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Connectedness- its evolution and applications

Nicholas A. Scoville

Ursinus College, nscoville@ursinus.edu

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Connectedness: Its Evolution and Applications

Nicholas A Scoville*

December 2, 2022

1 Introduction

In 1872, Georg Cantor published a paper titled “Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen” (“On the extension of a theorem from the theory of trigonometric series”) [Cantor, 1872]. Cantor was studying a problem concerning the uniqueness of a Fourier Series representation, a problem in Fourier analysis or more generally, real analysis. His method of solving this problem, however, involved studying a new kind of object that he called a “point-set.” The key properties of this object lie not so much in properties of real numbers, but in the relationships that the values have to one another. In addition to point-sets, Cantor introduced concepts such as *limit points*, *interiors*, and *derived sets*. It is argued by some that this viewpoint marks the birth of point-set topology.

Although the exact moment when point-set topology became its own branch of mathematics will always be debated, what is not debated is the importance of the program that Cantor had begun. He published several follow-up papers [Cantor, 1879, 1880, 1882, 1883a,b, 1884] in the important German research journal *Mathematische Annalen* in which continued his investigations into ideas on point-sets. It is the goal of this project to start in the middle of Cantor’s work and follow his ideas as they are picked up by others throughout history. Specifically, we will study the concept of connectedness. We begin with Cantor’s definition in terms of a metric in [Cantor, 1883b], and then move to a work of Schoenflies, [Schoenflies, 1904], in which he gave a purely set theoretic or topological definition of connectedness. We next move to the work of Norwegian mathematician Lennes (who spent his career at the University of Montana) on a proof of the Jordan Curve Theorem [Lennes, 1911]. We then come to the work of Knaster and Kuratowski, who claimed to be the first ones to study connectedness for its own sake [Knaster and Kuratowski, 1921]. As an added bonus, we are able to piece together some of these important results to provide a proof of the Invariance of Domain Theorem for $n = 1$. This important result states that for any $m > 1$, \mathbb{R} is not homeomorphic to \mathbb{R}^m . As mentioned above, we start with the work of Cantor.

2 Cantor: The Dawn of Connectedness

Georg Cantor (1845–1918) was born in Saint Petersburg, Russia in 1845. At the age of eleven, he moved to Germany, where he completed his education at Darmstadt, Zürich, Berlin, and Göttingen,

*Mathematics and Computer Science; 601 E. Main Street; Ursinus College; Collegeville, PA 19426; nscoville@ursinus.edu.

before becoming a professor at the University of Halle. He was the first mathematician to develop a theory of different notions of infinity and to define it mathematically using the simple but powerful idea of a one-to-one correspondence. One of his most notable mathematical discoveries is his comparison of the infinite set of rational numbers with the infinite set of natural numbers, showing that the infinite cardinality of these two sets are actually equal. Before he developed the mathematics of infinity, he worked on concepts in point-set topology, one of which was connectedness. Cantor's motivation for defining connectedness seems to have arisen out of the idea of a continuum. Below we will study his 1883 paper "Über unendliche lineare Punktmannigfaltigkeiten, [Teil] 5" ("On infinite linear manifolds of points, Part 5") [Cantor, 1883b]. Before we read Cantor's words, it worth reflecting upon the colloquial meaning of a continuum.

Task 1 Before reading any further, come up with a definition of a continuum.

Task 2 Which of the following do you think ought to be considered a continuum?

- (a) \mathbb{R}
- (a) \mathbb{R}^2
- (a) \mathbb{N}
- (a) \mathbb{Q}
- (a) $[0, 1]$
- (a) $(0, 1)$
- (a) $[0, 1] - \left\{ \frac{1}{2} \right\}$

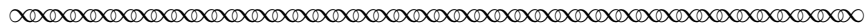
We can also think about popular definitions. Three definitions found on the internet are given below:

- a continuous extent, series, or whole.
- something that changes in character gradually or in very slight stages without any clear dividing points.
- a continuous sequence in which adjacent elements are not perceptibly different from each other, although the extremes are quite distinct.

Task 3 Compare the three above definitions with your answers in Task 2. That is, determine, according to each of the three definitions, if each of the sets in Task 2 is a continuum. Do any of the definitions agree with your answers? Would you now change your answer in Task 1? If so, how?

Having spent a little bit of time reflecting on a continuum, we now turn to what Cantor himself wrote about this concept.¹

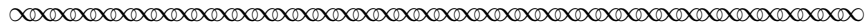
¹All translations of Cantor excerpts in this section were prepared by David Pengelley, New Mexico State University (retired), 2017.



[T]here remains for me nothing other than to attempt a most general possible purely arithmetical concept of a point continuum As foundation I am served, as cannot be otherwise, by the n -dimensional flat *arithmetical* space G_n , that is, the embodiment of all systems of values:

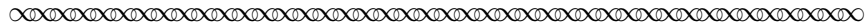
$$(x_1|x_2|\cdots|x_n),$$

in which each x can attain *all real* numerical values from $-\infty$ to $+\infty$, independent from the others.



Task 4

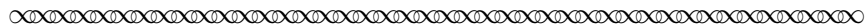
- (a) Rewrite in your own words what Cantor was attempting to do.
- (b) What is “arithmetical space”? What do we know it as today and what is our notation for it?
- (c) Why do you think Cantor chose “arithmetical space” as his foundation for what he was trying to accomplish?



Each particular system of values of this type I call an *arithmetical* point of G_n . The distance of two such points will be defined by the expression:

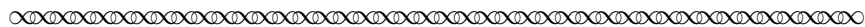
$$+\sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + \cdots + (x'_n - x_n)^2},$$

and by an *arithmetical* point-set P contained in G_n is understood each legitimate given embodiment of points in the space G_n . The investigation thus amounts to establishing a sharp and concurrently most general possible definition, for *when P is to be called a continuum*.

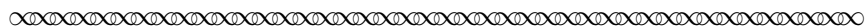


If it wasn't clear before, Cantor explicitly stated his goal in the last sentence (is your answer to Task 4 correct?). Furthermore, you might recognize that the formula which Cantor gave for distance above is a standard distance formula.

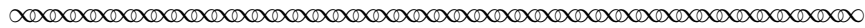
Next Cantor began to muse about how one might define a continuum.



In order to approach the general concept of a continuum situated in G_n , I recall the concept of the derivation $P^{(1)}$ of an arbitrary given point-set P , as it is first developed in the work: Math. Ann. vol. V, then in vols. XV, XVII, XX, and XXI, and is extended to the concept of a derivation $P^{(\gamma)}$, where γ can be any whole number



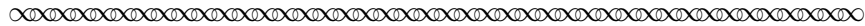
In the above excerpt, Cantor made reference to an earlier work of his, a paper he wrote in 1872. Below is a passage from Cantor's earlier paper in which he defined the derived set [Cantor, 1872].



I define a “limit point of a point-set P ” to be a point of the line situated in such a way that each neighborhood of it contains *infinitely* many points of P , and it may happen that the point itself belongs to the set. By a “neighborhood of a point” I mean any interval that has the point *in its interior*. It is easy to prove that a [“bounded”] point-set consisting of an infinite number of points has at least *one* limit point.

Every point of the line is now in a definite relation to a given set P , either being a limit point of P or not, and thereby along with the point-set P the set of limit points of P is *conceptually* given, a set which I wish to denote by P' and call the *first derived point-set of P* .

Unless the point-set P' contains only a finite number of points, it also has a derived set P'' , which I call the *second derived point-set of P* . By *v* such transitions one obtains the concept of the *v th derived set $P^{(v)}$ of P* .

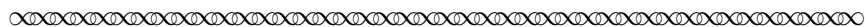


Let us gain some practice working through Cantor’s definition of limit point and derived set.

Task 5 In relation to the line \mathbb{R} , compute P' for each of the following sets.

- (a) $[0, 1]$
- (b) $\{(a, b), a < b \in \mathbb{R}\}$
- (c) $\{\frac{1}{n} : n = 1, 2, 3, \dots\}$.
- (d) \mathbb{Q}
- (e) \mathbb{R}
- (f) $\{\frac{m}{n} : m, n = 1, 2, 3, \dots, m \leq n\}$.
- (g) $[\mathbb{Q} \cap (0, 1)] \cup [-4, 2]$

In the previous task, you investigated the derived set of a subset of \mathbb{R} . However, there is nothing to stop us from working with derived sets more generally. In the following, Cantor divided a set P into two classes. Either there exists an α for which $P^{(\alpha)}$ vanishes (is the empty set), or



$P^{(1)}$ can always be broken down, and in fact only in one way, into two sets R and S , so that:

$$P^{(1)} \equiv R \cup S,$$

R is so constituted, such that through the repeated derivation process, it is capable of a continual reduction up until annihilation, so that there is always a first whole number $\gamma \dots$ for which:

$$R^{(\gamma)} = 0;$$

such point-sets R I call *reducible*.

S on the other hand is so constituted, such that for this point-set the derivation process yields no ending at all, in that:

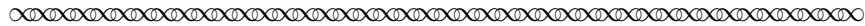
$$S \equiv S^{(1)}$$

and consequently also:

$$S \equiv S^{(\gamma)};$$

such sets S I call *perfect* point-sets

Although these two predicates “reducible” and “perfect” are incompatible in one and the same point-set, still on the other hand irreducible is not tantamount to perfect, any more than imperfect is exactly the same as reducible, as one can easily see by some attention.

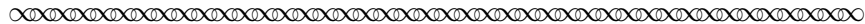


Task 6 Find the decomposition $P = R \cup S$ for each of the sets in Task 5.

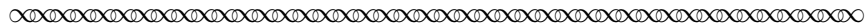
Let us study Cantor’s words here carefully. Let P be a point-set. Then either there exists a natural number α such that $P^{(\alpha)} = \emptyset$ or there does not. Cantor was not so much interested in the former case, but the latter. Hence assume that for all natural numbers α that $P^{(\alpha)} \neq \emptyset$. For the following exercise, refer to the Cantor quote above.

- Task 7**
- (a) Give a careful and rigorous definition of what it means for a point-set S to be **perfect**.
 - (b) Give a careful and rigorous definition of what it means for a point-set R to be **reducible**.
 - (c) Suppose that for all natural numbers α , $P^{(\alpha)} \neq \emptyset$. Prove that $P' = R \cup S$, where R is reducible and S is perfect.
 - (d) Prove that the decomposition you came up with above is unique; that is, if $P' = T \cup W$ where T is reducible and W is perfect, then $T = S$ and $W = R$.

Cantor had now defined point-sets with a certain kind of property; namely, either their derived set eventually vanishes or it can be broken down as a union of a reducible set (which is itself of the former kind of point-set mentioned) and a perfect point-set. He then began to muse about a perfect point-set and its possible relationship to other concepts, including a continuum.

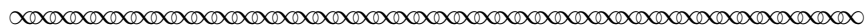


The *perfect* point-sets S are by no means always in themselves that which I in my previously mentioned work have called everywhere dense; therefore also they are not yet alone suitable for the complete definition of a point continuum, even if one must immediately admit, that the latter must always be a *perfect* set.



Task 8 Thinking back to your work in the first three tasks, do you agree that ‘one must immediately admit’ that a continuum must be perfect? Why or why not?

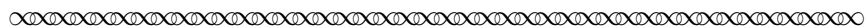
By S is everywhere dense in A , Cantor meant that for every point $a \in A$, and every open set U containing a , we have that U and S have nonempty intersection. He thus claimed that a point-set which is perfect is not the same thing as being everywhere dense. In fact, he relegated a counterexample to a footnote attached to the word “dense” in the above quote. We pull it out here.



As an example of a perfect point-set that is not everywhere dense in any interval, however small, I invoke the embodiment of all real numbers, that are contained in the formula:

$$z = \frac{c_1}{3} + \frac{c_2}{3^2} + \cdots + \frac{c_\nu}{3^\nu} + \cdots,$$

where the coefficients c_ν assume, at one’s discretion, the two values 0 and 2, and the series can consist of either a finite or an infinite number of terms.



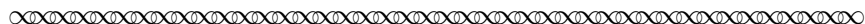
Today this set bears Cantor’s name and is referred to as the **Cantor set**, even though it was discovered in a more general form nearly a decade earlier, in 1874, by the Irish mathematician J. H. Smith (1826–1883).²

- Task 9** (a) Show that the Cantor set is not everywhere dense in \mathbb{R} .
 (b) Show that the Cantor set is perfect.

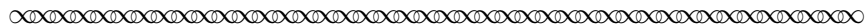
As Cantor noted above, in his view, being a perfect point-set is necessary but not sufficient for constituting a continuum.

Task 10 Let $X = [0, 1] \cup [2, 3]$. Show that this is a perfect point-set. Do you think X is a continuum? Why or why not?

Now Cantor defined the missing ingredient.



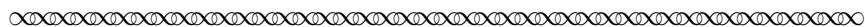
There is rather another essential concept, which together with the foregoing defines the continuum, namely the concept of a *connected* point-set T .



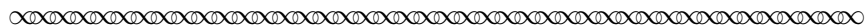
Task 11 Before reading Cantor’s definition, take a few moments to try and define what it means for a point-set to be “connected.” What conditions would a point-set need to satisfy in order to be considered connected?

²To explore Smith’s definition of the Cantor set through excerpts from his paper “On the Integration of Discontinuous Functions,” [Smith, 1874], see the project “The Cantor Set Before Cantor” (also by Nicholas A. Scoville), available at https://digitalcommons.ursinus.edu/triumphs_topology/2/.

Cantor continued.

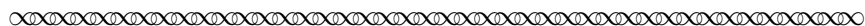


We call [a closed set] T a *connected* point-set, if for any two of its points t and t' , and an arbitrarily small given number ϵ , there is always a *finite* number of points t_1, t_2, \dots, t_ν of multiple forms in T , such that the distances $\overline{tt_1}, \overline{t_1t_2}, \overline{t_2t_3}, \dots, \overline{t_\nu t'}$ are all smaller than ϵ .

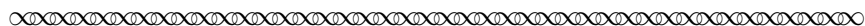


For our purposes, we will define a point-set A to be **closed** if its complement $X - A$ is open. It is worth pointing out here that Cantor limited his definition of connected to only closed point-sets T . We will explore this in Task 16.

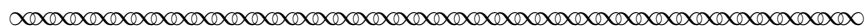
Task 12 How does your definition compare to Cantor's?



Now it is easy to see that all geometric point continua known to us fall under this concept of *connected* point-set; but I also believe in these two predicates "perfect" and "connected" to realize the necessary and *sufficient* characteristics of a point continuum, and therefore define a point continuum inside G_n as a *perfect-connected set*.

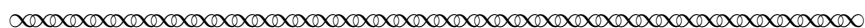


Cantor attached a footnote to the phrase "as a perfect-connected set" above. We pull it out here.



One notices that this definition of a continuum is free of any indication of what one calls the dimension of a continuous shape; the definition encompasses namely also such continua as consist of connected pieces of different dimensions, like lines, surfaces, solids, etc. ... According to my conception only a *perfect* and *connected* shape can be understood as a continuum. Accordingly for example, a straight line, from which one or both endpoints is missing, or equally a circular disk, from which the boundary is excluded, are not complete continua;

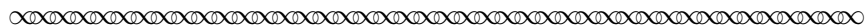
...
The *derivation* of a connected point-set is *always* a *continuum*



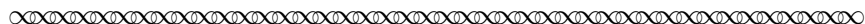
Task 13 Use Cantor's definition to show that a line is a continuum and that a circular disk which is missing the boundary is not a continuum.

Task 14 Prove that the first derived set or derivation of a connected point-set is a continuum.

Next, Cantor compared his definition of a continuum with a previous definition given by Bernard Bolzano (1781–1848), a Bohemian mathematician and Roman Catholic priest who held the chair of philosophy of religion at the Prague University. Here's what Cantor had to say:



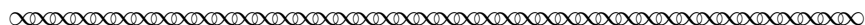
Bolzano's definition of the continuum ... is certainly not right; it expresses one-sidedly merely *one* property of the continuum, but which is also satisfied by sets which arise from G_n , such that one imagines some "isolated" point-set ... remote from G_n ; it is likewise satisfied by sets, which consist of multiple separated continua; clearly in such cases no continuum exists, even though according to Bolzano this is the case.



To see what Cantor had in mind, we give Bolzano's definition from his posthumously published book *Paradoxien des Unendlichen (Paradoxes of the Infinite)*, [Bolzano, 1851].



If we try to form a clear idea of what we call a '*continuous extension*' or '*continuum*,' we are forced to declare that a continuum is present when, and only when, we have an aggregate of simple entities (instances or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbor. When this does not obtain, when so much as a single point of the aggregate is not so thickly surrounded by neighbors as to have at least one at each individual and sufficiently small distance from it, then we call such a point *isolated*, and say for this reason that our aggregate does not form a continuum.



Task 15 Explain Cantor's critique of Bolzano's definition. Do you agree with this critique? Why or why not?

Task 16 Prove that if we apply Cantor's definition of connected without the condition of being closed, then $\mathbb{R} - \{0\}$ is connected.

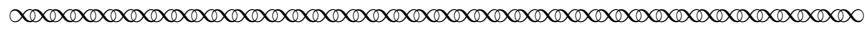
As mentioned above, a drawback of Cantor's definition is that it only applies to closed sets, and if we try to apply it to sets that are not closed, we obtain undesirable results. Another drawback of the definition is that it appeals to a notion of distance, a concept not necessary in topology.

Now that Cantor had set the stage, we turn to an important development in our understanding of connectedness. This took place roughly 20 years after Cantor's paper we have been studying, and it is found in the work of Schoenflies. As we will see, while Cantor did all his work in the context of a metric space with distance, Schoenflies abstracted this distance away, leaving us with a more general object.

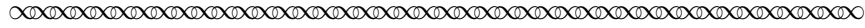
3 Schoenflies: The Abandonment of Distance

The next important contribution to connectedness that we study is in the work of another German mathematician, Arthur Schoenflies (1853–1928). Schoenflies's most well-known contribution to mathematics is hallowed in a theorem bearing his name, the so-called Jordan-Schoenflies Theorem.

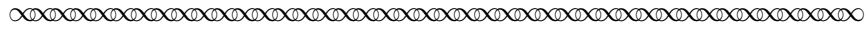
In his 1904 paper "Beiträge zur Theorie der Punktmenen I" ("Contributions to the Theory of Point-sets I") [Schoenflies, 1904], Schoenflies began by defining the kinds of point-sets he with which he was working.



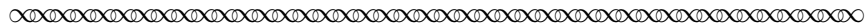
For each *closed* set \mathfrak{T} one can as known define *continuous* functions of position; all conclusions flowing from the continuity concept are valid for them in the same way as for the continuum. *Connectedness* can be defined both for a perfect set \mathfrak{T} and also for its complement \mathfrak{M} . Because we are dealing here with properties of the most general nature, I will take \mathfrak{T} for now to be an arbitrary [closed] perfect set.³



What does Schoenflies meant by closed? Just before this paragraph, he wrote:



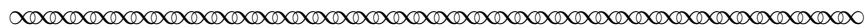
The point-sets with which we will for the time being occupy ourselves are *closed*, so that their boundary points belong to them.



We will say that a point x is a **boundary point** of X if every neighborhood of x contains at least one point of X and one point not in X . As mentioned above, Cantor was also only concerned with defining connectedness for closed sets. However, notice that this definition is different from the one we saw just before Task 12. Let's prove that these definitions are equivalent.

Task 17 Prove that a set is closed if and only if it contains all its boundary points.

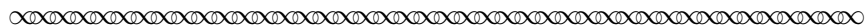
Having defined what he meant by a perfect set, Schoenflies gave his definition for connectedness and contrasted it with that given by Cantor.



Mr. G. Cantor, who first methodically did justice to all these things, uses the *distance concept* for defining connectedness. He defined an arbitrary point-set T as *connected*, "if for any pair of points t and t' of the set, and a given arbitrarily small number ϵ , there always exists a *finite* number of points t_1, t_2, \dots, t_ν of T such that the distances $\overline{tt_1}, \overline{t_1t_2}, \overline{t_2t_3}, \dots, \overline{t_\nu t'}$ are all smaller than ϵ ."

Now if the distance between two points forms an axiomatic geometric foundational concept for the investigations presented here, it appears useful to me to prefer a purely set-theoretical definition everywhere it is possible, especially though, when the advantage of theoretical simplicity is obtained. Such a definition is possible for a perfect set \mathfrak{T} ; namely I define:

A perfect set \mathfrak{T} is called connected, if it cannot be decomposed into subsets, each of which is perfect.

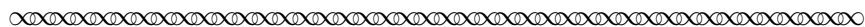


Task 18 Rewrite Schoenflies's definition of connected with modern terms and notation.

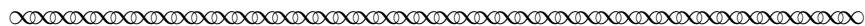
³All translations of Schoenflies excerpts in this section were prepared by David Pengelley, New Mexico State University (retired), 2017.

Task 19 Does the Schoenflies's definition give us any better indication of whether or not $\mathbb{R} - \{0\}$ is connected? Why or why not?

He then discussed the invariance of connectedness.



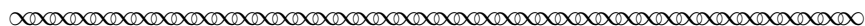
By contrast one comments that the definition possesses not only formal advantages. It suffices to point out the following. Connectedness is an important and foundational feature for the entire *Analysis situs*. Since one can take *Analysis situs* as being that science, which *behaves invariantly in the case of functions that are reversible, unambiguous, and continuous*, then the connectedness of shapes must remain *invariant* in the case of such functions. Indeed this property can be most simply concluded from the definition above, which I wish to deliver here.



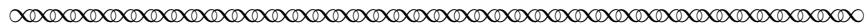
Task 20 Give a formal statement of Schoenflies's claim. That is, define what he meant by a function that is "reversible, unambiguous, and continuous." Give a formal definition of what it means for connectedness to "remain invariant."

Note that if we add the additional constraint that the "reverse" of the function must also be continuous, then we obtain the modern notion of a **homeomorphism**. This concept will be used in Section 6.

Schoenflies now provided a proof that connectedness is always preserved under a continuous function.



Namely, suppose the two perfect sets \mathfrak{T} and \mathfrak{T}_1 are reproduced in a one to one fashion and continuously on each other, and suppose the set \mathfrak{T} is connected. Now were the set \mathfrak{T}_1 not connected, it must decompose into two subsets \mathfrak{T}'_1 and \mathfrak{T}''_1 , which are both perfect. The corresponding subsets \mathfrak{T}' and \mathfrak{T}'' of \mathfrak{T} via the continuous reproduction must therefrom likewise be perfect, which however is impossible, because \mathfrak{T} is connected.

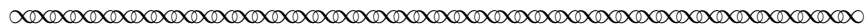


Task 21 Rewrite Schoenflies's proof using modern language and notation. Determine whether or not he has justified all of his claims; that is, evaluate his argument to our modern standard of rigor.

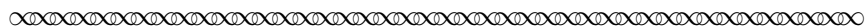
We have just seen that Schoenflies abstracted away the concept of distance from Cantor's definition of connectedness in order to come up with a purely set-theoretic definition. Since perfect may be defined without any notion of distance, we have a purely set-theoretic understanding of connectedness. Yet Schoenflies's definition of connectedness is not equivalent to the modern definition of connectedness that we use today.

4 Lennes: The Modern Definition

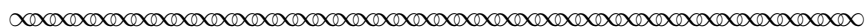
We now turn to Nels Johann Lennes (1874–1951). Lennes was aware of the mathematical thought on connectedness up to this point. In his 1911 paper “Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations” [Lennes, 1911], he wrote:



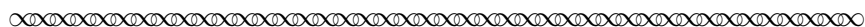
It is apparent that in a geometry possessing linear order and continuity, curves and limit curves exist independently of metric properties Schoenflies used metric hypotheses in the proof of practically every important theorem dealing with curves and the regions defined by them . . . his treatment makes full use of metric properties



Lennes gave his own definition of limit point.



A point ℓ is a **limit-point** of a set of points P if there are points of P other than ℓ within every [neighborhood] of which ℓ is an interior point.



Task 22 Compare Cantor’s definition of limit point with that of Lennes. Are they always equivalent? If so, prove it. If not, give a counterexample.

Lennes was interested in the Jordan Curve Theorem, one of the most important and difficult theorems of the late-19th and early-20th century mathematics. This theorem states:

Let J be a closed curve in \mathbb{R}^2 which does not self-intersect.
Then $\mathbb{R}^2 - J$ is disconnected with exactly two open, connected components.

Although easy to state and intuitively obvious, a rigorous and satisfying proof of this fact eluded mathematicians for many years. In order to attempt a rigorous proof then, Lennes needed a careful and precise definition of connectedness.

A set of points is a **connected set** if at least one of any two complementary subsets contains a limit point of points in the other set.

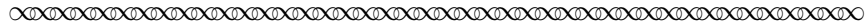
This definition given by Lennes turns out to be equivalent to the modern definition. To substantiate our claim, here is a definition of connected from a modern classic book on point-set topology [Kelley, 1975, p. 53].

Definition. A topological space is **connected** if and only if X is not the union of two non-void separated, open subsets.

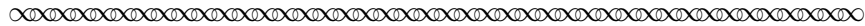
Task 23 Show that this definition and the one given by Lennes are equivalent. Then determine whether $\mathbb{R} - \{0\}$ is connected.

5 Knaster and Kuratowski: Connectedness *qua* Connectedness

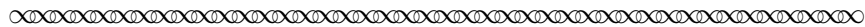
The last work that we study is an excerpt from the 1921 paper “Sur les ensembles connexes” (“On connected sets”) [Knaster and Kuratowski, 1921], written jointly by the Polish mathematicians Bronisław Knaster (1893–1980) and Kazimierz Kuratowski (1896–1980). Their paper was among the first to be published in the Polish journal *Fundamenta Mathematicae*, founded in the period between the two World Wars during which Poland made a remarkable leap forward in mathematical teaching and research. According to Knaster and Kuratowski, their 1921 paper was also the first to deal exclusively with the study of connected sets studied for their own sake.⁴



Connected sets have not yet been systematically studied. The purpose of this Note is to give a draft [of such a study] by methodically examining some fundamental problems concerning these sets, without claiming to have exhausted the subject.



The authors began with a definition that they will use in order to define what it means for sets to be connected and separated.



Definition. Given two sets of points A and B , we call the *junction of A and B* the union

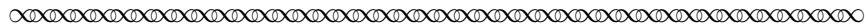
$$A \cap \bar{B} \cup \bar{A} \cap B$$

⁽¹⁾ [Knaster’s and Kuratowski’s Footnote] \bar{A} designates the set composed of the points of A and their limit points; it is easily shown that

$$A \subset \bar{A}, \quad \overline{A \cup B} = \bar{A} \cup \bar{B},$$

and hence,

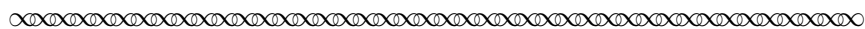
$$A \subset B \text{ implies } \bar{A} \subset \bar{B}.$$



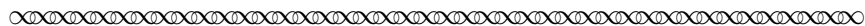
Task 24 Identify the three claims made in the footnote just quoted, and prove them.

⁴All translations of Knaster & Kuratowski excerpts in this section were prepared by Janet Heine Barnett, 2022, using the set notation \cup, \cap in place of the boolean algebra notation $+, \times$ that Kuratowski and Knaster employed for the operations of union and intersection respectively.

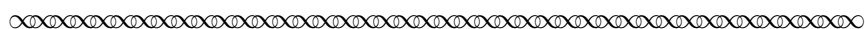
Knaster and Kuratowski next used the notion of junction to provide two further definitions.



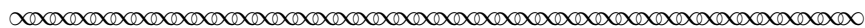
Two sets for which the junction is empty are called ... *separated*. A set that contains more than one point and which is not the union of two separated nonempty sets is called connected. We will call "*connected in the larger sense*" any set that is either connected, or else contains a single point or contains none.



According to Knaster and Kuratowski, we now have two notions of being connected: one definition considers both the empty set and a set with a single element to be connected sets, the other does not. Again in a footnote, the authors mentioned that



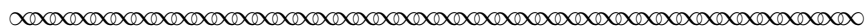
The notion of a set "connected in the larger sense" coincides with that of a "connected set" of Mr. Lennes, which he introduced in his Note "Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations," *American Journal of Mathematics*, XXIII, 1911. This is precisely the same sense that Mr. Hausdorff assigns to his "connected sets" (if we ignore the empty set) in the *Fundamentals of Set Theory*, Leipzig 1914, p. 244.



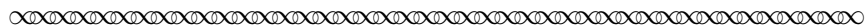
Task 25 Prove that "connected in the larger sense" is equivalent to the definition of connected given by Lennes.

Task 26 Which definition do you think is more appropriate? That is, do you think the empty set and a single point should or should not be considered connected? Why?

The main result of this section will be that the union of a collection of connected sets with another connected set which has nonempty intersection with all the sets of the collection is connected (Corollary 4). This result will be important in Section 6 where we prove the Invariance of Domain Theorem for $n = 1$. We now follow Knaster and Kuratowski through three results which are needed in order to prove Corollary 4.⁵

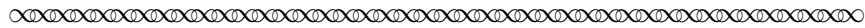


Lemma 1. If $A_1 \subset A$ and $B_1 \subset B$, the junction of A_1 and B_1 is contained in the junction of A and B .



Task 27 Prove Lemma 1.

⁵In [Knaster and Kuratowski, 1921], these results are numbered II–V; for convenience in referencing them, they are instead numbered 1–4 in this project.



Several simple properties of complex sets have been proven by Mr. Hausdorff. The following will be necessary for us ([Hausdorff, 1914], p. 246, IV):

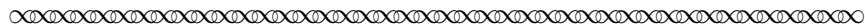
S being connected, so is every set T that satisfies the condition

$$S \subset T \subset \overline{S};$$

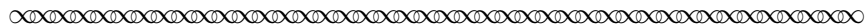
in particular, \overline{S} is a continuum.

This established, we move on to the general theorems on connected sets.

Theorem 2. If a connected set S is contained in the union of two separated sets, one of them contains S entirely, while the other is separated from S .

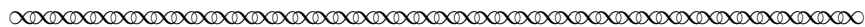


Task 28 Prove Theorem 2.



Theorem 3. If the connected sets S_1 and S_2 are not separated, their union $S_1 \cup S_2$ is also connected.

Corollary 4. When a family of connected sets contains a set that is not separated from any of the others, the union of all the sets in the family is connected.



Task 29 Prove Theorem 3 and Corollary 4.

6 Invariance of Domain for $n = 1$

Putting many of the pieces of our work together, we are able to prove a very powerful and celebrated theorem, the Invariance of Domain Theorem for $n = 1$. A more general form of this theorem is due to the Dutch mathematician L. E. J. Brouwer (1888–1961), who studied and taught at the University of Amsterdam. In his paper “Beweis der Invarianz des n -dimensionalen gebiets” (“Proof of the Invariance of n -dimensional domains”) [Brouwer, 1911], Brouwer used techniques from algebraic topology in order to prove invariance of domain holds in full generality; that is, if $m \neq n$, then $\mathbb{R}^m \not\cong \mathbb{R}^n$. For our purposes, since we only show that \mathbb{R} is not homeomorphic to \mathbb{R}^n for any $n > 1$, we can simply use the ideas of connectedness without having to resort to any higher ideas from algebraic topology.

The importance of this result cannot be overemphasized. The fact that there is a bijection between \mathbb{R} and \mathbb{R}^n meant that more structure was needed in order to distinguish between \mathbb{R} and \mathbb{R}^n . While a vector space structure does distinguish them, vector spaces are on the opposite extreme in terms of structure; that is, there is a lot of structure! Between all the structure of a vector space and no structure of a set, it is reasonable to ask how little structure you need to distinguish \mathbb{R} and \mathbb{R}^n . This is precisely the accomplishment of the Invariance of Domain Theorem.

We first need a technical lemma. The notation $f|A$ is the function X **restricted to** A ; that is, if $f: X \rightarrow Y$ and $A \subseteq X$, then $f|A: A \rightarrow Y$ is defined by $f|A(a) := f(a)$.

Lemma 5. *If $f: X \rightarrow Y$ is a homeomorphism with $x \in X$, then $f|(X - \{x\}): X - \{x\} \rightarrow Y - \{f(x)\}$ is a homeomorphism.*

Task 30 Prove Lemma 5 using the definition of homeomorphism given just below Task 20.

With this lemma established, it is simply a matter of putting together the proper pieces in order to establish the Invariance of Domain Theorem for $n = 1$.

Theorem 6. \mathbb{R} is not homeomorphic to \mathbb{R}^n for any $n > 1$

Task 31 Prove Theorem 6 using the following steps.

(a) Define $Y = \{(x_1, 0, 0, \dots, 0) : x_1 > 0\}$. Prove that

$$\mathbb{R}^n - \{(0, 0, \dots, 0)\} = \bigcup_{r>0} rS^{n-1} \cup Y,$$

where $rS^{n-1} = \{(a_1, \dots, a_n) \in \mathbb{R}^n : a_1^2 + \dots + a_n^2 = r^2\}$.

(b) Use Task 23 to determine whether $\mathbb{R} - \{0\}$ is connected.

(c) Use Corollary 4 to determine whether $\mathbb{R}^n - \{(0, 0, \dots, 0)\}$ is connected.

(d) Use Lemma 5 to conclude the truth of the Invariance of Domain Theorem for $n = 1$.

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Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed to introduce students to connectedness and some of its applications. The main applications are the fact that connectedness is a topological invariant (Task 16) and the Invariance of Domain Theorem for $n = 1$ (Task 26). Along the way, students will learn about limit points, derived sets, perfect sets, homeomorphisms, everywhere dense sets, and other topics typically covered in point-set topology. In addition to the content goals, there is the theme of seeing how definitions can change over time, and how mathematicians struggle to find the right concepts to express ideas. After Cantor's initial attempts at defining connectedness, students see that Schoenflies took this definition and attempted to make it more general by removing any notion of distance. This is an important conceptual step. Then the students see the definition tweaked slightly by Lennes who gave the definition we use today. After the definition seems to have been accepted by mathematicians, students see that Knaster and Kuratowski studied connectedness for its own sake, in a systematic way.

Student Prerequisites

This project is intended for a first course in point-set topology (or possibly analysis), and as such, students should be exposed to the standard concepts of topology — a topological space, open sets, and closed sets. But aside from the basic familiarity of what one is doing when one does topology, no other prerequisites are needed.

PSP Design and Task Commentary

Section 2 starts with the question of what it means to be a continuum and builds the necessary point-set ideas from there. Cantor's definition of connectedness appeals to a metric and is furthermore only defined on a closed set. An interesting running example is first proposed in Task 11. Here we ask if we were to extend Cantor's definition to any set, whether $\mathbb{R} - \{0\}$ would be connected. In this case, the answer is "yes." This is counterintuitive, and in the author's opinion, a good opportunity to have a discussion on what makes a good mathematical definition. How do we know whether a definition is "the right" definition? In fact, this conversation can begin earlier when discussing Cantor's musings on how one ought to define a continuum. See the next section for more detail. This same question is the asked in Task 19, using Schoenflies' definition. Task 18 again asks this question but now under Lennes's definition. This should give students the feeling that Lennes was on to something.

Suggestions for Classroom Implementation

Before the PSP begins, the instructor can assign pages 1–2 as reading, along with completion of Tasks 1, 2, and as homework. In fact, the instructor can hand out only pages 1–2 to students to keep them from reading ahead or "finding the answer" to what a continuum is. Class can then begin with a discussion where the instructor asks for examples of sets that the students think should and should not be considered a continuum. They can use either Cantor's words or their own ideas as justification. The idea is then to see how Cantor defined a continuum and determine whether the sets they said should be considered a continuum are and if the sets they said shouldn't be aren't. This can be a great discussion about the nature of mathematical definitions, whether a definition is a good one, and how it is that mathematicians come up with all the subtleties and nuances they do in definitions. This discussion can ensue after students work in groups on Tasks 7 and 8. Task 9 is

optional and part of illustrating the definition of everywhere dense, but it can be skipped. Task 10 illustrates that “perfect” is not enough to define a continuum and that there is still a piece missing. This leads into the definition of connectedness. Students then get practice with this definition in Tasks 13 and 14.

L^AT_EX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or ‘in-class worksheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

The following outline provides a schedule for implementing this project in 5 days. The actual number of class periods spent on each section naturally depends on the instructor’s goals and on how the PSP is actually implemented with students.

- Have students read and work through Task 2 as homework in preparation for the first day you start this PSP. Class can begin by immediately putting students in groups and having them share answers with each other. After some time discussing their answers, have the class regroup to share out their ideas. Task 3 can be assigned as homework. Students should then spend the rest of the class period working on Tasks 4–7 in groups.
- The next day’s class can begin with a discussion about the answers to Task 7. Ensure that students agree about the definitions given in this exercise. Have students work on Tasks 8–12 in groups for the remainder of the class with Tasks 13–16 given as homework.
- Class should begin the next day with a discussion and wrap up of the work of Cantor. In particular, a student can present Task 16 to the class, as it acts as a running example for the project. Next have students spend the majority of class on Section 3 (Schoenflies). Have students complete any tasks they were not able to complete in class for homework.
- Begin class the next day emphasizing the important aspects of Schoenflies; that is, the abstraction away of distance as well as the introduction of the notion of invariance of connectedness. The section on Lennes is short and can probably be done in class as a whole-class discussion with a few minutes for students to individually work on Task 23. Equipped with the current definition of connected, students can work on Tasks 24–26 in groups for the remainder of class with Tasks 27–29 for homework.
- The last section is the culmination of this project and a full day should be spent on it. The Invariance of Domain Theorem can be discussed by the class as a whole before breaking up into groups to work on Task 30. After working in groups, the class should share out their ideas for Task 30 to make sure everyone is on the same page. A discussion of Task 31 can ensue, making sure students understand the outline of the proof of the main theorem. Students can spend the rest of the class working in groups on this task with the formal write up as homework.

Connections to other Primary Source Projects

The author has a shorter version of this project titled *Connecting Connectedness*, designed to be completed in in 1–2 class periods. It begins by discussing Cantor’s interest in defining a continuum, but quickly moves from there to the work of Jordan, Schoenflies, and Lennes. Although this is a much quicker way to build up to the definition of connectedness, much of the content covered in this PSP is missed in the abbreviated PSP.

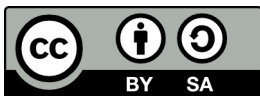
The following additional primary source-based projects by the author are also freely available for use in teaching courses in point-set topology. The first two projects listed are full-length PSPs that require 10 and 3 class periods respectively to complete. All others are designed for completion in 2 class periods.

- *Nearness without Distance*
- *From Sets to Metric Spaces to Topological Spaces*
- *Topology from Analysis* (Also suitable for use in Introductory Analysis courses.)
- *The Cantor set before Cantor* (Also suitable for use in Introductory Analysis courses.)
- *The Closure Operation as the Foundation of Topology*
- *A Compact Introduction to a Generalized Extreme Value Theorem*

Classroom-ready versions of these projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_topology. They can also be obtained (along with their L^AT_EX code) from the author.

Acknowledgments

The development of this project has been partially supported by the National Science Foundation's Improving Undergraduate STEM Education Program under Grants No. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily reflect the views of the National Science Foundation.



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