain directly to the Darbuox-Hoüel exchange concerning the definition of derivative. Although students find the related primary source passages difficult to parse, allowing them to grapple with the questions in these tasks is essential to the PSP goals of helping students to (a) develop a more rigorous and critical view of the basic ideas of calculus and (b) develop an understanding of the language, techniques and theorems of

elementary analysis. In this regard, having students get 'correct' answers to these items is less essential than ensuring that they engage as fully as possible with the ideas in the excerpts. These tasks are thus highly recommended for small-group discussion, supplemented by whole-class discussion as the instructor deems appropriate.

  – Task 5(c) and Task 6(b) ask students to use symbolic logic to write down $\epsilon - \delta$ definitions of 'differentiable on $[a, b]$' and 'uniformly differentiable on $[a, b]$' respectively. As a check on the symbolic statements obtained, the answer to Task 5(c) is incorporated into Task 6(a), and the answer to Task 6(b) is incorporated into Task 7(a).
  – Task 7 is a fairly standard exploration of the definition and basic theory of uniformly differentiable functions. *Note that this task does not refer directly to the primary source excerpts that appear in this section, and that only part (a) makes reference to the earlier tasks in this section.* The remaining parts of this task can provide useful practice with writing $\epsilon - \delta$ arguments based on the definition of uniform differentiability. These include working through the details related to one specific example of a function that is uniformly differentiable on $\mathbb{R}$, as well as the details related to one specific non-example of such a function; the details related to this non-example further call upon students to negate a quantifier-heavy $\epsilon - \delta$ definition. Each part of this task could be assigned either for small-group discussion, or as individualized homework; parts (d) and (e) are especially well-suited for assignment as individualized homework.

- Section 4: Monsters in Darboux's Published Works in Analysis
  This section examines the theorem known today as *Darboux's Theorem*, which states that every derivative function possesses the Intermediate Value Property, through a guided reading of Darboux's own original proof of this result. The project narrative reminds students about the modern formal statement of the Intermediate Value Property and its relationship to the property of continuity. Interestingly, Darboux's proof falls short of today's standards of rigor in some respects. The function $g(x) = \frac{1}{2}x + x^2 \sin\frac{1}{x}$, which is closely related to the function family $f_\alpha$ explored in Section 2, is employed as a concrete basis from which to critique Darboux's proof. Task 10(c) then guides students through the proof of two lemmas that can be used to revise his proof to meet today's standards of rigor. **Tasks 8, 9, 10(a) and 11 in this section are core tasks of the PSP.** Part c (subpart iv) of Task 9 assumes familiarity with the Interior Extremum Function (also called Fermat's Theorem), which students should have encountered in Calculus I and which they (or the instructor) could prove rigorously as an adjunct exercise to the PSP. Task 10(b) assumes familiarity with uniform differentiability, and should only be assigned if Task 7(d) from Section 3 is also assigned.

- Section 5: The monster debate revisited
  In this brief culminating section, Poincaré's comments from the PSP's introduction are re-stated (but now fully quoted), and a quotation from Darboux's student Borel is added. Emphasis is placed on the surprising fact that monster functions (e.g., discontinuous nowhere differentiable functions) are more common than their non-

monster counterparts: the monsters truly are everywhere! **The questions in Task 12 are central to the PSP's general goal of promoting student reflection on and understanding of the (changing) nature of rigor in analysis, and of the role of counterexamples in analysis as a tool to refine our mathematical understanding.** Even if student written responses for Task 12 are not collected, a whole group discussion of the questions posed in that task is recommended.

## Suggestions for Classroom Implementation

Classroom implementation of this and other PSPs may be accomplished through individually assigned work, small-group work and/or whole-class discussion. A combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project. To reap the full pedagogical and mathematical benefits offered by the PSP approach, students should be required to read assigned sections and complete advance work on tasks related that reading prior to in-class discussions. The author's method of ensuring that advance reading takes place is to require student completion of "Reading Guides" (or "Entrance Tickets") for which students receive credit for completion, but with no penalty for errors in solutions. See the Appendix to these Notes for a sample guide based on this particular PSP and more detail about their general design.

LaTeX code of the entire PSP is available from the author by request to facilitate preparation of reading guides or 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

## Sample Implementation Schedule (based on a 50-minute class period)

The following sample schedule assumes completion of the entire PSP, including Section 3 on uniform differentiability. Instructors who choose to omit some or all of Section 3 may wish to allot at least a portion of an additional class day to small-group work on Tasks 3 and 4 in Section 2, instead of assigning these only as individualized follow-up homework.

- **Advance Preparation Work for Day 1** (to be completed before class)
  Read pages 1–3 of the Introduction and Section 1 (stopping above Task 3), and complete parts of Task 1 and Task 2 for class discussion, per the sample Reading Guide in the Appendix to these Notes.

- **Day 1 of Class Work**
  - (Optional) Whole-class discussion of historical and mathematical ideas from Section 1.
  - Small-group discussion of the following:
    - Quick review of answers to advance preparation work on Task 1, parts (a)–(d).
    - Complete Task 1, part (e).
    - Quick review of answers to advance preparation work on Task 2, parts (a)–(b).
    - Complete the rest of Task 2.
  - Time permitting, begin work on Task 3. Completion of Task 3 can also be assigned as part of the Advance Preparation Work for Day 2 (see below), with the goal of providing students with informal feedback on their work before assigning Task 4 as formal homework.
  - **Homework**: A complete formal write-up of Task 4, to be due at a later date (e.g., one week after completion of the in-class work).

- **Advance Preparation Work for Day 2**
  As a follow-up to Day 1 Class Work, complete the table in Task 3. In Section 3, read pages 5–7, (stopping above Task 6); complete Task 5 for class discussion along the way.

- **Day 2 of Class Work**
  - Small-group discussion (supplemented as desired by whole group discussion) of the following:
    * Review answers to advance preparation work on Task 5.
    * Complete as much of Task 6 as possible.
    * Time permitting, begin individual or small-group work on Task 7, part (a).

- **Advance Preparation Work for Day 3** (to be completed before class)
  Re-read pages 5–7 of Section 3 as needed; prepare notes for class discussion of Task 7, Parts (a) and (b).

- **Day 3 of Class Work**
  - Small-group work on Task 7, Parts (a), (b) and (c).
  - **Homework**: A complete formal write-up of student work on Tasks 7(b) and 7(c) could be assigned, to be due at a later date (e.g., one week after completion of the in-class work). Depending on instructor goals, a formal write-up of Tasks 7(d) and 7(e) may also be assigned at this point, or postponed for a later date.

- **Advance Preparation Work for Day 4** (to be completed before class)
  In Section 4, read pages 9–11; complete Tasks 8 and 9 for class discussion along the way.

- **Day 4 of Class Work**
  - Small-group comparison of answers from Advanced Preparation Work on Task 8.
  - Whole-class or small-group discussion of Intermediate Value Property, and its connection to Darboux's Theorem, including a discussion of students' advanced preparation work on Task 9.
  - Small-group work on Task 10, completing as much as time permits.
  - **Homework**: A complete formal write-up of student work on Task 8 and/or Task 9(c) could be assigned, to be due at a later date (e.g., one week after completion of the in-class work).

- **Advance Preparation Work for Day 5** (to be completed before class)
  In Section 4, re-read the Darboux excerpt on the bottom of page 11, and continue working on Task 10. Specific parts of Task 10 should be assigned based on what was completed during Day 4 Class Work.

- **Day 5 of Class Work**
  - Whole or small-group discussion of the remaining parts of Task 10.

    A summarizing whole group discussion of Lemma 1 [Task 10(c-iii)] and of Lemma 2 [Task 10(c-iv)] can be especially valuable here.

  - Time permitting, begin individual or small-group work on Task 11.
  - **Homework**: A complete formal write-up of student work on portions of Task 10 could be assigned, to be due at a later date (e.g., one week after completion of the in-class work). Task 11 could also be assigned as formal homework, or assigned more informally as part of the Advance Preparation Work for Day 6 (see below).

- **Advance Preparation Work for Day 6** (to be completed before class)
  Complete Task 11 as a follow-up to the proof of Darboux's Theorem in Section 4 (unless this is to be assigned for homework); also read Section 5, pages 13–14, including a preliminary reading of Task 12.

- **Day 6 of Class Work** - The following may not take an entire class period, and could possibly be omitted altogether, depending on instructor's approach to Section 5.
  - As needed: Summarizing whole group discussion of proof of Darboux's Theorem from Section 4.
  - (Optional) Whole or small-group discussion of ideas in Section 5, possibly including preliminary answers to Task 12.
  - **Homework**: A complete formal write-up of Task 11(b) and/or Task 12 could be assigned, to be due at a later date (e.g., one week after completion of the in-class work).

### Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs designed to be completed in 1–2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_topology`; all other listed projects are available at `https://digitalcommons.ursinus.edu/triumphs_analysis`.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Heine Barnett)
- *Investigations into Bolzano's Bounded Set Theorem* (David Ruch)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)// Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)// A second version of this prjoect suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *The Mean Value Theorem*(David Ruch)
- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Heine Barnett)
- *Euler's Rediscovery of* $e^*$ (David Ruch; sequence convergence, series and sequence expressions for $e$)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)

- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

### Recommendations for Further Reading

Instructors who wish to know more about the history of analysis in the nineteenth and early twentieth centuries will find the following articles of interest: [Chorlay, 2016], [Hochkirchen, 2003], [Lützen, 2003]. See the reference list of the student portion of this PSP for bibliographic details.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.

# APPENDIX

This appendix provides a 'Sample Reading Guide' that illustrates the author's method for assigning advance preparation work in connection with classroom implementation of primary source projects. As described in the subsection "Suggestions for Classroom Implementation" of the Notes to Instructors for this project, students receive credit for completion of these guides, but with no penalty for errors in solutions. Students are asked to strive to answer each question correctly, but to think of Reading Guides as preparatory work for class, not as a final product (e.g., formal polished write-ups are not expected). Students who arrive unprepared to discuss assignments on days when group work is conducted based on advance reading are not allowed to participate in those groups, but are allowed to complete the in-class work independently. Guides are collected at the end of each class period for instructor review and scoring prior to the next class period.

A typical guide (such as the one that follows) will include "Classroom Preparation" exercises (generally drawn from the PSP Tasks) for students to complete prior to arriving in class, as well as "Discussion Questions" that ask students only to read a given task and jot down some notes in preparation for class work. Students are also encouraged to record any questions or comments they have about the assigned reading on their guide and are sometimes explicitly prompted to write 1–3 questions or comments about a particular primary source excerpt; their responses to such prompts are especially useful as starting points for in-class discussions. On occasion, tasks are also assigned as follow-up to a prior class discussion.

Experience has proven the value of reproducing the full text of any assigned project task on the guide itself, with blank space for students' responses deliberately left below each question. This not only makes it easier for students to jot down their thoughts as they read, but also makes their notes more readily available to them during in-class discussions. It also makes it easier for the instructor to efficiently review each guide for completeness (or to skim responses during class for a quick assessment of students' understanding), and allows students to make more effective use of their Reading Guide responses and instructor feedback on them at a later date.

The primary goal of the reading and tasks assigned in the Sample Reading Guide that follows is to familiarize students with the historical and mathematical background of the project of this PSP, and to prepare them for in-class small-group work on Tasks 1–3 (per the "Sample Implementation Schedule" found in the Notes to Instructors of this project).

**Day 1 Reading Guide:** *Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis*
**Reading Assignment:** pp. 1–4 through Task 1(d).

1. Read the Introduction and Section 1, pages 1–2.

   *Questions or comments?*

2. Read the start of Section 2, including the first excerpt from Darboux.

   **Write at least one comment OR one question about this excerpt:**

3. **Complete Task 1, part (a):**

   **What is the name that is usually used in a current US calculus or analysis textbook for what Darboux called 'the rule for composition functions'?**

   **Use this rule to verify Darboux's claim about the derivative of** $y = x^2 \sin \dfrac{1}{x}$ **for** $x \neq 0$**.**

   **Why is this derivative function indeterminate for** $x = 0$**?**

4. **DISCUSSION: Answer the following question from Task 1, part (b):**

Notice that the function $y = x^2 \sin \dfrac{1}{x}$ given by Darboux is undefined at $x = 0$.
**What did Darboux say in the preceding excerpt that gives us reason to believe that he was implicitly assuming that $y$ is continuous at $x = 0$?**

5. **Complete Task 1, part (c)**

In order to make the assumption that $y$ is continuous at $x = 0$ explicit, we can stipulate a value for $y(0)$ and define $y$ as the piecewise function

$$y = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ A & \text{if } x = 0 \end{cases},$$

where $A$ is a well-chosen real number.
**What value must be assigned to $A$ in order to ensure that $y$ is continuous at $0$?** $A =$ ______ **Justify your response.**

6. **DISCUSSION: Jot down your preliminary thoughts about Read Task 1, part d, below.**

**Now verify Darboux's claim that the 'true value of' of $y'(0)$ is $0$ by computing $\lim_{x \to 0} \dfrac{y}{x} = 0$.**

**Describe how this particular limit relates to the standard Calculus textbook definition(s) for the derivative at a specific point in order to explain why $\lim_{x \to 0} \dfrac{y}{x} = 0$ gives us the value of $y'(0)$.**

**Note that Darboux himself did not specify that $x \to 0$ in his letter.**
**What did he write that tells us that this is what he meant?**