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Connectedness- its evolution and applications

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1 Introduction

In 1872, Georg Cantor published a paper titled "Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen" (On the extension of a theorem from the theory of trigonometric series) [Cantor, 1872]. Cantor was studying a problem concerning the uniqueness of a Fourier Series representation, a problem in Fourier analysis or more generally, real analysis. His method of solving this problem, however, involved studying a new kind of object that he called a "point-set." The key properties of this object lie not so much in properties of real numbers, but in the relationships that the values have to one another. In addition to point-sets, Cantor introduced concepts such as *limit points, interiors,* and *derived sets*. It is argued by some that this viewpoint marks the birth of point-set topology.

Although the exact moment when point-set topology became its own branch of mathematics will always be debated, what is not debated is the importance of the program that Cantor had begun. He published several follow-up papers [Cantor, 1879, 1880, 1882, 1883a,b] in which continued his investigations into ideas on point sets. It is the goal of this PSP to start in the middle of Cantor's work and follow his ideas as they are picked up by others throughout history. Specifically, we will study the concept of connectedness. We begin with Cantor in 1883 [Cantor, 1883b] who gave a definition in terms of a metric, and then move to a 1904 work of Schoenflies [Schoenflies, 1904] in which he gives a purely set theoretic or topological definition of connectedness. We next move to the Norwegian mathematician Lennes (who spent his career at the University of Montana) who was working on a proof of the Jordan curve theorem [Lennes, 1911]. We then come to the 1921 work of Knaster and Kuratowski, who claimed to be the first ones to study connectedness for its own sake. As an added bonus, we are able to piece together some of these important results to provide a proof of the Invariance of Domain theorem for $n = 1$. This important result states that for any $m > 1$, $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^m$. As mentioned above, we start with the work of Cantor.

2 Cantor: the dawn of connectedness

Georg Cantor (1845 –1918) was born in Saint Petersburg, Russia in 1845. At the age of eleven, he moved to Germany, where he completed his education at Darmstradt, Zürich, Berlin and Göttingen,

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before becoming a professor at the University of Halle. He was the first mathematician to develop a theory of different notions of infinity and to define it mathematically using the simple but powerful idea of a one-to-one correspondence. One of his most notable mathematical discoveries is his comparison of the infinite set of rational numbers with the infinite set of natural numbers, showing that the infinite cardinality of these two sets are actually equal. Before he developed the mathematics of infinity, he worked on concepts in point-set topology, one of which was connectedness. Cantor's motivation for defining connectedness seems to have arisen out of the idea of a continuum. Below we will study the his 1883 paper "Über unendliche lineare Punktmannigfaltigkeiten 5" (On linear point-sets 5) [Cantor, 1883b]. Before we read Cantor's words, it worth reflecting upon the colloquial meaning of a continuum.

Task 1 Before reading any further, come up with a definition of a continuum.

Task 2 Which of the following do you think ought to be considered a continuum?

1. $\mathbb{R}$
2. $\mathbb{R}^2$
3. $\mathbb{N}$
4. $[0, 1]$
5. $(0, 1)$
6. $[0, 1] - \{\frac{1}{2}\}$
7. $\mathbb{Q}$

We can also think about popular definitions. Three definitions found on the internet are given below:

- a continuous extent, series, or whole.
- something that changes in character gradually or in very slight stages without any clear dividing points.
- a continuous sequence in which adjacent elements are not perceptibly different from each other, although the extremes are quite distinct.

Task 3 Compare the three above definitions with your answers in Task 2. That is, determine, according to each of the three definitions, if each of the sets in Task 2 is a continuum. Do any of the definitions agree with your answers? Would you now change your answer in Task 1? If so, how?

After having spent a little bit of time reflecting on a continuum, we now turn to Cantor himself. Whenever we quote someone in this project, we will place the quote between the following symbol.

[T]here remains for me nothing other than to attempt a most general possible purely arithmetical concept of a point continuum... As foundation I am served, as cannot be otherwise, by the n-dimensional flat *arithmetic* space G_n, that is, the embodiment of all systems of values:

$$(x_1|x_2|\cdots|x_n),$$

in which each x can attain *all real* numerical values from $-\infty$ to $+\infty$, independent from the others.[1]

Task 4

1. Rewrite in your own words what Cantor is attempting to do.
2. What is "arithmetic space"? What do we know it as today and what is our notation for it?
3. Why do you think Cantor chose "arithmetic space" as his foundation for what he is trying to accomplish?

Each particular system of values of this type I call an *arithmetical* point of G_n. The distance of two such points will be defined by the expression:

$$+\sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2 + \cdots + (x_n' - x_n)^2},$$

and by an *arithmetical* point set P contained in G_n is understood each legitimate given embodiment of points in the space G_n. The investigation thus amounts to establishing a sharp and concurrently most general possible definition, for *when P is to be called a continuum.*

[1] Translation of Section 1 by David Pengelley, 2017.

If it wasn't clear before, Cantor explicitly stated his goal in the last sentence (is your answer to Task 4 correct?). Furthermore, you might recognize Cantor's distance given above a standard distance formula.

Next Cantor begins to muse about how one might define a continuum.

> In order to approach the general concept of a continuum situated in G_n, I recall the concept of the derivation $P^{(1)}$ of an arbitrary given point set P, as it is first developed in the work: Math. Ann. vol. V, then in vols. XV, XVII, XX, and XXI, and is extended to the concept of a derivation $P^{(\gamma)}$, where γ can be any whole number...

In the above excerpt, Cantor makes reference to an earlier work of his, a paper he wrote in 1872. Below is a passage from Cantor's earlier paper which defined the derived set [Cantor, 1872].

> I define a "limit point of a point set P" to be a point of the line situated in such a way that each neighborhood of it contains *infinitely* many points of P, and it may happen that the point itself belongs to the set. By a "neighborhood of a point" I mean any interval that has the point *in its interior*. It is easy to prove that a ["bounded"] point set consisting of an infinite number of points has at least *one* limit point.
>
> Every point of the line is now in a definite relation to a given set P, either being a limit point of P or not, and thereby along with the point set P the set of limit points of P is *conceptually* given, a set which I wish to denote by P' and call the *first derived point set of* P.
>
> Unless the point set P' contains only a finite number of points, it also has a derived set P'', which I call the *second derived point set of* P. By v such transitions one obtains the concept of the vth derived set $P^{(v)}$ of P.

Let us gain some practice working through Cantor's definition of limit point and derived set.

Task 5 In relation to the line $\mathbb{R}$, compute P' for each of the following sets.

1. $[0, 1]$
2. $\{(a, b),\ a < b \in \mathbb{R}\}$
3. $\{\frac{1}{n} : n = 1, 2, 3, \ldots\}$.
4. $\mathbb{Q}$
5. $\mathbb{R}$
6. $\{\frac{m}{n} : m, n = 1, 2, 3, \ldots, m \leq n\}$.

7. $[\mathbb{Q} \cap (0,1)] \cup [-4, 2)$

In the previous task, you investigated the derived set of a subset of $\mathbb{R}$. However, there is nothing to stop us from working with derived sets more generally. In the following, Cantor divides a set P into two classes. Either there exists an α for which $P^{(\alpha)}$ vanishes (is the empty set), or

$P^{(1)}$ can always be broken down, and in fact only in one way, into two sets R and S, so that:

$$P^{(1)} \equiv R \cup S,$$

R is so constituted, such that through the repeated derivation process, it is capable of a continual reduction up until annihilation, so that there is always a first whole number γ ... for which:

$$R^{(\gamma)} = 0;$$

such point sets R I call *reducible*.

S on the other hand is so constituted, such that for this point set the derivation process yields no ending at all, in that:

$$S \equiv S^{(1)}$$

and consequently also:

$$S \equiv S^{(\gamma)};$$

such sets S I call *perfect* point sets...

Although these two predicates "reducible" and "perfect" are incompatible in one and the same point set, still on the other hand irreducible is not tantamount to perfect, any more than imperfect is exactly the same as reducible, as one can easily see by some attention.

Task 6 Find the decomposition $P = R \cup S$ for each of the sets in Task 5.

Let us study Cantor's words here carefully. Let P be a point set. Then either there exists a natural number α such that $P^{(\alpha)} = \emptyset$ or there does not. Cantor is not so much interested in the former case, but the latter. Hence assume that for all natural numbers α that $P^{(\alpha)} \neq \emptyset$. For the following exercise, refer to Cantor's above quote.

Task 7

1. Give a careful and rigorous definition of what it means for a point set S to be **perfect**.
2. Give a careful and rigorous definition of what it means for a point set R to be **reducible**.
3. Suppose that for all natural numbers α, $P^{(\alpha)} \neq \emptyset$. Prove that $P' = R \cup S$, where R is reducible and S is perfect.

4. Prove that the decomposition you came up with above is unique; that is, if $P' = T \cup W$ where T is reducible and W is perfect, then $T = S$ and $W = R$.

Cantor has now defined point sets with a certain kind of property; namely, either their derived set eventually vanishes or it can be broken down as a union of a reducible set (which is itself of the former kind of point set mentioned) and a perfect point set. He then begins to muse about a perfect point set and its possible relationship to other concepts, including a continuum.

The *perfect* point sets S are by no means always in themselves that which I in my previously mentioned work have called everywhere dense; therefore also they are not yet alone suitable for the complete definition of a point continuum, even if one must immediately admit, that the latter must always be a *perfect* set.

Task 8 Thinking back to your work in the first three tasks, do you agree that 'one must immediately admit' that a continuum must be perfect? Why or why not?

By S **is everywhere dense in** A, Cantor means that for every point $a \in A$, and every open set U containing a, we have that U and S have nonempty intersection. He thus claims that a point set which is perfect is not the same thing as being everywhere dense. In fact, he relegates a counterexample to a footnote attached to the word "dense" in the above quote. We pull it out here.

As an example of a perfect point set that is not everywhere dense in any interval, however small, I invoke the embodiment of all real numbers, that are contained in the formula:

$$z = \frac{c_1}{3} + \frac{c_2}{3^2} + \cdots + \frac{c_\nu}{3^\nu} + \cdots ,$$

where the coefficients c_v assume, at one's discretion, the two values 0 and 2, and the series can consist of either a finite or an infinite number of terms.

Today this set bears Cantor's name and is referred to as the **Cantor set**, even though it was discovered in a more general form a couple years earlier by J.H. Smith [Smith, 1874/75].

Task 9

1. Show that the Cantor set is not everywhere dense in $\mathbb{R}$.
2. Show that the Cantor set is perfect.

As Cantor noted above, in his view, being a perfect point set is necessary but not sufficient for constituting a continuum.

Task 10 Let $X = [0,1] \cup [2,3]$. Show that this is a perfect point set. Do you think X is a continuum? Why or why not?

Now Cantor defines the missing ingredient.

There is rather another essential concept, which together with the foregoing defines the continuum, namely the concept of a *connected* point set T.

Task 11 Before reading Cantor's definition, take a few moments to try and define what it means for a point set to be "connected." What conditions would a point set need to satisfy in order to be considered connected?

Cantor continues.

We call [a closed set] T a *connected* point set, if for any two of its points t and t', and an arbitrarily small given number ϵ, there is always a *finite* number of points $t_1, t_2, \cdots t_\nu$ of multiple forms in T, such that the distances $\overline{tt_1}, \overline{t_1t_2}, \overline{t_2t_3}, \cdots \overline{t_\nu t'}$ are all smaller than ϵ.

For our purposes, we will define a point set A to be **closed** if its complement $X - A$ is open. It is worth pointing out here that Cantor limited his definition of connected to only closed point sets T. We will explore this in Task 16.

Task 12 How does your definition compare to Cantor's?

Now it is easy to see that all geometric point continua known to us fall under this concept of *connected* point set; but I also believe in these two predicates "perfect" and "connected" to realize the necessary and *sufficient* characteristics of a point continuum, and therefore define a point continuum inside G_n as a *perfect-connected set*.

Cantor has a footnote attached to the phrase "as a perfect-connected set" above. We pull it out here.

One notices that this definition of a continuum is free of any indication of what one calls the dimension of a continuous shape; the definition encompasses namely also such continua as consist of connected pieces of different dimensions, like lines, surfaces, solids, etc. ... According to my conception only a *perfect* and *connected* shape can be understood as a continuum. Accordingly for example, a straight line, from which one or both endpoints is missing, or equally a circular disk, from which the boundary is excluded, are not complete continua; ... The *derivation* of a connected point set is *always* a *continuum*...

Task 13 Use Cantor's definition to show that a line is a continuum and that a circular disk which is missing the boundary is not a continuum.

Task 14 Prove that the first derived set or derivation of a connected point set is a continuum.

Next, Cantor compares his definition of a continuum with a previous definition given by Bernard Bolzano (1781– 1848), a professor at the Prague University .

Bolzano's definition of the continuum ... is certainly not right; it expresses one-sidedly merely *one* property of the continuum, but which is also satisfied by sets which arise from G_n, such that one imagines some "isolated" point set ... remote from G_n; it is likewise satisfied by sets, which consist of multiple separated continua; clearly in such cases no continuum exists, even though according to Bolzano this is the case.

To see what Cantor has in mind, we give Bolzano's definition [Bolzano, 1950, p. 129] from his book "Paradoxes of the Infinite".

If we try to form a clear idea of what we call a '*continuous extension*' or '*continuum*,' we are forced to declare that a continuum is present when, and only when, we have an aggregate of simple entities (instances or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbor. When this does not obtain, when so much as a single point of the aggregate is not so thickly surrounded by neighbors as to have at least one at each individual and sufficiently small distance from it, then we call such a point *isolated*, and say for this reason that our aggregate does not form a continuum.

Task 15 Explain Cantor's critique of Bolzano's definition. Do you agree with this critique? Why or why not?

Task 16 Prove that if we apply Cantor's definition of connected without the condition of being closed, then $\mathbb{R} - \{0\}$ is connected.

As mentioned above, a drawback of Cantor's definition is that it only applies to closed sets, and if we try to apply it to sets that are not closed, we obtain undesirable results. Another drawback of the definition is that it appeals to a notion of distance, a concept not necessary in topology.

Now that Cantor has set the stage, we turn to an important development in our understanding of connectedness. This took place roughly 20 years after Cantor's paper we have been studying, and it is found in the work of Schoenflies. As we will see, while Cantor did all his work in the context of a metric space with distance, Schoenflies abstracts this distance away, leaving us with a more general object.

3 Schoenflies: the abandonment of distance

The next important contribution to connectedness that we study is in the work of another German mathematician, Arthur Schoenflies (1853 - 1928). Schoenflies's most well-known contribution to mathematics is hallowed in a theorem bearing his name, the so-called Jordan–Schoenflies theorem.

In his 1904 paper *Beiträge zur Theorie der Punktmengen I* (Contributions to the theory of point sets I) [Schoenflies, 1904], Schoenflies begins by defining the kinds of point sets he will be working with.

For each *closed* set $\mathfrak{T}$ one can as known define *continuous* functions of position; all conclusions flowing from the continuity concept are valid for them in the same way as for the continuum. *Connectedness* can be defined both for a perfect set $\mathfrak{T}$ and also for its complement $\mathfrak{M}$. Because we are dealing here with properties of the most general nature, I will take $\mathfrak{T}$ for now to be an arbitrary [closed] perfect set.[2]

What does Schoenflies mean by closed? Just before this paragraph, he writes

The point sets with which we will for the time being occupy ourselves are *closed*, so that their boundary points belong to them.

[2]Translation of Section 3 by David Pengelley, 2017.

We will say that a point x is a **boundary point** of X if if every neighborhood of x contains at least one point of X and one point not in X. As mentioned above, Cantor was also only concerned with defining connectedness for closed sets. However, notice that this definition is different from the one we saw just before Task 12. Let's prove that these definitions are equivalent.

Task 17 Prove that a set is closed if and only if it contains all its boundary points.

Mr. G. Cantor, who first methodically did justice to all these things, uses the *distance concept* for defining connectedness. He defines an arbitrary point set T as *connected*, "if for any pair of points t and t' of the set, and a given arbitrarily small number ϵ, there always exists a *finite* number of points $t_1, t_2, \ldots t_\nu$ of T such that the distances $\overline{tt_1}, \overline{t_1t_2}, \overline{t_2t_3}, \ldots \overline{t_\nu t'}$ are all smaller than ϵ."

Now if the distance between two points forms an axiomatic geometric foundational concept for the investigations presented here, it appears useful to me to prefer a purely set-theoretical definition everywhere it is possible, especially though, when the advantage of theoretical simplicity is obtained. Such a definition is possible for a perfect set $\mathfrak{T}$; namely I define:

A perfect set $\mathfrak{T}$ is called connected, if it cannot be decomposed into subsets, each of which is perfect.

Task 18 Rewrite Schoenflies's definition of connected with modern terms and notation.

Task 19 Does the Schoenflies's definition give us any better indication of whether or not $\mathbb{R} - \{0\}$ is connected? Why or why not?

He then discusses the invariance of connectedness.

By contrast one comments that the definition possesses not only formal advantages. It suffices to point out the following. Connectedness is an important and foundational feature for the entire *Analysis situs*. Since one can take *Analysis situs* as being that science, which *behaves invariantly in the case of functions that are reversible, unambiguous, and continuous*, then the connectedness of shapes must remain *invariant* in the case of such functions. Indeed this property can be most simply concluded from the definition above, which I wish to deliver here.

Task 20 Give a formal statement of Schoenflies's claim. That is, define what he means by a function that is "reversible, unambiguous, and continuous." Give a formal definition of what it means for connectedness to "remain invariant."

Note that if we add the additional constraint that the "reverse" of the function must also be continuous, then we obtain the modern notion of a **homeomorphism.** This concept will be used in Section 6.

Scheonflies now provides a proof that connectedness is always preserved under a continuous function.

> Namely, suppose the two perfect sets $\mathfrak{T}$ and $\mathfrak{T}_1$ are reproduced in a one to one fashion and continuously on each other, and suppose the set $\mathfrak{T}$ is connected. Now were the set $\mathfrak{T}_1$ not connected, it must decompose into two subsets $\mathfrak{T}_1'$ and $\mathfrak{T}_1''$, which are both perfect. The corresponding subsets $\mathfrak{T}'$ and $\mathfrak{T}''$ of $\mathfrak{T}$ via the continuous reproduction must therefrom likewise be perfect, which however is impossible, because $\mathfrak{T}$ is connected.

Task 21 Rewrite Schoenflies's proof using modern language and notation. Determine whether or not he has justified all of his claims; that is, evaluate his argument to our modern standard of rigor.

We have just seen that Schoenflies abstracted away the concept of distance from Cantor's definition of connectedness in order to come up with a purely set-theoretic definition. Since perfect may be defined without any notion of distance, we have a purely set-theoretic understanding of connectedness. Yet Schoenflies's definition of connectedness is not equivalent to the modern definition of connectedness that we use today.

4 Lennes: the modern definition

We now turn to Nels Johann Lennes (1874–1951). Lennes was a Norwegian–born mathematician who earned his Ph.D. at the University of Chicago and lived out most of his mathematics career at Montana State University. Lennes was aware of the mathematical thought on connectedness up to this point. He wrote:

> It is apparent that in a geometry possessing linear order and continuity, curves and limit curves exist independently of metric properties... Schoenflies uses metric hypotheses in the proof of practically every important theorem dealing with curves and the regions defined by them... his treatment makes full use of metric properties... [Lennes, 1911].

Lennes gave his own definition of limit point.

A point ℓ is a **limit-point** of a set of points P if there are points of P other than ℓ within every [neighborhood] of which ℓ is an interior point.

Task 22 Compare Cantor's definition of limit point with that of Lennes. Are they always equivalent? If so, prove it. If not, give a counterexample.

Lennes was interested in the Jordan curve theorem, one of the most important and difficult theorems of the late 19^{th} and early 20^{th} century mathematics. This theorem states "Let J be a closed curve in $\mathbb{R}^2$ which does not self intersect. Then $\mathbb{R}^2 - J$ is disconnected with exactly two open, connected components." Although easy to state and intuitively obvious, a rigorous and satisfying proof of this fact eluded mathematicians for many years. In order to attempt a rigorous proof then, Lennes needed a careful and precise definition of connectedness.

A set of points is a **connected set** if at least one of any two complementary subsets contains a limit point of points in the other set.

This definition given by Lennes turns out to be equivalent to the modern definition. To substantiate our claim, here is a definition of connected from a modern classic book on point-set topology [Kelley, 1975, p. 53].

A topological space is **connected** iff X is not the union of two non-void separated, open subsets.

Task 23 Show that this definition and the one given by Lennes are equivalent. Then determine whether $\mathbb{R} - \{0\}$ is connected.

5 Knaster and Kuratowski: connectedness *qua* connectedness

Our last work that we study is an excerpt from Knaster and Kuratowski's 1921 paper *Sur les ensembles connexes* (On connected sets) [Knaster and Kuratowski, 1921]. According to Knaser and Kuratowski, this is the first paper that dealt exclusively with connected sets studied for their own sake.

Connected sets have not yet been the subject of a systematic study. The goal of this Note is to provide a rough draft of one by methodically examining some fundamental problems concerning these sets, without claiming to have exhausted the subject.[3]

Although it is beyond the scope of this project to investigate the connection, it is interesting to note w that the first paper of Cantor's that we studied in this project comes full circle in Knaster and Kuratowski's study of the relationship connectedness and a continuum.

At the same time, we draw the reader's attention to the connections between the properties of connected sets considered here and the corresponding properties in the study of continua.

The authors begin with their definitions of both connected and separated. In order to avoid confusion and use the notation that is adopted today, we have chosen to replace the author's use of $+$ to $\cup$ when the authors intend to communicate a union of sets.

Given two sets of points A and B, the *junction* of A and B is the union

$$A \times \overline{B} \cup \overline{A} \times B$$

In a footnote attached to the above definition, the authors mention that

[3]Translation of Section 5 by Gwyneth Harrison-Shermoen, 2017.

$\overline{A}$ denotes the set composed of the points of A and their limit points; one can easily show that

$$A \subset \overline{A},\ \overline{A \cup B} = \overline{A} \cup \overline{B},$$

and hence,

$$A \subset B \text{ implies } \overline{A} \subset \overline{B}.$$

Task 24 Identify the three claims made in the footnote, and prove them.

Two sets whose junction is empty are called ... *separated*. A set containing more than one point and which is not a union of two nonempty separated sets is called *connected*. We say that a set is "*connected in the larger sense*" if it is either connected, empty, or contains exactly one point.

According to Knaster and Kuratowski, we now have two notions of being connected - one definition considers the empty set, and a set with a single element, to each be connected sets, the other does not. Again in a footnote, the authors mention that

The notion of a set "connected in the larger sense" coincides with Lennes' notion of "connected set" [Lennes, 1911]. Hausdorff defines his "zusammenhängende Mengen" similarly (excluding the empty set) in [Hausdorff, 1914], p. 244.

Task 25 Prove that "connected in the larger sense" is equivalent to the definition of connected given by Lennes.

Task 26 Which definition do you think is more appropriate? That is, do you think the empty set and a single point should or should not be considered connected? Why?

The main result of this section will be that the union of a collection of connected sets with another connected set which has nonempty intersection with all the sets of the collection is connected (Corollary 1). This result will be important in Section 6 where we prove the Invariance of Domain theorem for $n = 1$. We now follow Knaster and Kuratowski through three results which are needed in order to prove Corollary 1.

Lemma 1. *If $A_1 \subset A$ and $B_1 \subset B$, the junction of A_1 and B_1 is contained in the junction of A and B.*

Task 27 Prove Lemma 1.

Hausdorff proved several simple properties of connected sets. The following will be necessary for us ([Hausdorff, 1914], p. 246, IV):

If S is connected, so is any set T such that

$$S \subset T \subset \overline{S}. \tag{1}$$

In particular, $\overline{S}$ is a continuum.

Having established this, we move on to general theorems about connected sets.

Theorem 1. *If a connected set S is contained in the union of two separated sets, one of them contains S entirely, while the other is separated from S.*

Task 28 Prove Theorem 1.

Theorem 2. *If the connected sets S_1 and S_2 are not separated, their union $S_1 \cup S_2$ is also connected.*

Corollary 1. *When a class of connected sets contains a set that is not separated from any of the others, the union of all the sets in the class is connected.*

Task 29 Prove Theorem 2 and Corollary 1.

6 Invariance of domain for $n = 1$

Putting many of the pieces of our work together, we are able to prove a very powerful and celebrated theorem, the Invariance of Domain theorem for $n = 1$. It was originally in generality due to Brouwer [Brouwer, 1911]. He used techniques from algebraic topology in order to prove the more general version; that is, if $m \neq n$, then $\mathbb{R}^m \ncong \mathbb{R}^n$. For our purposes, since we only show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n > 1$, we can simply use the ideas of connectedness without having to resort to any higher ideas ifrom algebraic topology.

The importance of this result cannot be overemphasized. The fact that there is a bijection between $\mathbb{R}$ and $\mathbb{R}^n$ meant that more structure was needed in order to distinguish between $\mathbb{R}$ and $\mathbb{R}^n$. While a vector space structure does distinguish them, vector spaces are are on the opposite extreme in terms of structure; that is, there is a lot of structure! Between all the structure of a vector space and no structure of a set, it is reasonable to ask how little structure you need to distinguish $\mathbb{R}$ and $\mathbb{R}^n$. This is precisely the accomplishment of the Invariance of Domain theorem.

We first need a technical lemma. The notation $f|A$ is the function X **restricted to** A; that is, if $f\colon X \to Y$ and $A \subseteq X$, then $f|A\colon A \to Y$ by $f(|A(a)) := f(a)$.

Lemma 3. *If $f\colon X \to Y$ is a homeomorphism with $x \in X$, then $f|(X - \{x\})\colon X - \{x\} \to Y - \{f(y)\}$ is a homeomorphism.*

Task 30 Prove Lemma 3 using the definition of homeomorphism given just below Task 20.

With this lemma established, it is simply a matter of putting together the proper pieces in order to establish the Invariance of domain theorem for $n = 1$.

Theorem 4. *$\mathbb{R}$ is not homeomporphic to $\mathbb{R}^n$ for any $n > 1$*

Task 31 Prove Theorem 4 using the following steps.

1. Define $Y = \{(x_1, 0, 0, \ldots, 0) : x_1 > 0\}$. Prove that

$$\mathbb{R}^n - \{(0, 0, \ldots, 0)\} = \bigcup_{r>0} rS^{n-1} \cup Y,$$

   where $rS^{n-1} = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : a_1^2 + \ldots + a_n^2 = r^2\}$.
2. Use Task 4 to determine whether $\mathbb{R} - \{0\}$ is connected.
3. Use Corollary 1 to determine whether $\mathbb{R}^n - \{(0, 0, \ldots, 0)\}$ is connected.
4. Use Lemma 3 to conclude the truth of the Invariance of Domain theorem for $n = 1$.

References

L. D. Ames. An Arithmetic Treatment of Some Problems in Analysis Situs. *Amer. J. Math.*, 27(4): 343–380, 1905.

B. Bolzano. *Paradoxes of the Infinite*. Routledge and Kegan Paul, London, 1950. Translated from the German of the posthumous Edition by Dr. Fr. Přihonský and furnished with a historical introduction by Donald A. Steele.

E. Borel. Quelques remarques sur les ensembles de droites ou de plans. *Bulletin de la S. M. F.*, 31: 272–275, 1903.

L.E.J Brouwer. Beweis der Invarianz des n-dimensionalen gebiets. *Math. Ann.*, 3(71):305–313, 1911.

G. Cantor. Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen. *Math. Ann.*, 5(1):123–132, 1872.

G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten 1. *Math. Ann.*, 15:1–7, 1879.

G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten 2. *Math. Ann.*, 17:355–358, 1880.

G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten 3. *Math. Ann.*, 20:113–121, 1882.

G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten 4. *Math. Ann.*, 21:51–58, 1883a.

G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten 5. *Math. Ann.*, 21:545–586, 1883b.

G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten 6. *Math. Ann.*, 23:453–488, 1884.

J. W. Dauben. The trigonometric background to Georg Cantor's theory of sets. *Arch. History Exact Sci.*, 7(3):181–216, 1971.

M. Frechet. Généralisation d'un théorème de Weierstrass. *C.R. Acad. Soi.*, 139:848–850, 1909.

F. Hausdorff. *Grundzüge der Mengenlehre.* Leipzig, Von Veit, 1914.

F. Hausdorff. *Set theory.* Chelsea Publishing Company, New York, 1957. Translated by John R. Aumann, et al.

C. Jordan. *Cours d'Analyse.* vol. 1, 1893.

J. L. Kelley. *General topology.* Springer-Verlag, New York, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27.

B. Knaster and K. Kuratowski. Sur les ensembles connexes. *Fund. Math.*, 2, 1921.

K. Kuratowski. Sur la notion de l'ordre dans la théorie des ensembles. *Fund. Math.*, 2:161–171, 1921.

N. J. Lennes. Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations. *Amer. J. Math.*, 33(1-4):287–326, 1911.

A. Schoenflies. Beiträge zur Theorie der Punktmengen I. *Math. Ann.*, 58:195–238, 1904.

H. J. S. Smith. On the Integration of Discontinuous Functions. *Proc. Lond. Math. Soc.*, 6:140–153, 1874/75.

O. Veblen. Theory on plane curves in non-metrical analysis situs. *Trans. Amer. Math. Soc.*, 6(1): 83–98, 1905.

Notes to Instructors

PSP Content: Topics and Goals

This PSP is designed to introduce students to connectedness and some of its applications. The main applications are the fact that connectedness is a topological invariant (Task 16) and the Invariance of domain Theorem for $n = 1$ (Task 26). Along the way, students will learn about limit points, derived sets, perfect sets, homeomorphisms, everywhere dense sets, and other topics typically covered in point set topology. In addition to the content goals, there is the theme of seeing how definitions can change over time, and how mathematicians struggle to find the right concepts to express ideas. After Cantor's initial attempts at defining connectedness, students see that Schoenflies takes this definition and attempts to make it more general by removing any notion of distance. This is an important conceptual step. Then the students see the definition tweaked slightly by Lennes who gave the definition we use today. After the definition seems to have been accepted by mathematicians, students see that Knaster and Kuratowski study connectedness for its own sake, in a systematic way.

Student Prerequisites

This project is intended for a first course in point set topology (or possibly analysis), and as such, students should be exposed to the standard concepts of topology – a topological space, open sets, and closed sets. But aside from the basic familiarity of what one is doing when one does topology, no other prerequisites are needed.

PSP Design, and Task Commentary

Section 2 starts with the question of what it means to be a continuum and builds the necessary point-set ideas from there. Cantor's definition of connectedness appeals to a metric and is furthermore only defined on a closed set. An interesting running example is first proposed in Task 11. Here we ask if we were to extend Cantor's definition to any set, whether $\mathbb{R} - \{0\}$ would be connected. In this case, the answer is "yes." This is counter-intuitive, and in the author's opinion, a good opportunity to have a discussion on what makes a good mathematical definition. How do we know whether a definition is "the right" definition? In fact, this conversation can begin earlier when discussing Cantor's musings on how one ought to define a continuum. See the next section for more detail. This same question is the asked in Task 19, using Schoenflies definition. Task 18 again asks this question but now under Lennes's definition. This should give students the feeling that Lennes is onto something.

The author has a shorter version of this project titled *Connecting Connectedness* which may be downloaded from the TRIUMPHS website. This is a mini-PSP and is designed to be completed in 1–2 class periods. It begins by discussing Cantor's interest in defining a continuum, but quickly moves from there to the work of C. Jordan, Schoenflies, and Lennes. Although this is a much quicker way to build up to the definition of connectedness, much of the content covered in this PSP is missed in the mini-PSP.

Suggestions for Classroom Implementation

Before the PSP begins, the instructor can assign pages 1–2 as reading along with completion of Tasks 1, 2, and as homework. In fact, the instructor can hand out only pages 1-2 to students to keep them from reading ahead or "finding the answer" to what a continuum is. Class can then begin with a discussion where the instructor asks for examples of sets that the students think should and should not be considered a continuum. They can use either Cantor's words or their own ideas as justification. The idea is then to see how Cantor defines a continuum and determine whether the sets they said should be considered a continuum are and if the sets they said shouldn't be aren't. This can be a great discussion about the nature of mathematical definitions, whether a definition is a good one, and how it is that mathematicians come up with all the subtleties and nuances they do in definitions. This discussion can ensue after students work in groups on Tasks 7 and 8. Task 9 is optional and part of illustrating the definition of everywhere dense, but it can be skipped. Task 10 illustrates that "perfect" is not enough to define a continuum and that there is still a piece missing. This leads into the definition of connectedness. Students then get practice with this definition in Tasks 13 and 14.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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For more information about TRIUMPHS, visit `http://webpages.ursinus.edu/nscoville/TRIUMPHS.html`.