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The Trigonometric Functions Through Their Origins: Varahamihira and the Poetry of Sines

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Trigonometry is concerned with the measurements of angles about a central point (or of arcs of circles centered at that point) and quantities, geometrical and otherwise, which depend on the sizes of such angles (or the lengths of the corresponding arcs). It is one of those subjects that has become a standard part of the toolbox of every scientist and applied mathematician. Today an introduction to trigonometry is normally part of the mathematical preparation for the study of calculus and other forms of mathematical analysis, as the trigonometric functions make common appearances in applications of mathematics to the sciences, wherever the mathematical description of cyclical phenomena is needed. This project is one of a series of curricular units that tell some of the story of where and how the central ideas of this subject first emerged, in an attempt to provide context for the study of this important mathematical theory. Readers who work through the entire collection of units will encounter six milestones in the history of the development of trigonometry. In this unit, we encounter a few lines of Vedic verse by the Hindu scholar Varāhamihira (sixth century, CE) that contain the "recipe" for a table of sines as well as some of the methods used for its construction.

1 Ancient Astronomy from Babylonia to India

Varāhamihira (505-587 CE) was a prolific astronomer/astrologer of the sixth century who lived in central India near the end of the two-century-long rule of the Gupta Empire. The reign of the Guptas was a time of general peace and prosperity throughout the northern part of the Indian subcontinent and is associated with a golden age of art, architecture, Sanskrit literature, mathematics, astronomy, and medicine [Katz, 1998]. It was around this time that decimal positional numeration with a system of ten digits, including a symbol for zero, was first used to perform arithmetic. This system evolved over the centuries and was picked up by Muslim scholars to the west of India to become the Indo-Arabic numeration system in use now throughout the world.

Some scholars think that Varāhamihira was a court astronomer and astrologer to the Maharajah of Daśapura (modern day Mandasor, India). Whether or not this is true, he was an expert both in Greek astronomy and in astrology in the Greek, Roman, and Egyptian traditions; he was the author of a number of works in mathematical astronomy. We will consider the first nine verses from part IV of his *Pañcasiddhāntikā* [Neugebauer and Pingree, 1970/1972], dating from around 575 CE. *Pañca* is Sanskrit for "five",[1] and *siddhānta* refers to a "dogma" or canonical text, so it makes sense to

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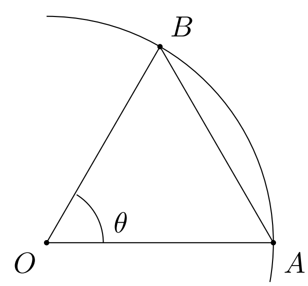
[1] Also the root of the English word 'punch,' a blow from a five-fingered hand!

translate the title of this work as *The Five Canons*. In it, Varāhamihira commented on five earlier Hindu works in mathematical astronomy and astrology [Plofker, 2008].

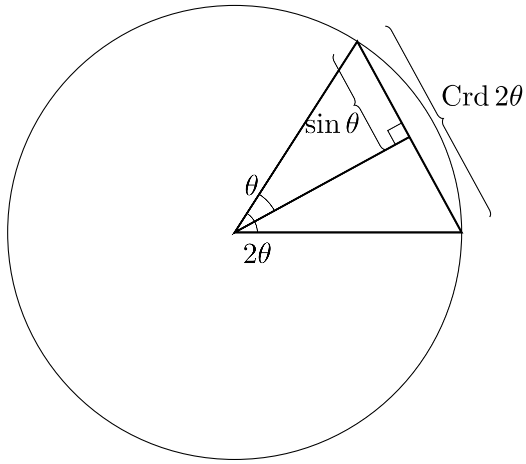
Sanskrit was the language of Hindu priest-scholars. This scholarship was founded on the long tradition of the Vedas, epic poems about mythology, theology, and Hindu practice whose creation dates from around 1500 BCE [Katz et al., 2007]. These texts were intended to be memorized and recited by devotees, and were only written down for the first time in about 500 BCE. As a member of this tradition, Varāhamihira also composed in Sanskrit verse.

One of our primary interests in this passage is that it corresponds to a key development in the history of trigonometry, namely, the shift in interest from tabulation of the chord of an arc or angle to that of its *sine* (and *cosine*). Greek geometer-astronomers like Hipparchus of Rhodes (second century BCE) and Claudius Ptolemy (second century CE) were among the first people we know who investigated what we today call trigonometry. They were interested in the relations between the lengths of arcs along a circle and the segments (called chords) that connect the endpoints of these arcs. More specifically, they created tables that measured various arcs of a circle – in degrees, minutes and seconds – along with the lengths of the chords that span those arcs, thereby relating circular distance to linear distance. The diagram below displays a portion of a circle centered at O with central angle θ and associated arc $\overset{\frown}{AB}$ together with the chord $\overline{AB}$.[2] Note that the length of the chord, denoted $\operatorname{Crd}\theta$, is always shorter than the length of the arc, which equals $OA \cdot \theta$, since a straight line must be the shortest distance between the two points.



In many applications of chord tables to problems in computational astronomy, the problems called for finding the measures of the sides of some right triangle, given one of its angles—call it θ—and the length of its hypotenuse, which for simplicity we will take to be the same radius used to prepare a table of chords. The obvious way to use the table to solve this sort of problem (see the figure below) is to look up the angle twice as large as the given angle (2θ) to find its chord, $\operatorname{Crd} 2\theta$ (in Sanskrit, the word for "chord" was *jyā*). Half this magnitude (in Sanskrit, the *jyā-ardha*) will give the side of the right triangle opposite the angle θ. Having to use this process repeatedly probably influenced Hindu astronomers to stop tabulating the chord length and start tabulating the *jyā-ardha* instead. After a time, the term for half-chord was shortened to simply *jyā*; centuries later, Arabic scholars transliterated this Sanskrit word into Arabic as *jiba*. Then in the twelfth century, a European Latinist mistranslated this Arabic word, thinking that it was the word *jaib* instead (in written Arabic, vowels appear only as diacritical marks), which means "bosom" or "fold", so he rendered it as the Latin word for "fold", namely *sinus* (the same word we use to describe the folded layers of the nasal cavity). This became the English word *sine*!

[2]See the two projects "Hipparchus' Table of Chords" and "Ptolemy Finds High Noon in Chords of Circles" at https://blogs.ursinus.edu/triumphs/; the first of these explores Greek astronomy and Hipparchus' early table of circular arcs and corresponding chord lengths, while the second illustrates how Ptolemy used a similar table to tell time using the lengths of shadows cast by the Sun.



2 A Table of Sines in Sanskrit Verse

In the passage below (taken from [Neugebauer and Pingree, 1970/1972]), Varāhamihira produced the beginning of a Table of Sines in Sanskrit verse. He began by orienting us to the geometric content of the *Pañcasiddhāntikā*, in which the basic facts of circle geometry are being laid out. It is important to note that for Varāhamihira, the word *Sine* (specifically written here with a capital S) referred to the side of the right triangle opposite the given angle in a triangle whose radius R has been given; in contrast, we use the word *sine* (with a lowercase s) to mean the length of the side of the right triangle opposite the given angle in a triangle whose radius has length 1. This is a distinction which may seem trivial, but is of real importance: for the ancients, the Sine was a geometrical line segment whose length was to be determined; for us moderns, a sine is a number which measures a certain distance. In particular, the Sine of an arc differs depending on the radius of the circle in which it lies, while the arc has only one sine.

[1]The square-root from the tenth part of the square [of a circle] whose circumference is 360 [degrees]
is the diameter.
In this [circle], by one establishing four parts [quadrants],
the Sine of an eighth part of a zodiacal sign [is to be determined].
[2]The square of the radius is called the *dhruva*.
A fourth part of this is [the square of the Sine] of Aries [i.e., of 30°].
The *dhruva*-square is diminished by [the square of the Sine] of Aries;
the square-root is the Sine for two zodiacal signs [i.e., of 60°].

Task 1 This passage is likely to look nothing like mathematical texts the reader has encountered before. Recognize that this an English translation of Sanskrit verse, so the meter and rhythm of the language are not longer evident to the reader. Identify at least three features of the passage you find confusing or which catch your interest (more than three is just as good). Identify your items as either "confusing" or "interesting."

The opening sentence of verse 1 of the text above presents a geometric identity: where the "[circle] whose circumference is 360 [degrees]" refers to the full circumference C of the circle, and D stands for its diameter, Varāhamihira stated here in verse that

$$\sqrt{\frac{C^2}{10}} = D.$$

Task 2 Use the standard formula you learned in school for the circumference of a circle, and substitute it for C in Varāhamihira's "formula" above. This will allow you to eliminate D from the formula as well (realizing that the radius of a circle is half its diameter). Solve the resulting equation for the fundamental constant π. This "formula" produces a reasonable approximation for π that was known to Hindu geometers. How many decimal places of accuracy does this approximation give?

The zodiacal signs mentioned in the text above referenced a system for organizing and describing the heavens that was developed by Babylonia astronomers by the eighth century BCE.[3] This system was adopted by Greek astronomers, who in turn passed it along for use by Indian practicioners. Based on their extensive observational records, Babylonian astronomers recognized that the Sun, Moon, Mercury, Venus, Mars, Jupiter, and Saturn—all the *planets*[4] visible to the naked eye—moved within a narrow band around the sky; the constellations found in this band were collectively called the *zodiac*.[5] They also knew that the Sun moves along a circular path against the background stars, called the *ecliptic circle*, about one 360th of its full annual course every day. In other words, it takes somewhat more than twelve lunar months for the Sun to make its yearly trek around the heavens along the ecliptic circle and return to the same place relative to the background stars. This was the mechanism whereby the Babylonians used the sky as a calendar. Since their year began with the spring equinox, the Sun would pass through the first of these constellations, Aries, during the first month of the year.[6] During the second month, the Sun would be in Taurus the Bull , then Gemini the Twins, etc.[7] By the end of the following winter, the Sun would complete its journey through the twelfth zodiacal band, for Pisces the Fish, returning to its starting point for the next year's course.

[3]See the project "Babylonian Astronomy and Sexagesimal Notation" at https://blogs.ursinus.edu/triumphs/ for more about this.

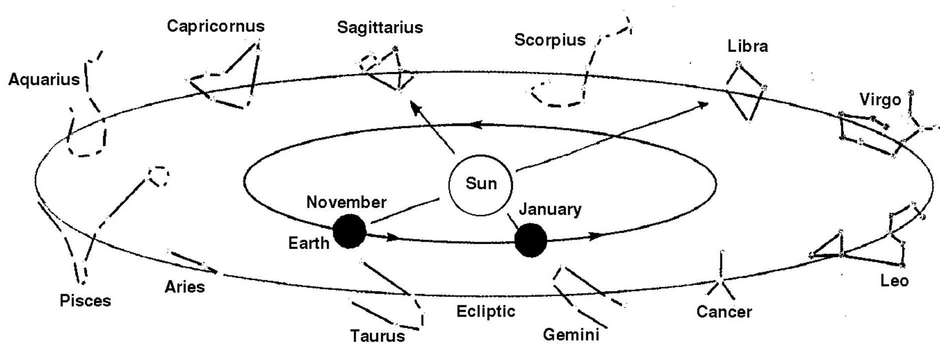
[4]The word *planet* comes from the Greek *planetos* = English *wanderer*. Since both the Sun and the Moon are seen to wander through the sky, both were considered to be planets for many centuries. In contrast, the apparently stationary Earth (like the fixed stars) was not considered a planet.

[5]From the Greek *zodiakos* = English *little animals*. The names of the zodiacal signs, and of the other constellations in the sky, were often those of animals suggested by the shapes that came to mind when viewing those collections of stars.

[6]Today, because the axis of the earth's rotation performs a slow gyroscopic rotation relative to the plane of its orbit around the Sun (a motion called the *precession of the equinoxes* by astronomers), the spring equinox no longer finds the Sun in the constellation Aries, but rather in Pisces. In fact, this slow movement will soon push the Sun at equinox into the next zodiacal constellation, Aquarius. That is, we are inexorably moving into the Age of Aquarius, as announced in the famous 1967 Broadway musical *Hair!*.

[7]The zodiacal constellations, in the order through which the Sun passes through them from the beginning of spring, through the summer, fall and winter months, are: Aries (the Ram), Taurus (the Bull), Gemini (the Twins), Cancer (the Crab), Leo (the Lion), Virgo (the Maiden), Libra (the Scales), Scorpio (the Scorpion), Sagittarius (the Archer), Capricorn (the Sea-Goat), Aquarius (the Waterbearer) and Pisces (the Fish). The reader will recognize that these are also the astrological signs to which modern-day horoscopes are associated. One's horoscope is cast depending on where the Sun was found in the sky on the day one was born.

(The diagram below sketches how this works, but it is drawn from the modern perspective, showing the Earth orbiting the Sun.)



Task 3 Use the fact that the twelve signs of the zodiac partitioned the ecliptic circle into 12 equal arcs of 30° each to determine how much "an eighth part of a zodiacal sign" is. This will be the first arc whose Sine is tabulated in Varāhamihira's table.

The traditional layout of the signs of the zodiac lists the constellations that line up from east to west following the path of the Sun across the background stars during the course of a year along the ecliptic circle, starting in the constellation Aries. During the first millennium BCE, the position of the Sun at the spring equinox was in the constellation Aries; this was the point along the ecliptic set as the 0° mark. Following the yearly path of the Sun from east to west, 30° later it passes into the constellation Taurus; at the 60° mark, it comes into Gemini, then Cancer at 90°, and on through the other twelve constellations, until it finally moves into Pisces, at 330° around the ecliptic. This explains why, in the text, when Varāhamihira referred to Aries, he was actually talking about an arc of 30°, and later, when he referred to Taurus, he meant an arc of 60°. The constellation names were stand-ins for the arc measures they represented along the ecliptic.

Task 4 Explain what Varāhamihira meant in the first line of verse 2 when he said that "a fourth part of [the *dhruva*] is [the square of the Sine] of Aries." What does this say about the value of the Sine of 30°? Use this same result to justify the claim made in the second line of verse 2; what does it say the Sine of 60° is equal to?

The next three verses of the text present rules for determining other Sine values besides those already considered.

[3]When the remaining [Sines] are desired, the radius is diminished by the Sine of the remainder
of the subtraction of twice the arc from a quadrant;
the square of half of that [remainder] is to be added to the square
of half [the Sine] of double [the arc].
[4]The square-root of that is the desired Sine.
The *dhruva* diminished by that [square is the square] of the remainder [of the] sum.
Half of the *dhruva*-square is called *adhyardha*, [the square of the Sine of] one and a half [signs, or 45°].
Here another rule is described.
[5]The Sine of the arc of three [signs] is diminished by Sine of three signs diminished by twice the given degrees; [the remainder] multiplied by sixty is the square [of the Sine of the given arc].
The *dhruva* diminished by that square is the square of the remainder [i.e., of the Cosine].

In verses 3–4, Varāhamihira stated a rule equivalent to the following formula, in which we recognize that by "quadrant" he meant an arc of 90° :

$$\sqrt{\left(\frac{R - \mathrm{Sin}(90^\circ - 2\theta)}{2}\right)^2 + \left(\frac{\mathrm{Sin}\, 2\theta}{2}\right)^2} = \mathrm{Sin}\,\theta.$$

The reader is urged to confirm that the text is faithfully represented in this rather complicated-looking formula. We will not attempt to justify the result in detail, but we do wish to note that this result is one way to propose what will later be called a *half angle formula*, since it allows the computation of the Sine of a given angle, $\mathrm{Sin}\,\theta$, given knowledge of the Sine of its double, $\mathrm{Sin}\,2\theta$.

In particular, the expression $\mathrm{Sin}(90^\circ - 2\theta)$ which appears in the first term under the radical in the above formula, is the Sine of the *complementary arc* to 2θ (arcs being complementary when they sum to a right angle); this Sine of the complement can be found from the Sine of the given angle easily using the Pythagorean Theorem. The Sine of the complementary arc frequently appears in the same calculations as the Sine of the original angle (as both appear as sides in the same right triangle), and the term *Sine-complement* was often used to refer to it. Later, it became known as the *Cosine*.

Task 5 Varāhamihira's next assertion in verse 4 was that "The *dhruva* diminished by that [square is the square] of the remainder [of the] sum." Explain why this is yet another version of the Pythagorean Theorem, which in symbolic form, can be expressed as

$$R^2 - (\mathrm{Sin}\,\theta)^2 = (\mathrm{Cos}\,\theta)^2.$$

In the case where the radius of the circle has unit length ($R = 1$), the *cosine*, $\cos\phi$, of the angle ϕ is the sine of the complementary angle to ϕ, that is, the other acute angle in a right triangle of hypotenuse with unit length. By definition then,

$$\cos\phi = \sin(90^\circ - \phi).$$

Moreover, the rule from Task 5 then takes the very simple form

$$\sin^2\theta + \cos^2\theta = 1.$$

This algebraic identity is often called the *Trigonometric Pythagorean Theorem.*

Task 6 Continue the thread of Task 5 by considering what happens when we apply the result quoted there to the arc equal to "one and a half" zodiacal signs. First, explain why the Sine and Cosine of this angle/arc are the same value, then apply the rule to show that

$$\text{Sin}\, 45^\circ = \sqrt{\frac{R^2}{2}}.$$

3 Populating Varāhamihira's Table

Finally, in verses 6–9, Varāhamihira began the recitation of the entries of his table of Sine values. He fixed the radius of his circle at $R = 120$, or two units of sixty minutes each, as will become clear.

6 The Sines in Aries are 7, 15, 20 plus 3, plus 11, and plus 18, 45, 50 plus 3, and 60 minutes;

7 in Aries 50 plus 1, 5 times 8, 5 squared, 4, 30 plus 4, 56, 5, and 0 [seconds].

8 In Taurus [they are] 6, 13, 19, 3 times 8, and 30 – plus 0, plus 5, plus 9, and plus 13 minutes;

9 in Taurus 40, 3, 7, 50 plus 1, 13, 12, and 60 minus 14 and minus 5 seconds.

Recall from verse 1 (and Task 3) that Varāhamihira wanted to build a table of Sine values for arcs which are multiples of $3^\circ 45'$ of arc. So when he began by reciting "the Sines in Aries," he meant the Sines of the arcs which are the first few multiples of this smallest reference angle.

Task 7 By the end of this task, you will have extracted Varāhamihira's full Table of Sines from the above text. (A more complete table can be constructed by continuing this analysis in the optional section at the end of this project.)

(a) List the multiples of $3^\circ 45'$ up to 30° and enter them into the first column of a table.

(b) The second column should list the Sines of the eight corresponding Arcs. In verse 6, Varāhamihira first listed only the parts, in units of minutes, for these values; in verse 7, he listed their smaller parts, in units of seconds. Thus, the first entry gave the Sine of $3°45'$ as 7 minutes, $50+1 = 51$ seconds (or $7'51''$), and the second entry gives $\text{Sin}\,7°30' = 15'40''$. Readers will notice that some of these numbers were identified with rather unusual language: in verse 6, "20 plus 3, plus 11" identifies the two values 23 and 31; and in verse 7, "5 times 8, 5 squared" refer to the two values 40 and 25. Varāhamihira used this odd language so as to fit the poetic meter of the Sanskrit text, a sense that is entirely lost in our English translation! Now, with this in mind, fill in the rest of this first part of your table with the Sine values reported in verses 6-7.

(c) In verses 8–9, Varāhamihira moved on to the next eight entries, filling out the Arcs corresponding to the sign of Taurus. Enter all these Arc and Sine values into your table. All the Sine values are to be added to the 60 minutes which is the Sine of 30°. Thus, $\text{Sin}\,33°45' = 66'40''$. (Warning: the last four entries are spelled out in the verse as adding values to $30'$, using a phrase of the form "plus x".)

(d) Check that the values in Varāhamihira's table are correct as follows: his Sine values correspond to line segments drawn relative to a circle of radius $R = 2$ units $= 120'$. Modern sine values (lowercase s) correspond to choosing a radius of $R = 1$ unit. So add a third column to your table. Place here the (lowercase s) sines of these same Arcs: to determine these, convert the sexagesimal Sine values in minutes/seconds which appear in the second column into sine values (in minutes/seconds) by cutting them in half so that they correspond to a circle with a one unit radius.

(e) Finally, prepare a decimal version of Varāhamihira's table which we can use to check his values against values computed with a calculator. Begin a new table by translating the 16 arcs/angles from degree/minute form into decimal degrees to fill a first column of Angles. Then, in the next column of this table, convert the Sines we found from Varāhamihira's values in part (d) into decimal quantities as well. Finally, in a third column, use a scientific calculator to compute the sines (lowercase s!) of the Angles in the first column to three-place accuracy. How well do the values in the second and third columns agree?

(f) In a coordinate plane, plot the points given by the first two columns of this last table: the plot is a graph of Varāhamihira's sine function with domain including arcs/angles up to 60°.

Task 8 In this task, we will explore what can be accomplished using Varāhamihira's half-angle formula, from verses 3–4 above.

(a) In Task 7(c) above, you tabulated Varāhamihira's Sines for Arcs which are the multiples of $3°45'$, up to $60°$ in a circle of radius $R = 2$ units, or $120'$ (sixtieths of a unit). Recall from the table the value of $\text{Sin}\,30°$, then use the half-angle formula,

$$\text{Sin}\,\theta = \sqrt{\left(\frac{R - \text{Sin}(90° - 2\theta)}{2}\right)^2 + \left(\frac{\text{Sin}\,2\theta}{2}\right)^2},$$

to compute the Sine for an arc of 15°; note that you also need the Sine-complement of arc $\theta = 30°$, i.e., $\text{Cos}\,30°$, for this calculation, a value that already appears in the table! Do you get the same value for $\text{Sin}\,15°$ as you found in Task 22(c)?

(b) Next, use the Pythagorean Theorem (see Task 5) to compute the Cosine for arc 15° and incorporate this computation into one for the Sine of arc $7°30'$ using the half-angle formula; compare also with the value you found in Task 7(c).

(c) Repeat the procedure of part (b) to determine the Sine of the arc $3°45'$, then compare your computation with the value as you found in Task 7(c).

4 Conclusion

The brief nine lines of Varāhamihira's Sanskrit verse which we have examined here were packed with information, providing any who understood them a wealth of geometric and computational information about building a Table of Sines. It also provides a nice illustration of how Hindu science preserved and transmitted information from one generation to the next, wherein poetic verse becomes a powerful data compression tool. But it also marks a key innovation in the evolution of trigonometry from its earlier Greek version: we recognize the shift from interest in tabulating chord lengths for arcs (or central angles) of a circle to instead tabulating Sines, the half-chords which are segments dropped perpendicularly from one end of an arc of a circle to the radius at the other end of the arc.

In our next episode, we will roll the calendar forward another 500 years to an era in which the dominant scientific global culture is Islam. In the work of the eleventh century scholar al-Bīrūnī, who was also an expert in Hindu astronomy and mathematics, we will mark the emergence of other trigonometric quantities besides Sines which become important to astronomers to identify and measure.

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5 Appendix: More of Varāhamihira's Table

Students interested in building a more complete Table of Sines based on Varāhamihira's text are invited to consider the next two verses of the *Pañcasiddhāntikā* and the subsequent extension of Task 7 from above.

[10]The minutes of the Sines for the intervals [in Gemini] are: 3, 6, 9, 12,
13, 3 times 5 twice, and 16;
this is added to [the Sine for] the second sign [Taurus]. Then the seconds:
[11]60 minus 18, minus 3, and minus 18, 0, 50 minus 3, 4, 50 minus 1, and 5.

Task 9 In this task, we will extend the Table of Sines constructed in Task 7 above from 16 to 24 entries. This allows for a full tabulation of the Sines of arcs up to 90°.

(a) As in verses 8–9, Varāhamihira sets out in verses 10–11 the next eight entries of the table. Continuing your work from Task 7(c), fill into the table the next eight Arc measures following 60°, which was the sixteenth of the Arcs. These eight new Arcs are those in the sign of Gemini (from 60° to 90°).

(b) Consult verses 10–11 of the source text to determine the Sines of these eight Arcs; as with the Sines in Taurus tabulated earlier, the numbers from the text are to be "added to [the Sine for] the second sign [Taurus]," that is, to $\mathrm{Sin}\,60° = 103'55''$. In other words, add the minute values in verse 10 to $103'$ and the seconds values in verse 11 to $55''$ to obtain the correct Sines for the table.

(c) Continue the work from Task 7(d) to convert the Sines of these angles into their corresponding sine values.

(d) Continue the work from Task 7(e) by transferring the eight sexagesimal Arc and sine values into a new table of three-place decimal values, then comparing the sines taken from Varāhamihira's verses to those produced by a scientific calculator. Do you find close agreement of the sine computations?

(e) Extend the plot of the points from Task 7(f) with these eight new values. Superimpose on the plot of points the graph of the (modern) sine function? Do the points appear to lie on the curve?

Notes to Instructors

This project is the fourth in a collection of six curricular units drawn from a Primary Source Project (PSP) titled *A Genetic Context for Understanding the Trigonometric Functions*. The full project is designed to serve students as an introduction to the study of trigonometry by providing a context for the basic ideas contained in the subject and hinting at its long history and ancient pedigree among the mathematical sciences. Each of the individual units in that PSP looks at one of the following specific aspects of the development of the mathematical science of trigonometry:

- the emergence of sexagesimal numeration in ancient Babylonian culture, developed in the service of a nascent science of astronomy;
- a modern reconstruction (as laid out in [Van Brummelen, 2009]) of a lost table of chords known to have been compiled by the Greek mathematician-astronomer Hipparchus of Rhodes (second century, BCE);
- a brief selection from Claudius Ptolemy's *Almagest* (second century, CE) [Toomer, 1998], in which the author (Ptolemy) showed how a table of chords can be used to monitor the motion of the Sun in the daytime sky for the purpose of telling the time of day;
- a few lines of Vedic verse by the Hindu scholar Varāhamihira (sixth century, CE) [Neugebauer and Pingree, 1970/1972], containing the "recipe" for a table of sines as well as some of the methods used for its construction;
- passages from *The Exhaustive Treatise on Shadows* [Kennedy, 1976], written in Arabic in the year 1021 by Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī, which include precursors to the modern trigonometric tangent, cotangent, secant, and cosecant;
- excerpts from Regiomontanus' *On Triangles* (1464) [Hughes, 1967], the first systematic work on trigonometry published in the West.

This collection of units is not meant to substitute for a full course in trigonometry, as many standard topics are not treated here. Rather, it is the author's intent to show students that trigonometry is a subject worthy of study by virtue of the compelling importance of the problems it was invented to address in basic astronomy in the ancient world. Each unit may be incorporated, either individually or in various combinations, into a standard course in College Algebra with Trigonometry, a stand-alone Trigonometry course, or a Precalculus course. These lessons have also been used in courses on the history of mathematics and as part of a capstone experience for pre-service secondary mathematics teachers.

In this unit, students will examine nine lines of Sanskrit Vedic verse which provide in compressed form 16 lines of a Table of Sines as well as statements of a handful of trigonometric theorems! Because the context of the source text is astronomical, students also receive a brief introduction to how circular arcs, and equivalently, the central angles the span, are measured, as well as a treatment of some basic astronomy having to do with the motion of the Sun through the zodiacal constellations. The highpoint of this PSP is to help readers understand how the text describes a Table of Sines for arcs up to 60° in multiples of $3^\circ 45'$. In the process, there is a discussion of the trigonometric innovation by Hindu astronomers which replaced the chord with the Sine as the quantity of chief interest.

Student Prerequisites

Very little mathematical machinery is needed to work on this PSP. But it would help for students to have been exposed to sexagesimal numeration (to handle the arithmetic of angle measures in degrees and minutes) and to a reasonably good high school geometry course, especially with regard to the geometry of the circle and the measure of arcs and angles. A simple introduction to sexagesimal numeration is provided by the first mini-PSP in this series, "Babylonian Astronomy and Sexagesimal Numeration", and a presentation of the geometry of the circle and the measure of arcs and angles that serve as the foundation of trigonometry will be obtained by working the second mini-PSP in this series, "Hipparchus' Table of Chords". A familiarity with some basic astronomy can also be helpful, and the third mini-PSP in this series, "Ptolemy Finds High Noon in Chords of Circles" addresses this.

Suggestions for Classroom Implementation

This project can be implemented in two 50-minute class periods. (There is rather too much to do to expect that it can also be accomplished in a single 75-minute period, so instructors who teach twice a week will need to adapt what is described here to a two-75-minute-period schedule.) For a detailed plan for how to structure the periods, see the next subsection below. Students are not likely to come with much knowledge of observational astronomy and the variation in the positions of the Sun over the course of a year. These are described in this project, but the careful instructor may consider enlisting the help of colleagues who teach introductory astronomy for resources and classroom models that could assist students in making sense of the geometry of the Sun-Earth system. For instance, the author made use of a model of the celestial sphere (a transparent globe with constellations, the ecliptic circle, and celestial coordinates printed on its surface) borrowed from the Physics department to show the geometry of these astronomical elements for Earth-based observers.

Students have likely never encountered writing similar to this source text, so they will probably find making sense of it quite difficult; Task 1 gives them an opportunity to express this surprise/frustration. Allow classroom discussion about this to set the stage for the rest of the work they will do in this project.

The high point of the project comes with Task 7, when students build a partial table of sines (for angles up to $60°$ in multiples of $3°45'$) based on the numerical information given them by Varāhamihira's verses; they will then compare these entries with what a scientific calculator tells them.

Sample Implementation Schedule (based on two 50-minute class periods)

In preparation for the first class period, students should be asked to read through Section 2 of the project and write up their responses for the first two Tasks. Inform students that it is not expected that everything they read will make sense to them immediately, and that the classroom discussion should help to clear up many of these issues.

Begin the first class with a quick review of the geometric diagrams on pages 2 and 3 to ensure that they understand the notions of arc, chord and sine. Then begin a discussion of Task 1 to get a sense of what they have gleaned about the first four verses in the text from their reading, first in small groups and then as a whole class. The focused attention of the students on certain portions of the project, like this bit of the source text, can be facilitated by having someone read these sections aloud, with generous pauses and time for questions of clarification.

After devoting a few minutes to their answers for Task 2, see if the students can make sense of the diagram of the zodiac on p. 5 by asking questions like: "In what constellation is the Sun in November as seen by an observer on Earth? in January? in other months of the year? What constellations are visible in the night sky in these same months?" This should be done to emphasize that the context motivating the Hindu astronomer-priests who recited these texts 1500 years ago was the use of the position of the Sun along the ecliptic circle as a functioning calendar.

Task 3 should be straightforward to answer; Task 4 will be more challenging. The remainder of the period can be given over to a reading of verses 3–5 and a processing of what these verses say. Guide the students to recognize that verses 3–4 give a representation of the half-angle formula (see page 6). Then have them begin work on Tasks 5 and 6 with what time remains. For homework have them write up formal solutions for Tasks 4, 5 and 6.

In the second class period, after airing any issues that may have arisen in their homework, devote the bulk of the hour to a careful reading of verses 6–9 of the source text and then work in small groups on Task 7.The (optional) final section of the project provides in verses 10–11 of the source text the information needed to extend the Table of Sines from to arcs of measure 90° (in Task 9). Assign completion of Task 8, and formal write-ups of these last two Tasks for homework.

LaTeX code of this entire PSP is available from the author by request to facilitate implementation. The PSP itself may also be modified by instructors as desired to better suit the goals for their course.

Recommendations for Further Reading

Instructors who want to learn more about the history of trigonometry are recommended to consult Glen Van Brummelen's masterful *The Mathematics of the Heavens and the Earth: The early history of trigonometry* [Van Brummelen, 2009], from which much of this work took inspiration.

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