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Fall 2017

### Generating Pythagorean Triples: A Gnomonic Exploration

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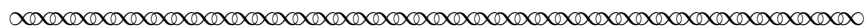
# Generating Pythagorean Triples: A Gnomonic Exploration

Janet Heine Barnett\*

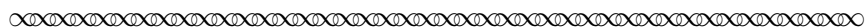
November 27, 2017

You no doubt remember hearing about the so-called ‘Pythagorean Theorem’ — something about right triangles and squares of their sides, right? But you may not know that this theorem was actually known to mathematicians in various cultures long before the Greek mathematician Pythagoras (569–500 BCE) was born! For example, the 4000-year old clay tablet known as ‘Plimpton 322’ (c. 1900 BCE) tells us that Babylonian mathematicians both knew the theorem *and* had a method for generating triples of numbers  $(a, b, c)$  that satisfy the equation  $a^2 + b^2 = c^2$ , subject to the additional condition that all three of the numbers  $a$ ,  $b$  and  $c$  must be positive *whole* numbers. Perhaps not surprisingly, such integer triples are today called ‘Pythagorean triples.’

In this project, we will study two methods for generating Pythagorean triples that were known to ancient Greek mathematicians and are still used in number theory today. Our historical source for these methods will be a 1970 English translation of the text *Commentary on Euclid’s ‘Elements’* [Proclus, 1970]. Written by the Greek mathematician Proclus (c. 411 – 485 CE) approximately 700 years after Euclid (c. 300 BCE), the *Commentary* is also the principal source of our knowledge about the early history of Greek mathematics.<sup>1</sup> The excerpts that we will read from this text are taken specifically from Proclus’ commentary on Euclid’s proof of one part of the Pythagorean Theorem.<sup>2</sup> Here is how Proclus stated Euclid’s version of this result [Proclus, 1970, p. 337]:<sup>3</sup>



XLVII: In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.



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<sup>1</sup>In fact, very few mathematical texts survive from the period before Euclid’s *Elements*, a testimony in part to the greatness of Euclid’s work. The lack of original source material about early Greek mathematics also lends itself to the mystery that surrounds Pythagoras and his followers, about whom little is known with any certainty.

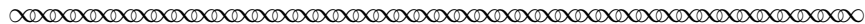
<sup>2</sup>Today, the Pythagorean Theorem is generally stated as a single ‘if and only if’ statement. In Euclid’s *Elements*, the two directions were stated in two separate theorems. The first of these appeared as Theorem XLVII in Book I of the *Elements*, the penultimate theorem of that book. The second, or converse, direction of the Pythagorean Theorem appeared as the final theorem of that same book [Proclus, 1970, p. 341]:

XLVIII: If in a triangle the square on one of the sides is equal to the squares on the remaining two sides, the angle contained by the remaining two sides of the triangle is right.

<sup>3</sup>To set them apart from the project narrative, all original source excerpts are set in **sans serif font** and bracketed by the following symbol at their beginning and end:

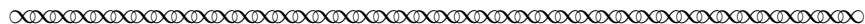
**Task 1** How does Proclus' statement of Theorem XLVII compare to the way that we state the Pythagorean Theorem today? In particular, what names do we use today for what Proclus called 'the side subtending the right angle' and 'the sides containing the right angle'?

Reading Proclus' commentary on this theorem further, we see that he was well aware that not all right triangles satisfy the Pythagorean triple requirement of whole number solutions [Proclus, 1970, pp. 339–340]:



There are two sorts of right-angled triangles, isosceles and scalene. In isosceles triangles you cannot find numbers that fit the sides; for there is no square number that is the double of a square number, if you ignore approximations, such as the square of seven which lacks one of being double the square of five. But in scalene triangles it is possible to find such numbers, and it has been clearly shown that the square on the side subtending the right angle may be equal to the squares on the sides containing it. Such is the triangle in the *Republic*,<sup>4</sup> in which sides of three and four contain the right angle and five subtends it, so that the square on five is equal to the squares on those sides. For this is twenty-five, and of those the square of three is nine and that of four sixteen. The statement, then, is clear for numbers.

Certain methods have been handed down for finding such triangles, one of them attributed to Plato, the other to Pythagoras. ....



**Task 2** This task examines the examples given by Proclus in the preceding excerpt.

- (a) As an illustration that isosceles right triangles never give us a Pythagorean triple, Proclus stated that 'the square of seven ... lacks one of being double the square of five.' Explain how this relates to an isosceles right triangle with legs of (equal) length  $a = b = 5$ . What is the exact length of the hypotenuse in this example?

*Bonus* Explain in general why an isosceles right triangle never gives a Pythagorean triple.

- (b) Proclus gave (3, 4, 5) as an example of a Pythagorean triple associated with a scalene right triangle. Give another such example.
- (c) Do all scalene right triangles give us a Pythagorean triple? Justify your answer.

In Sections 2 and 3 of this project, we will explore the methods for generating Pythagorean triples attributed by Proclus to Plato and to Pythagoras in the previous excerpt.<sup>5</sup> To prepare for those explorations, Section 1 first introduces some background on the Greek geometrical concept of a *gnomon*.<sup>6</sup>

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<sup>4</sup>The *Republic* is an important philosophical text written by the famous Greek philosopher Plato (c. 428 – c. 348 BCE).

<sup>5</sup>Although a major proponent of the study of mathematics, Plato himself was not a mathematician. It is thus unlikely that the Pythagorean triple method that Proclus attributed to Plato was actually developed by Plato; more likely, this method was developed by one of the many mathematicians who were associated with Plato's Academy. Although Pythagoras was considered a mathematician, it is also likely that some member of the Pythagorean community other than Pythagoras himself actually developed the Pythagorean triple method that Proclus attributed to Pythagoras.

<sup>6</sup>The initial *g* in the word *gnomon* is silent, as is the case in the words *gnat* and *gnarly*. Words that rhyme with *gnomon* include *omen* and *snowmen*.

# 1 The Casting of Shadows: Gnomons and Figurate Numbers

The literal meaning of the Greek word *gnomon*<sup>7</sup> is “that which allows one to know.” In its original sense, one of the things that gnomons allowed one to know was the time! In astronomy, a gnomon is the part of a sundial that casts a shadow. In the ancient world, a vertical stick or pillar often served as the gnomon on a sundial. The term *gnomon* was also associated with an L-shaped instrument, sometimes called a ‘set square,’ that was used in architecture for the construction of (or ‘knowing of’) right angles. Eventually, the geometric shape shown in Figure 1, formed by cutting a smaller square from a larger one, came to be associated with the word *gnomon* as well.

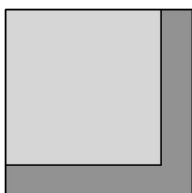


Figure 1

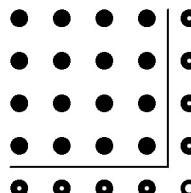


Figure 2

Figure 2 shows a different type of gnomonic diagram that is brought to mind by reading certain passages in the *Physics* where Aristotle<sup>8</sup> (c. 384 – c. 322 BCE) discussed the mathematical beliefs of Pythagoras and his followers. Historians of mathematics believe that diagrams like the one shown in Figure 2 developed from the Pythagoreans’ use of pebbles to represent various figurate numbers: not just square numbers, but also triangular numbers, pentagonal numbers, and so on.<sup>9</sup> Note that figurate number diagrams are especially well suited to a number theory perspective that allows the use of whole numbers only. For example, the diagram in Figure 2 can be viewed as a representation of the Pythagorean triple (3, 4, 5), since adding the gnomon 9 to the smaller square 16 gives the larger square 25 — in other words,  $3^2 + 4^2 = 5^2$ !

### Task 3

This task examines the connection between gnomons and Pythagorean triples.

- Figure 3 shows the square number  $12^2 = 144$ . Add a gnomon to that square. What Pythagorean triple  $(a, b, c)$  is represented by the completed diagram?
- Figure 4 shows the square number  $8^2 = 64$ . Does adding a gnomon to that square give us a Pythagorean triple  $(a, b, c)$ ? Explain why or why not.

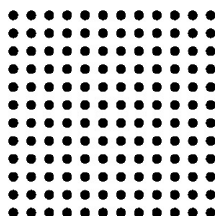


Figure 3

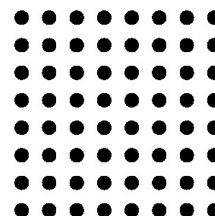


Figure 4

<sup>7</sup>Etymologically, the word *gnomon* shares roots with the English words *gnostic*, *agnostic* and *ignorance*.

<sup>8</sup>Like his teacher Plato, Aristotle was not himself a mathematician, but made important contributions to philosophy that influenced the study of mathematics, logic and physics for centuries to come.

<sup>9</sup>The primary source project *Construction of the Figurate Numbers* written by Jerry Lodder (New Mexico State University) explores properties of various figurate numbers through excerpts from the text *Introduction to Arithmetic* by the neo-Pythagorean mathematician Nicomachus of Gerasa (60 – 120 CE); that project is available at <http://webpages.ursinus.edu/nscoville/TRIUMPHS.html>.

## 2 Gnomons and the Method of Pythagoras

Before we go back to our reading of Proclus' *Commentary*, let's see what we can learn from gnomons about generating Pythagorean triples.

**Task 4** This task uses a figurate number diagram to develop a method for generating Pythagorean triples.

Consider the diagram shown in Figure 5.

Notice again that the larger square is obtained by adding a gnomon to the smaller square.

Let  $b$  denote the length of the side of the smaller square.

Let  $c$  denote the length of the side of the larger square, so that  $c = b + 1$ .

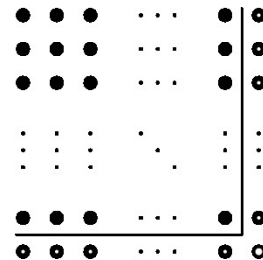


Figure 5

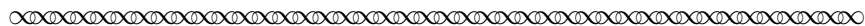
Complete the following to determine what the diagram tells us about a Pythagorean triple  $(a, b, c) = (a, b, b + 1)$  in this case.

- Use the fact that  $c = b + 1$  to write a formula for  $c^2$  in terms of  $b$ . Explain your work in terms of the geometry of the diagram. (If you use any algebra for this, your explanation should also relate that algebra back to the geometry of the diagram.)
- Now denote the total number of dots in the gnomon by  $n$ . Use the diagram to write an equation for  $n$  in terms of  $b$ .
- Explain why  $n$  must be a square number in order to get the desired Pythagorean triple.
- Based on part (c), we know there is some number  $a$  such that  $n = a^2$ . Substitute this in your equation from part (b). Then solve the result to get a formula for  $b$  in terms of  $a$ . How does the algebra that you did here relate to the geometry of the diagram?
- Are there any restrictions needed on the values of  $a$  or  $b$  in order for this method to work? Explain your answer in terms of the geometry of the diagram.

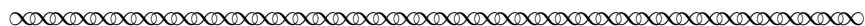
**Task 5** This task looks at specific numerical examples of the method for finding Pythagorean triples  $(a, b, c)$  that you developed in Task 4.

- Follow the procedure you developed in Task 4(d) beginning with the value  $a = 7$ . After computing the values of  $b$  and  $c$ , verify directly that  $a^2 + b^2 = c^2$ .
- Repeat part (a) with the starting value of  $a = 11$ .
- Does your method work the starting value of  $a = 16$ ? Explain why or why not.

Let's now go back to our reading of Proclus' *Commentary*, to see what he had to say about the method for generating Pythagorean triples that is attributed to Pythagoras [Proclus, 1970, p. 340]:



The method of Pythagoras begins with odd numbers, positing a given odd number as being the lesser of the two sides containing the angle, taking its square, subtracting one from it, and positing half of the remainder as the greater of the sides about the right angle; then adding one to this, it gets the remaining side, the one subtending the angle. For example, it takes three, squares it, subtracts one from nine, takes the half of eight, namely, four, then adds one to this and gets five; and thus is found the right-angled triangle with sides of three, four, and five.



Notice that, even though Proclus referred to '3' and '4' in his example as 'the sides containing the angle' and to '5' as 'the side subtending the angle,' there are absolutely no triangles involved in his computation of the Pythagorean triple (3, 4, 5)! Proclus simply began with the odd number  $a = 3$ , then computed  $b = 4$  and  $c = 5$  by following a step-by-step arithmetical procedure. Let's see how this method for obtaining a Pythagorean triple relates to the formulas you found in Task 4.

**Task 6**

This task translates the Method of Pythagoras described by Proclus into algebraic symbolism.

Assume that  $a$  is an odd number and that  $(a, b, c)$  is a Pythagorean triple obtained by the Method of Pythagoras.

- (a) Follow the procedure indicated by Proclus to write a formula for  $b$  in terms of  $a$ .
- (b) Next follow the procedure indicated by Proclus to write a formula for  $c$  in terms of  $b$ .
- (c) Now use algebra to verify that the formulas you found in parts (a) and (b) necessarily satisfy the relationship  $a^2 + b^2 = c^2$ . *Hint: First solve the formula in part (a) for  $a^2$ . Alternatively, you can use parts (a) and (b) to write  $c$  in terms of  $a$ .*
- (d) Where was the assumption that  $a$  is odd actually used in these formulas — or was it?
- (e) How does the Method of Pythagoras described by Proclus compare to the formulas you found in Task 4?

**Task 7**

Why did Proclus specify that the odd number with which we begin the procedure will be 'the lesser of the two sides containing the angle'? Was this necessary?

### 3 The Platonic Method and Double Gnomons

In Section 2, we considered a method for generating a Pythagorean triple starting from any odd whole number. In the next task, we will develop a similar method that instead begins with an even whole number.

**Task 8** This task uses a double gnomon figurate number diagram to develop a second method for generating Pythagorean triples.

Consider the diagram shown in Figure 6.

Notice here that the larger square is obtained by adding a *double* gnomon to the smaller square.

Let  $b$  denote the length of the side of the smaller square.

Also let  $c$  denote the length of the side of the larger square, so that  $c = b + 2$ .

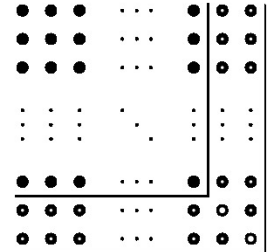


Figure 6

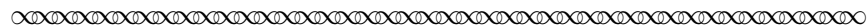
Complete the following to determine what the diagram tells us about a Pythagorean triple  $(a, b, c) = (a, b, b + 2)$  in this case.

- Use the fact that  $c = b + 2$  to write a formula for  $c^2$  in terms of  $b$ . Explain your work in terms of the geometry of the diagram. (If you use any algebra for this, your explanation should also relate that algebra back to the geometry of the diagram.)
- Now denote the total number of dots in the *double* gnomon by  $n$ . Use the diagram to write an equation for  $n$  in terms of  $b$ .
- Explain why  $n$  must be a square number in order to get the desired Pythagorean triple.  
*Bonus:* Explain also why  $b + 1$  must be a square number as well.
- Based on part (c), we know there is some number  $a$  such that  $n = a^2$ . Substitute this in your equation from part (b). Then solve the result to get a formula for  $b$  in terms of  $a$ . How does the algebra that you did here relate to the geometry of the diagram?
- Recalling that  $c = b + 2$ , also write a formula for  $c$  in terms of  $a$ . Explain your work in terms of the geometry of the diagram. (If you use any algebra for this, your explanation should also relate that algebra back to the geometry of the diagram.)
- Explain why  $a$  must be an even number. (If you use any algebra for this, your explanation should also relate that algebra back to the geometry of the diagram.)

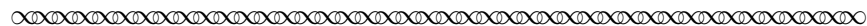
**Task 9** This task looks at specific numerical examples of the method for finding Pythagorean triples  $(a, b, c)$  that you developed in Task 8.

- Follow the procedure you developed in Task 8 beginning with the value  $a = 8$ . After computing the values of  $b$  and  $c$ , verify directly that  $a^2 + b^2 = c^2$ .
- Repeat part (a) with the starting value of  $a = 12$ .

Let's go back now to our reading of Proclus, and his description of a second method for generating Pythagorean triples [Proclus, 1970, p. 340]:



The Platonic method proceeds from even numbers. It takes a given even number as one of the sides about the right angle, divides it in two and squares the half, then by adding one to the square gets the subtending side, and by subtracting one from the square gets the other side about the right angle. For example, it takes four, halves it and squares the half, namely, two, getting four; then subtracting one it gets three and adding one gets five, and thus it has constructed the same triangle that was reached by the other method. For the square of this number is equal to the square of three and the square of four taken together.



Notice that Proclus' example is again the famous (3, 4, 5) Pythagorean triple, but this time starting with the even-valued side length  $a = 4$ . Let's look at how this method for obtaining a Pythagorean triple  $(a, b, c)$  compares in general to the one that you developed in Task 8.

**Task 10** This task translates Proclus' description of the Platonic Method into algebraic symbolism.

Assume that  $a$  is an even number.

Also assume that  $(a, b, c)$  is a Pythagorean triple obtained by the Method of Plato.

- Follow the procedure indicated by Proclus to give a formula for  $b$  in terms of  $a$ .
- Next follow the procedure indicated by Proclus to give a formula for  $c$  in terms of  $a$ .
- Now use algebra to verify that the formulas you found in parts (a) and (b) necessarily satisfy the relationship  $a^2 + b^2 = c^2$ . *Hint: First write  $a = 2k$  for some number  $k$ .*
- Where was the assumption that  $a$  is even actually used in these formulas — or was it?
- How does the Method of Pythagoras described by Proclus compare to the formulas you found in Task 8?

## 4 Comparisons and Conjectures

Now that we have met both methods for generating Pythagorean triples, let's put them to work to see what else we can learn about these methods in particular, and Pythagorean triples more generally.

**Task 11** This task compares the two methods described by Proclus for generating Pythagorean triples.

We know from Proclus' examples that both these methods produce the Pythagorean triple (3, 4, 5). Will these two methods eventually produce the same list of Pythagorean triples? What other similarities or differences are there between these methods and the Pythagorean triples that they produce? Give some form of mathematical evidence or explanation in support of your answer.



**Task 12** This task explores other properties of Pythagorean triples, starting from a list of known triples.

The following table gives a list of the Pythagorean triples that you found in earlier project tasks, using one of the two generation methods described by Proclus.

$a$	$b$	$c$
3	4	5
5	12	13
7	24	25
8	15	17
11	60	61
12	35	37

Extend this list by using each of the two methods to compute at least three new Pythagorean triples; your extended list should contain at least 12 different triples.

Use the data in your table to write three observations/conjectures about Pythagorean triples.

*Bonus* Prove your conjectures are true — or find a counterexample for any that are false!

## 5 Conclusion

Although Proclus described the two methods for generating Pythagorean triples studied in this project completely in words, we can also represent these methods using algebraic symbols as follows:<sup>10</sup>

- Given  $a$  odd,  $\left(a, \frac{a^2 - 1}{2}, \frac{a^2 - 1}{2} + 1\right)$ , or simplifying the third term,  $\left(a, \frac{a^2 - 1}{2}, \frac{a^2 + 1}{2}\right)$ .
- Given  $a$  even,  $\left(a, \left(\frac{a}{2}\right)^2 - 1, \left(\frac{a}{2}\right)^2 + 1\right)$ , or setting  $a = 2k$  and simplifying,  $(2k, k^2 - 1, k^2 + 1)$ .

You may be wondering if these two formulas give every possible Pythagorean triple — or, if not, whether there is some other formula, or short list of formulas, that does. This is the kind of question that the Greeks asked as well, and the answer to at least one of them can be found in Euclid's *Elements*, as well as in most current number theory textbooks. And now that you've met some of the fascinating shadows cast by gnomons in the study of numbers, you're well equipped to generate a Pythagorean triples formula or two of your own!

## References

Proclus. *A Commentary on the First Book of Euclid's Elements*. Princeton University Press, Princeton, NJ, 1970. English translation by Glenn R. Morrow.

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<sup>10</sup>Make sure you agree that these are correct representations!

## Notes to Instructors

### Overview: Topics, Goals, Task Commentary and PSP Design

This Primary Source Project (PSP) is designed to provide students an opportunity to explore the number-theoretic concept of a Pythagorean triple, with a focus on developing an understanding of two now-standard formulas for such triples and how to develop/prove those formulas via figurate number diagrams involving gnomons.

Two versions of the project are available, both of which begin with some basic historical and mathematical background (Introduction and Section 1). Both versions also include two open-ended “comparisons and conjectures” tasks (in Section 4) that could be omitted (or expanded upon) depending on the instructor’s goals for the course, as well as a brief conclusion (Section 5) that presents modern algebraic versions of the two formulas. The student tasks included in the two intermediary sections of the project are essentially the same in the two versions as well, but differently ordered in a fashion that renders one version somewhat more open-ended than the other. In light of their similarities (and differences), the two versions share the same title, “Generating Pythagorean Triples,” but can be distinguished by their respective subtitles.

- The less open-ended version is subtitled “The Methods of Pythagoras and of Plato via Gnomons.” In this version, students begin by completing tasks based on Proclus’ verbal descriptions of the two methods, and are presented with the task of connecting the method in question to gnomons in a figurate number diagram only after assimilating its verbal formulation. This version of the project may be more suitable for use in lower division mathematics courses for non-majors or prospective elementary teachers.<sup>11</sup>
- The more open-ended version of this mini-PSP is subtitled “A Gnomonic Exploration.” **This is the version you are currently reading.** In this version, students begin with the task of using gnomons in a figurate number diagram to first come up with procedures for generating new Pythagorean triples themselves, and are presented with Proclus’ verbal description of each method only after completing the associated exploratory tasks. This version of the project may be more suitable for use in upper division courses on number theory or capstone courses for prospective secondary teachers.

### Student Prerequisites

Beyond basic arithmetic and (high school level) algebraic skills, no mathematical content pre-requisites are required in either version, although more advanced students will naturally find the algebraic simplification involved in certain tasks to be more straightforward. The major distinction between the two versions of the project is the degree of general mathematical maturity expected.

### Suggestions for Classroom Implementation

Classroom implementation of this and other PSPs in the collection may be accomplished through individually assigned work, small-group work and/or whole-class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project. To reap the full mathematical benefits offered by the PSP approach, students should be required to read assigned sections in advance of in-class work, and to work through primary source excerpts together in small groups in class.

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<sup>11</sup>The task of translating those verbal formulations into symbolic formulas could also be omitted from this version of the project in courses where algebraic manipulation is not a particular focus.

The author’s method of ensuring that advance reading takes place is to require student completion of “Reading Guides” (or “Entrance Tickets”), for which students receive credit for completion, but with no penalty for errors in solutions. A typical Reading Guide will include “Classroom Preparation” exercises (drawn from the PSP Tasks) for students to complete prior to arriving in class. They may also include “Discussion Questions” that ask students only to read a given task and jot down some notes in preparation for class work. On occasion, other tasks are assigned as follow-up to a prior class discussion.

In completing assigned Reading Guides, students are asked to strive to answer each question correctly, but to think of the guide as preparatory work for class, not as a final product (e.g., formal polished write-ups are not expected). Students who arrive unprepared to discuss assignments on days when group work is conducted based on advance reading are not allowed to participate in those groups, but are allowed to complete the in-class work independently. Reading Guides are collected at the end of each class period for instructor review and scoring (again, for completeness only), thereby providing helpful feedback to the instructor about individual and whole-class understanding of the material, in addition to supporting students’ advance preparation efforts.

A sample Reading Guide (based on the Day 1 Advanced Preparation Work suggested in the Sample Implementation Schedule given below) appears in the Appendix at the end of these Notes.

L<sup>A</sup>T<sub>E</sub>X code of the entire PSP is available from the author by request to facilitate preparation of reading guides or ‘in-class task sheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

### **Sample Implementation Schedule (based on a 75-minute class period)**

For this particular version of this particular mini-PSP, the following schedule allows for implementation in a single class period. Instructors who wish to do so may also use part or all of a second class period to discuss the two “compare and contrast” tasks in Section 4.

- **Advance Preparation Work** (to be completed before class): Read pages 1–4, completing Tasks 1–3 and preliminary notes for class discussion on Task 4 along the way, per the sample Reading Guide on pages 11–13 below.
- **One Period of Class Work**
  - Introduction & Section 1: Whole or small group comparison of answers to Tasks 1–3.
  - Section 2: Small group work on Tasks 4–7, supplemented by whole class discussion as deemed appropriate by the instructor.
  - Section 3: Small group work on Tasks 8–10, supplemented by whole class discussion as deemed appropriate by the instructor.
  - Time permitting, small group work on Tasks 11 and/or 12 in Section 4.
- **Follow-up Assignment** (to be completed for discussion during next class period or assigned as individual homework): As needed, continue work on the tasks in Section 3, plus Tasks 11 and/or 12 if these are to be assigned; read the brief Conclusion.
- **Homework (optional)**: Formal write-up of student work on some or all of the project tasks, to be due at a later date (e.g., one week after completion of the in-class work). For upper division students, a formal write-up of Task 12 could also require proofs of the main conjectures generated by students.

## Connections to other Primary Source Projects

The following additional projects based on primary sources (written by the indicated authors) are also freely available for use in teaching standard topics in an **introductory number theory course**. Shorter PSPs designed to be completed in 1–2 class periods, are designated with an asterisk (\*). The implementation time required for the other “Full-length PSPs” ranges from 3–9 classroom periods; the Notes to Instructors section of each project provides a detailed sample schedule and suggestions for possible modifications.

- *Gaussian Integers and Dedekind’s Creation of an Ideal: A Number Theory Project* (Janet Heine Barnett)
- *Greatest Common Divisor: Algorithm and Proof* (Mary K. Flagg)
- *Primes, Divisibility, and Factoring* (Dominic Klyve)
- *The Mobius Function and Mobius Inversion* (Carl Lienert)
- *The Origin of the Prime Number Theorem\** (Dominic Klyve)
- *The Pell Equation in India* (Toke Knudsen and Keith Jones)

For instructors who are instead interested in the “Generating Pythagorean Triples” project for use in a **course aimed at general education audiences or pre-service elementary teachers**, the following PSPs were explicitly designed with one of these two audiences in mind.

- *Babylonian Numeration* (Dominic Klyve)
- *Construction of Figurate Numbers* (Jerry Lodder)
- *Cross Cultural Comparisons: The Art of Computing the Greatest Common Divisor* (Mary K. Flagg)
- *Pascal’s Triangle and Mathematical Induction* (Jerry Lodder)

Classroom-ready versions of all of the projects listed above can be downloaded from [https://digitalcommons.ursinus.edu/triumphs\\_number/](https://digitalcommons.ursinus.edu/triumphs_number/), or obtained (along with their L<sup>A</sup>T<sub>E</sub>X code) from their authors.

## Acknowledgments

The development of this project has been partially supported by the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Project with funding from the National Science Foundation’s Improving Undergraduate STEM Education Program under Grant Number 1523494. Any opinions, findings, conclusions or recommendations expressed in this project are those of the author and do not necessarily represent the views of the National Science Foundation.

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## APPENDIX

This appendix provides a ‘Sample Reading Guide’ that illustrates the author’s method for assigning advance preparation work in connection with classroom implementation of primary source projects. More detail concerning these guides is included in the subsection “Suggestions for Classroom Implementation” of the Notes to Instructors for this project.

Note that the full text of each assigned Task is reproduced on the guide, with blank space for students’ responses deliberately left below each question. This not only makes it easier for students to jot down their thoughts as they read, but also makes their notes more readily available to them during in-class discussions. This practice also makes it easier for the instructor to efficiently review each guide for completeness, and for students to review their own notes and instructor feedback once it is returned to them.

## SAMPLE READING GUIDE

**Background Information:** The goal of the advance reading and tasks assigned in this guide is to familiarize students with the definition and examples of Pythagorean triples and gnomons in order to prepare them for in-class small group work on Tasks 4–10. This sample is designed with an upper division audience in mind. When using this version of the mini-PSP with a lower division course for non-mathematics majors, it is possible that two full class periods will be needed to complete the entire project. With either audience, the optional ‘bonus’ exercise in part (a) of Task 2 could be omitted from the Study Guide, and need not be assigned at all for the purpose of completing the rest of the tasks in this project.

\*\*\*\*\*

### Reading Assignment:

*Generating Pythagorean Triples: Gnomonic Explorations*, pages 1–4

1. Read the introduction on page 1.

*Any questions or comments?*

2. **Class Prep** Complete **Task 1** from page 2 here:

**Task 1** How does Proclus’ statement of Theorem XLVII compare to the way that we state the Pythagorean Theorem today? In particular, what names do we use today for what Proclus called ‘the side subtending the right angle’ and ‘the sides containing the right angle’?

3. Read the rest of page 2.

*Any questions or comments?*

4. **Class Prep** Complete **Task 2** from page 2 here:

**Task 2** This task examines the examples given by Proclus in the excerpt on page 2.

- (a) As an illustration that isosceles right triangles never give us a Pythagorean triple, Proclus stated that 'the square of seven ... lacks one of being double the square of five'.

Explain how this relates to an isosceles right triangle with legs of (equal) length  $a = b = 5$ . What is the exact length of the hypotenuse in this example?

*Bonus* Explain in general why an isosceles right triangle never gives a Pythagorean triple.

- (b) Proclus gave (3, 4, 5) as an example of a Pythagorean triple associated with a scalene right triangle. Give another such example.

- (c) Do all scalene right triangles give us a Pythagorean triple? Justify your answer.

5. In Section 1, read page 3.  
*Any questions or comments?*

6. **Class Prep** Complete **Task 3** on page 3 here:

**Task 3** This task examines further the connection between gnomons and Pythagorean triples.

- (a) Figure 3 shows the square number  $12^2 = 144$ . Add a gnomon to that square.  
 What Pythagorean triple  $(a, b, c)$  is represented by the completed diagram?

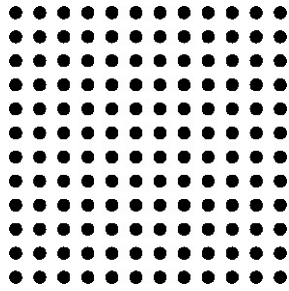


Figure 3

- (b) Figure 4 shows the square number  $8^2 = 64$ .  
 Does adding a gnomon to that square give us a Pythagorean triple  $(a, b, c)$ ?  
 Explain why or why not.

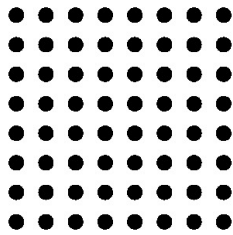


Figure 4

7. In Section 2, read through (but do not complete) **Task 4** on page 4.  
**Write at least one question or comment about this task.**