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### Abel and Cauchy on a Rigorous Approach to Infinite Series

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# Abel and Cauchy on a Rigorous Approach to Infinite Series

David Ruch\*

November 22, 2021

## 1 Introduction

Infinite series were of fundamental importance in the development of calculus by Newton, Euler and other mathematicians during the late 1600s and 1700s. Questions of rigor and convergence were of secondary importance in these times, but attitudes began to change in the early 1800s. When the brilliant young mathematician Niels Abel (1802–1829) moved to Paris in 1826 at age 24, he was aware of many paradoxes with infinite series and wanted big changes. Indeed, in a letter to his teacher Bernt Holmboe (1795–1850), Abel wrote:

I shall devote all my efforts to bring light into the immense obscurity that today reigns in Analysis. It so lacks any plan or system, that one is really astonished that there are so many people who devote themselves to it — and, still worse, it is absolutely devoid of rigor.<sup>1</sup>

Abel was born and raised in Norway, far from the centers of mathematical activity in his time. His work was largely unrecognized during his lifetime through a series of misfortunes. Nevertheless, he managed to get to Paris and attend lectures by mathematical stars such as Adrien-Marie Legendre (1752–1833) and Augustin Louis Cauchy (1789-1-857). Abel was particularly taken by Cauchy and his efforts to introduce rigor into analysis, and wrote to Holmboe that Cauchy “is the only man who knows how mathematics should be treated. What he does is excellent.”<sup>2</sup>

In this project, we will read excerpts from 1820s work by Abel and Cauchy as we rigorously develop infinite series and examine some of the tough infinite series problems of their day.

The study and use of series go back to antiquity. The Greek mathematician Archimedes used series to help calculate the area under the arc of a parabola, and geometric series such as

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$

were well known and used extensively in the development of calculus. The notion of divergent series was not clearly understood and somewhat controversial in the times leading up to Abel and Cauchy.

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<sup>1</sup>As quoted in [Hairer and Wanner, 2008, p. 188].

<sup>2</sup>As quoted in [Hairer and Wanner, 2008, p. 188].

For a simple example, here are two groupings and “sums” of the series  $1 - 1 + 1 - 1 + 1 - 1 + \dots$

$$(1 - 1) + (1 - 1) + (1 - 1) + \dots = 0 + 0 + 0 + \dots = 0$$

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots = 1 + 0 + 0 + 0 + \dots = 1$$

which suggests that  $1 = 0$  (*oops!*). Some mathematicians in the 1700’s suggested that the sum of this series should split the difference and be  $1/2$ , and others argued that the series did not converge and had no sum.

The following short excerpt from Abel comes from another 1826 letter written home to Holmboe from Paris.<sup>3</sup> In it, Abel referenced a much more sophisticated series example (equation (1) below) that Euler had discussed in 1750.<sup>4</sup> This series was quite important historically, as Joseph Fourier used it in his development of Fourier series and his model of heat transfer during the early 1800s.



Divergent series are in their entirety an invention of the devil and it is a disgrace to base the slightest demonstration on them. You can take out whatever you want when you use them, and they are what has produced so many failures and paradoxes. . . . The following example shows how one can err. One can rigorously demonstrate that

$$\frac{x}{2} = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \tag{1}$$

for all values of  $x$  smaller than  $\pi$ . It seems that consequently the same formula must be true for  $x = \pi$ ; but this will give

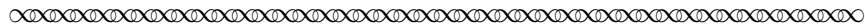
$$\frac{\pi}{2} = \sin \pi - \frac{1}{2} \sin 2\pi + \frac{1}{3} \sin 3\pi - \text{etc.} = 0.$$

One can find innumerable examples of this kind.

In general the theory of infinite series, up to the present, is very poorly established. One performs every kind of operation on infinite series, as if they were finite, but is it permissible? Never at all. Where has it been demonstrated that one can obtain the derivative of an infinite series by taking the derivative of each term? It is easy to cite examples where this is not right . . . . By taking derivatives [of (1)], one has

$$\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \text{etc.} \tag{2}$$

A completely false result, because this series is divergent.



**Task 1** Find a few values of  $x$  less than  $\pi$  which, substituted into (2), produce strange results and support Abel’s contention that the series in (2) is divergent.

In Abel’s day, there was no standard terminology for “absolute value”<sup>5</sup> and mathematicians were

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<sup>3</sup>As quoted in [Bottazzini, 1986, p. 87–89].  
<sup>4</sup>See [Bottazzini, 1986] and [Jahnke, 2003] for more on this series, which appears in discussions between Euler and d’Alembert on how to model the motion of a vibrating string.  
<sup>5</sup>The absolute value symbol  $|x|$  was introduced by Weierstrass later in the 1800s.

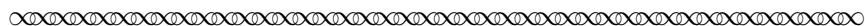
not always clear whether they meant the absolute value of a number or the number itself.

**Task 2** In the excerpt above, Abel claimed that the series in equation (1) is valid “for all values of  $x$  smaller than  $\pi$ .” Do you think he meant  $x$  itself or  $|x|$ ?

**Task 3** Let’s try to visualize equation (1). Use a Computer Algebra System (CAS) to graph  $y = x/2$  and  $y = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots - \frac{1}{10} \sin 10x$  together for  $-2\pi \leq x \leq 2\pi$ . What do you observe at  $x = \pm\pi$ ? If you were to plot even more terms from the infinite series, the “wiggly” parts of the sine curve would grow even closer to straight line for  $|x| < \pi$ . What do you think of Abel’s comments about this series and its derivative?

We won’t try to tackle all the issues Abel raised with this example. However, we can see why mathematicians of his time were struggling with infinite series at the same time that they were amazed by their power!

Another series that bothered Abel comes from the Binomial Theorem. Newton had discovered that the standard *finite* binomial expansion  $(1 + x)^m$  for positive integer  $m$  could be generalized to an *infinite* series for *non-integer* values of  $m$  (equation (3) below), and he was able to use this series to produce a number of new results. While Newton thought this series converged only for  $|x| < 1$ , no one had produced a convergence proof that fully convinced Abel, and he set out to do so in an 1826 paper.<sup>6</sup> Here is an excerpt from the introduction of that paper [Abel, 1826].



1.

If one subjects to a more precise examination the reasoning that one generally uses when dealing with infinite series, one will find that, taken as a whole, it is not very satisfactory, and that consequently the number of theorems concerning infinite series that may be considered rigorously based is very limited. One normally applies the operations of analysis to infinite series as if the series were finite. This does not seem to me permissible without special proof.

... ..

One of the most remarkable series in algebraic analysis is

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \dots + \frac{m(m-1)(m-2)\dots[m-(n-1)]}{1 \cdot 2 \cdot 3 \dots n}x^n \text{ etc..} \quad (3)$$

When  $m$  is a positive whole number, one knows that the sum of this series, which in this case is finite, may be expressed as  $(1 + x)^m$ . When  $m$  is not a whole number, the series becomes infinite, and it will be convergent or divergent, according to different values that one gives to  $m$  and  $x$ . In this case one writes in the same way

$$(1 + x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \dots \text{ etc.;} \quad (4)$$

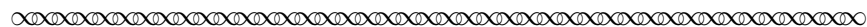
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<sup>6</sup>This paper was appropriately entitled “Untersuchungen über die Reihe:  $1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 \dots$ ,” or “Investigations on the Series:  $1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 \dots$ ”

... One assumes that numerical equality will always hold when the series is convergent; but this is what until now has not yet been proved. No one has even examined all the cases where the series is convergent . . . .

The aim of this memoir is to try to fill a gap with the complete solution of the following problem:

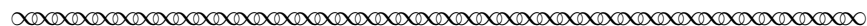
“Find the sum of the series (3) for all real or imaginary values of  $x$  and  $m$  for which the series is convergent.”



**Task 4** To get a sense of the binomial series equality for whole numbers  $m$ , verify (4) with  $m = 3$ .

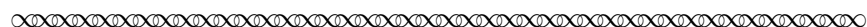
**Task 5** To get a visual idea of Abel’s concerns about the binomial series with non-integer  $m$ , use a CAS to graph  $y = \sqrt{1+x}$  and the first five terms of series (3) together for  $-1.2 \leq x \leq 1.2$ . Newton claimed this series converges for  $|x| < 1$ . Does your plot suggest this is correct? Does the plot suggest the series converges at  $x = \pm 1$ ? What about for  $|x| > 1$ ?

Abel continued his introduction as follows.



## 2.

We are first going to establish some necessary theorems on series. The excellent work of Cauchy *Cours d'analyse de l'école polytechnique*, which must be read by every analyst who loves rigor in mathematical research, will serve as our guide.

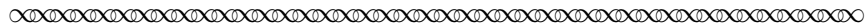


We will now take Abel’s advice and read Cauchy on infinite series in the next part of this project. Then we will return to Section 2 of Abel’s paper, where he developed some new infinite series results and tackled a controversial theorem of Cauchy. This work is important in its own right, independent of the Binomial Theorem, and will serve as the primary focus of our project.

## 2 Cauchy on Infinite Series

Augustin Louis Cauchy was a renowned mathematician in 1826 Paris. After graduating in 1810 from the École Polytechnique in Paris, he published much impressive mathematics and became a professor at this same institution. Cauchy loved pure mathematics and was convinced of the need for a rigorous approach to analysis. He wrote his *Cours d'Analyse* [Cauchy, 1821] for his teaching, and he constructed it with his philosophy of rigor. Abel had read this text before coming to Paris and was inspired to use its methods and spirit in his own research. One radical aspect of Cauchy’s book was his study of convergence of series without necessarily finding the sum of the series, which was quite a departure from the eighteenth century tradition of focusing on series sums with little attention to convergence issues.

We now start reading Chapter 6 on infinite series of Cauchy’s *Cours d'Analyse*.



### 6.1 General Considerations on series.

We call a series an indefinite sequence of quantities,

$$u_0, u_1, u_2, u_3, \dots,$$

which follow from one to another according to a determined law. These quantities themselves are the various terms of the series under consideration. Let

$$s_n = u_0 + u_1 + u_2 + \dots + u_{n-1} \tag{5}$$

be the sum of the first  $n$  terms, where  $n$  denotes any integer number. If, for ever increasing values of  $n$ , the sum  $s_n$  indefinitely approaches a certain limit  $s$ , the series is said to be convergent, and the limit in question is called the sum of the series. On the contrary, if the sum  $s_n$  does not approach any fixed limit as  $n$  increases indefinitely, the series is divergent, and does not have a sum. In either case, the term which corresponds to the index  $n$ , that is  $u_n$ , is what we call the general term. For the series to be completely determined, it is enough that we give this general term as a function of the index  $n$ .

One of the simplest series is the geometric progression

$$1, x, x^2, x^3, \dots,$$

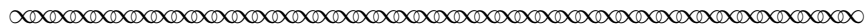
which has  $x^n$  for its general term, that is to say the  $n$ th power of the quantity  $x$ . If we form the sum of the first  $n$  terms of this series, then we find

$$1 + x + x^2 + \dots + x^{n-1} = \frac{1}{1-x} - \frac{x^n}{1-x} \tag{6}$$

As the values of  $n$  increase, the numerical value of the fraction  $\frac{x^n}{1-x}$  converges towards the limit zero, or increases beyond all limits, according to whether we suppose that the numerical value of  $x$  is less than or greater than 1. Under the first hypothesis, we ought to conclude that the progression

$$1, x, x^2, x^3, \dots,$$

is a convergent series which has  $\frac{1}{1-x}$  as its sum, whereas, under the second hypothesis, the same progression is a divergent series which does not have a sum.



Cauchy's terminology and notation are close to what we use today, but there are some differences. Cauchy began by calling a series "an indefinite sequence of quantities,  $u_0, u_1, u_2, u_3, \dots$ " but then says "These quantities themselves are the various terms of the series under consideration." This may seem confusing, especially if you remember the distinction between a sequence and a series from your introductory calculus courses. Fortunately, Cauchy elaborated by defining and discussing the  $s_n$  sums. The  $s_n$  expression Cauchy defined in (5) above is still used in modern terminology and is nowadays called the *n*th partial sum. Notice that we can form a sequence of partial sums ( $s_n$ ), and observe that the convergence of the **series** is equivalent to convergence of the **sequence** ( $s_n$ ). Most modern texts formally define the infinite series *generated by the  $u_k$  terms* to be the sequence ( $s_n$ )

of partial sums. Informally, we often write both the infinite series and its sum (when defined) as  $u_0 + u_1 + u_2 + u_3 + \cdots$ . Cauchy did start writing the sum of a *convergent* series this way later in his Section 6.1, as we shall see later in the project.

Cauchy's definition that a divergent series does not have a sum was not universally accepted in his day, but is now standard.

**Task 6** As an example, let  $u_k = (-1)^k$  for all integers  $k \geq 0$ .

- Find  $s_0, s_1, s_2, s_3$  for the infinite series generated the  $u_k$  terms.
- Do you think the sequence of partial sums  $(s_n)$  converges or diverges? Explain.
- Do you think the infinite series generated by the  $u_k$  terms converges or diverges? Explain.
- Discuss the pros and cons of writing this infinite series informally as

$$1 - 1 + 1 - 1 + \cdots$$

**Task 7** Using modern notation, rewrite Cauchy's definition of series convergence and the sum  $s$  in terms of the sequence of *partial sums*  $(s_n)$ .

**Task 8** Verify the algebra in (6). This is often called the **finite** geometric series formula. For what  $x$  values is this formula valid?

**Task 9** In Cauchy's discussion of the convergence of the geometric series, note his language "whether we suppose that the numerical value of  $x$  is less than or greater than 1."

- Explain why the geometric series diverges for  $x = -2$ .
- Does the series converge or diverge when  $x = 1, -1$ ?
- In modern terminology, what do you think Cauchy meant by the "numerical value of  $x$ "? Cauchy frequently used the term "*numerical value*" with this meaning.

Notice that Cauchy did not use sigma summation notation  $\sum_{k=0}^{\infty} u_k$  in this 1821 work, nor did Abel use it in his 1826 paper. The sigma summation did not come into common use until later in the 1800s. Since it is now conventional to denote both the infinite series and its sum using the symbols  $\sum u_i$  or  $\sum_{i=0}^{\infty} u_i$ , we will do so in this project.

**Task 10** Rewrite Cauchy's proof for the geometric series when  $|x| < 1$  using modern notation and results from a modern treatment of sequences.

**Task 11** Suppose a series  $\sum a_i$  converges and  $c \in \mathbb{R}$ . Prove the series  $\sum (ca_i)$  converges with sum  $c \sum a_i$ . If  $\sum a_i$  diverges, what can you say about  $\sum (ca_i)$ ?

**Task 12** Suppose series  $\sum a_i$  and  $\sum b_i$  converge to  $A$  and  $B$ , respectively. Prove the series  $\sum (4a_i + 7b_i)$  converges. Write its sum in terms of  $A$  and  $B$ . Generalize the results of this task.

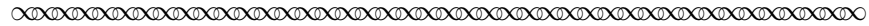
**Task 13**

Use the results above to determine convergence and sum, or divergence, of the following series.

$$(a) \ 6 - \frac{2}{3} + \frac{2}{27} - \frac{2}{243} + \dots$$

$$(b) \ \sum_{k=2}^{\infty} \frac{5^{k-1}}{4^{k+1}}$$

Let's return to Cauchy. As you read this next excerpt, pay careful attention to how Cauchy used the terms "necessary" and "sufficient" in his claims.



(Cauchy Section 6.1 continued)

Following the principles established above, in order that the series

$$u_0, u_1, u_2, \dots, u_n, u_{n+1}, \dots \tag{7}$$

be convergent, it is necessary and it suffices that increasing values of  $n$  make the sum

$$s_n = u_0 + u_1 + u_2 + \dots + u_{n-1}$$

converge indefinitely towards a fixed limit  $s$ . In other words, it is necessary and it suffices that, for infinitely large values of the number  $n$ , the sums

$$s_n, s_{n+1}, s_{n+2}, \dots$$

differ from the limit  $s$ , and consequently from one another, by infinitely small quantities. Moreover, the successive differences between the first sum  $s_n$  and each of the following sums are determined, respectively, by the equations

$$s_{n+1} - s_n = u_n$$

$$s_{n+2} - s_n = u_n + u_{n+1}$$

$$s_{n+3} - s_n = u_n + u_{n+1} + u_{n+2}$$

Hence, in order for series (7) to be convergent, it is first of all necessary that the general term  $u_n$  decrease indefinitely as  $n$  increases. But this condition does not suffice, and it is also necessary that, for increasing values of  $n$ , the different sums,

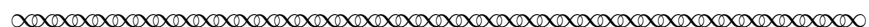
$$u_n + u_{n+1}$$

$$u_n + u_{n+1} + u_{n+2}$$

that is to say, the sums of as many of the quantities

$$u_n, u_{n+1}, u_{n+2}, \dots,$$

as we may wish, beginning with the first one, eventually constantly assume numerical values less than any assignable limit. Conversely, whenever these various conditions are fulfilled, the convergence of the series is guaranteed.





**Task 14** List all the “necessary” claims in this excerpt, expressing each as an implication. Then list all the “sufficient” claims, expressing each as an implication.

**Task 15** Carefully re-read Cauchy’s sentence beginning with “In other words . . .” and notice that he was making two separate equivalence claims, from a modern viewpoint. Rewrite each equivalence claim in Cauchy’s sentence with modern  $\epsilon$ - $N$  terminology.

Cauchy’s statements that

“... in order for series (7) to be convergent, it is first of all necessary that the general term  $u_n$  decrease indefinitely as  $n$  increases. But this condition does not suffice.”

are worth a clarification, a proof and some examples.

**Task 16** First, clarify what Cauchy meant by “the general term  $u_n$  decrease indefinitely as  $n$  increases.” Second, write the claim “it is first of all necessary that the general term  $u_n$  decrease indefinitely as  $n$  increases ” as a theorem, and give a modern proof using Cauchy’s equation  $s_{n+1} - s_n = u_n$  and modern sequence limit laws.

**Task 17** Write down the contrapositive of your theorem implication from Task 16. This result should remind you of an infinite series “test” from your Introductory Calculus course. What is the test called?

**Task 18** Apply your result in Task 16 to the following series, where possible. Then interpret Cauchy’s statements that “in order for series (7) to be convergent, it is first of all necessary that the general term  $u_n$  decrease indefinitely as  $n$  increases. But this condition does not suffice” for each series.

(a)  $1 - 1 + 1 - 1 + 1 - 1 + \dots$

(b)  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

(c)  $\frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \dots$

**Task 19** Consider the statement “the sums of as many of the quantities

$$u_n, u_{n+1}u_{n+2}, \dots,$$

as we may wish, beginning with the first one, eventually constantly assume numerical values less than any assignable limit.” Using finite sums, convert this statement into modern  $\epsilon$ - $N$  terminology. What is this saying about the sequence  $(s_n)$ , in modern terminology?

**Task 20** Consider the statement “Conversely, whenever these various conditions are fulfilled, the convergence of the series is guaranteed.” What modern theorem about sequences of real numbers justifies this statement?

It is interesting that Cauchy, and many of his contemporaries, thought this last necessary and sufficient condition for convergence of a series was obvious and did not need a proof. As we shall see, Cauchy and Abel used this criterion, nowadays named after Cauchy, to prove some convergence results.

**Task 21** Rewrite this new “Cauchy criterion” for series convergence in modern  $\epsilon$ - $N$  terminology.

The next result will come in handy when we read Abel.

**Task 22** Suppose an infinite series  $\sum u_n$  is convergent and  $\epsilon_0 > 0$ , and define  $Q_m$  by

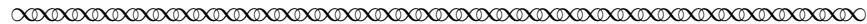
$$Q_m = \sup \left\{ \left| \sum_{k=m}^{m+n} u_k \right| : n \in \mathbb{N} \right\}.$$

Prove there exists  $N \in \mathbb{N}$  such that for any  $m \geq N$ ,  $Q_m < \epsilon_0$ .

Hint: Use a property of the convergent sequence of partial sums  $(s_n)$  of  $\sum u_n$ .

**Task 23** Suppose series  $\sum |x_k|$  converges. Use the Cauchy criterion to prove that  $\sum x_k$  must converge.

Now that we have carefully analyzed some fundamental results by Cauchy, let's return to his discussion, where he considered two important examples.



(Cauchy Section 6.1 continued)

Let us take, for example, the geometric progression

$$1, x, x^2, x^3, \dots \tag{8}$$

If the numerical value of  $x$  is greater than 1, that of the general term  $x^n$  increases indefinitely with  $n$ , and this remark alone suffices to establish the divergence of the series. The series is still divergent if we let  $x = \pm 1$ , because the numerical value of the general term  $x^n$ , which is 1, does not decrease indefinitely for increasing values of  $n$ . However, if the numerical value of  $x$  is less than 1, then the sums of any number of terms of the series, beginning with  $x^n$ , namely

$$\begin{aligned} & x^n, \\ & x^n + x^{n+1} = x^n \frac{1 - x^2}{1 - x}, \\ & x^n + x^{n+1} + x^{n+2} = x^n \frac{1 - x^3}{1 - x}, \end{aligned}$$

are all contained between the limits

$$x^n \quad \text{and} \quad \frac{x^n}{1 - x},$$

each of which becomes infinitely small for infinitely large values of  $n$ . Consequently, the series is convergent, as we already knew.

As a second example, let us take the numerical series

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots \quad (9)$$

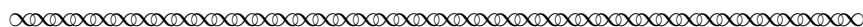
The general term of this series, namely  $\frac{1}{n+1}$ , decreases indefinitely as  $n$  increases. Nevertheless, the series is not convergent, because the sum of the terms from  $\frac{1}{n+1}$  up to  $\frac{1}{2n}$  inclusive, namely

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n}$$

is always greater than the product

$$n \frac{1}{2n} = \frac{1}{2}$$

whatever the value of  $n$ . As a consequence, this sum does not decrease indefinitely with increasing values of  $n$ , as would be the case if the series were convergent.



**Task 24** In this part of Section 6.1, Cauchy gave a proof that the geometric series is convergent for  $|x| < 1$ , using the new Cauchy criterion for series convergence that you put into modern form in Task 21. Notice that he was a bit cavalier for the negative  $x$  case when stating that terms are “all contained between the limits . . .” Write a careful modern version of his proof using the modern form of the Cauchy criterion for series convergence.

Now we turn to Cauchy’s second example, where he argued that the series (9) diverges.

**Task 25** Write a “Cauchy criterion” for series divergence in modern  $\epsilon$ - $N$  terminology.  
Hint: Negate your definition from Task 21.

**Task 26** Justify Cauchy’s claim after (9) that

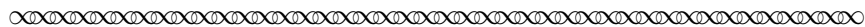
$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n} > \frac{1}{2}.$$

**Task 27** Rewrite Cauchy’s proof that series (9) diverges, using modern terminology, quantifiers, and the new Cauchy criterion.

Hint: Consider  $s_{2n} - s_n$ .

You may recognize this series (9) from introductory calculus as the *harmonic series*. It is interesting to note that this series was first shown to diverge by Nicole Oresme (c. 1323–1382), long before Cauchy’s time. Oresme’s proof was different from Cauchy’s, but used the same crucial observation that  $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} + \frac{1}{2n} > \frac{1}{2}$  for each  $n$ .

Let’s go back to Cauchy for another important example, which he analyzed with another useful technique. Also notice that Cauchy did start writing the sum of a *convergent* series as  $u_0 + u_1 + u_2 + u_3 + \dots$  in this excerpt.



(Cauchy Section 6.1 continued)

Let us further consider the numerical series

$$1, \frac{1}{1}, \frac{1}{1 \cdot 2}, \frac{1}{1 \cdot 2 \cdot 3}, \dots, \frac{1}{1 \cdot 2 \cdot 3 \dots n} \dots \quad (10)$$

The terms of this series with index greater than  $n$ , namely

$$\frac{1}{1 \cdot 2 \cdot 3 \dots n}, \frac{1}{1 \cdot 2 \cdot 3 \dots n(n+1)}, \frac{1}{1 \cdot 2 \cdot 3 \dots n(n+1)(n+2)}, \dots,$$

are, respectively, less than the corresponding terms of the geometric progression

$$\frac{1}{1 \cdot 2 \cdot 3 \dots n}, \frac{1}{1 \cdot 2 \cdot 3 \dots n n}, \frac{1}{1 \cdot 2 \cdot 3 \dots n n^2}, \dots,$$

As a consequence, the sum of however many of the initial terms as we may wish is always less than the sum of the corresponding terms of the geometric progression, which is a convergent series, and so *a fortiori*, it is less than the sum of this series, which is to say

$$\frac{1}{1 \cdot 2 \cdot 3 \dots n} \frac{1}{1 - \frac{1}{n}} = \frac{1}{1 \cdot 2 \cdot 3 \dots (n-1)} \frac{1}{n-1}. \quad (11)$$

Because this last sum decreases indefinitely as  $n$  increases, it follows that series (10) is itself convergent. It is conventional to denote the sum of this series by the letter  $e$ . By adding together the first  $n$  terms, we obtain an approximate value of the number  $e$ ,

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots + \frac{1}{1 \cdot 2 \cdot 3 \dots (n-1)}.$$

According to what we have just said, the error made will be smaller than the product of the  $n$ th term by  $\frac{1}{n-1}$ . Therefore, for example, if we let  $n = 11$ , we find as the approximate value of  $e$

$$e = 2.7182818 \dots, \quad (12)$$

and the error made in this case is less than the product of the fraction  $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}$  by  $\frac{1}{10}$ , that is  $\frac{1}{36,288,000}$ , so that it does not affect the seventh decimal place.

The number  $e$ , determined as we have just said, is often used in the summation of series and in the infinitesimal Calculus. Logarithms taken in the system with this number as its base are called Napierian, for Napier, the inventor of logarithms, or hyperbolic, because they measure the various parts of the area between the equilateral hyperbola and its asymptotes.<sup>7</sup>

In general, we denote the sum of a convergent series by the sum of the first terms, followed by an ellipsis. Thus, when the series

$$u_0, u_1 u_2, u_3, \dots,$$

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<sup>7</sup>Cauchy meant the area under the curve  $y = 1/x$ , using standard terminology of his time.

is convergent, the sum of this series is denoted

$$u_0 + u_1 + u_2 + u_3 + \dots$$

By virtue of this convention, the value of the number  $e$  is determined by the equation

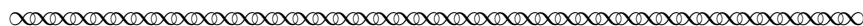
$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots, \quad (13)$$

and, if one considers the geometric progression

$$1, x, x^2, x^3, \dots$$

we have, for numerical values of  $x$  less than 1,

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1 - x}. \quad (14)$$



Euler derived this series expression (10) for  $e$  by 1748 from the sequence definition  $e = \lim (1 + 1/n)^n$  using an infinitesimal argument with the Binomial Theorem. Cauchy proved convergence of the  $e$  series by comparing each series term to a *larger* geometric series value, which he could sum precisely. You will generalize this method in Task 30 below.

**Task 28** Fill the algebraic details of Cauchy’s argument between (10) and (11) that, for arbitrary  $n, m \in \mathbb{N}$ , the difference  $s_{n+m} - s_n$  for the  $e$  series is less than  $\frac{1}{1 \cdot 2 \cdot 3 \dots (n-1)} \frac{1}{n-1}$ .

**Task 29** Use Cauchy’s approach with the Cauchy criterion to give a modern  $\epsilon$ - $N$  proof that the series  $\sum_{k=0}^{\infty} \frac{1}{k!}$  converges.

**Task 30** Use Cauchy’s comparison ideas for the  $e$  series to fill in the blanks below (with  $\sum a_i$  or  $\sum b_i$ ) and create a valid theorem.

**Theorem 1.** *Suppose  $(a_i)$  and  $(b_i)$  are sequences and there exists a  $K \in \mathbb{N}$  for which  $0 \leq a_i \leq b_i$  whenever  $i \geq K$ . If \_\_\_\_\_ converges, then \_\_\_\_\_ converges.*

This theorem might remind you of an infinite series “test” from your introductory calculus course. Do you remember the name of this test?

**Task 31** Prove Theorem 1.

Hint: Use partial sum sequences and the Monotone Convergence Theorem.

**Task 32** Use a contrapositive to state and prove a “divergence” version of Theorem 1.

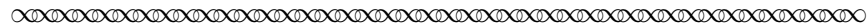
**Task 33** Use your results to determine convergence or divergence of the following series.

(a)  $\frac{4}{7} + \frac{5}{2} + \frac{6}{3} + \frac{1}{4} + \frac{8}{5} + \dots$

(b)  $\sum_{n=2}^{\infty} \frac{(3n+1)4^{2n-1}}{7^{n+5}}$

(c)  $\frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \frac{1}{162} + \dots$

In the next excerpt, we will see how Cauchy tried to extend his ideas on series of real numbers to series of a function  $x$ . His argument has problems from a modern point of view, so read it carefully.



Denoting the sum of the convergent series

$$u_0, u_1, u_2, u_3, \dots,$$

by  $s$  and the sum of the first  $n$  terms by  $s_n$ , we have

$$\begin{aligned} s &= u_0 + u_1 + u_2 + \dots + u_{n-1} + u_n + u_{n+1} + \dots \\ &= s_n + u_n + u_{n+1} + \dots, \end{aligned}$$

and, as a consequence,

$$s - s_n = u_n + u_{n+1} + \dots$$

From this last equation, it follows that the quantities

$$u_n, u_{n+1}, u_{n+2}, \dots$$

form a new convergent series, the sum of which is equal to  $s - s_n$ . If we represent this sum by  $r_n$ , we have

$$s = s_n + r_n,$$

and  $r_n$  is called the remainder of series (7) beginning from the  $n$ th term.

Suppose the terms of series (7) involve some variable  $x$ . If the series is convergent and its various terms are continuous functions of  $x$  in a neighborhood of some particular value of this variable, then

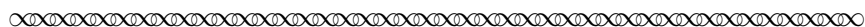
$$s_n, r_n \text{ and } s$$

are also three functions of the variable  $x$ , the first of which is obviously continuous with respect to  $x$  in a neighborhood of the particular value in question. Given this, let us consider the increments in these three functions when we increase  $x$  by an infinitely small quantity  $\alpha$ . For all possible values of  $n$ , the increment in  $s_n$  is an infinitely small quantity. The increment of  $r_n$ , as well as  $r_n$  itself, becomes infinitely small for very large values of  $n$ . Consequently, the increment in the function  $s$  must be infinitely small. From this remark, we immediately deduce the following proposition:

**Theorem I** — *When the various terms of series (7) are functions of the same variable  $x$ , continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum  $s$  of the series is also a continuous function of  $x$  in the neighborhood of this particular value.*

By virtue of this theorem, the sum of series (8) must be a continuous function of the variable  $x$  between the limits  $x = -1$  and  $x = 1$ , as we may verify by considering the values of  $s$  given by the equation

$$s = \frac{1}{1 - x}.$$



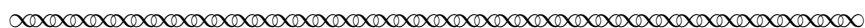
For simplicity we will take Cauchy’s meaning of the term “neighborhood” about a value  $x$  to be a small open interval  $(x - \delta, x + \delta)$  centered at the value  $x$ .

**Task 34** In his proof outline for Theorem I, Cauchy stated that  $s_n$  is “obviously continuous with respect to  $x$  in a neighborhood of the particular value in question.” Justify this statement using a theorem for continuous functions.

**Task 35** A key part of Cauchy’s argument is: “consider the increments in these three functions when we increase  $x$  by an infinitely small quantity  $\alpha$ . For all possible values of  $n$ , the increment in  $s_n$  is an infinitely small quantity. The increment of  $r_n$ , as well as  $r_n$  itself, becomes infinitely small for very large values of  $n$ .”

- (a) Rewrite this argument using modern terminology and quantifiers. Note: you are just translating, not **proving** his claim.
- (b) What part of this argument seems most difficult to justify?

Cauchy’s claim in this theorem seems pretty reasonable: if we add up some continuous functions that converge to a limit function at a point  $x$ , it seems plausible that the limit function is also continuous at  $x$ . Unfortunately, this is not always the case at all  $x$  values<sup>8</sup>. Indeed, Abel noticed this issue. Here is a footnote from Abel’s 1826 paper where he addressed this problem:

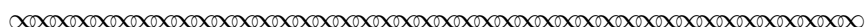


In the work by M. Cauchy one will find the following theorem: “When the various terms of series  $u_0 + u_1 + u_2 + \dots$  are functions of the same variable  $x$ , continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum  $s$  of the series is also a continuous function of  $x$  in the neighborhood of this particular value.”

But it seems to me that this theorem admits of exceptions. For example the series

$$\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \tag{15}$$

is discontinuous for any value  $(2m + 1)\pi$ , where  $m$  is a whole number.




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<sup>8</sup>Some historians argue that Cauchy meant the convergence was *uniform*, which makes his theorem valid. For more on this debate, see [Jahnke, 2003] and [Bottazzini, 1986].

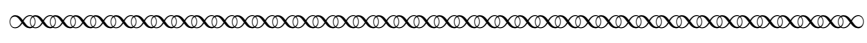
**Task 36**

Go back to the project introduction and re-read Abel’s discussion of this series in the excerpt on page 2. As Abel stated in that excerpt, this series (15) can be shown to converge to  $x/2$  for any value of  $x$  with  $|x| < \pi$ , although the proof method (Fourier series convergence), is beyond the scope of this project. Carefully explain why the conclusion to Cauchy’s Theorem I is not correct for the series (15) at  $x = \pi$ .

Note that Abel politely mentioned this example as an “exception” to Cauchy’s theorem. A blunter interpretation would be that Abel had found a counterexample to this theorem, rendering it invalid without stronger hypotheses. He did not identify the problem in Cauchy’s proof, but he did prove a correct variation of this theorem with significantly stronger hypotheses. We will examine Abel’s theorem in the next section of this project. Other major mathematicians worked hard during the mid-1800s to prove other corrected variations on Cauchy’s Theorem I. This indicates the subtlety of Cauchy’s error and the difficulty involved in fixing it!

### 3 Abel’s 1826 Paper

Abel was aware of the difficulties with Cauchy’s theorem on a series of continuous functions. He did not identify the specific problem in Cauchy’s proof, but he was able to prove an important theorem on the convergence of power series in his 1826 paper. Along the way, he proved other important results. These ideas are the focus of this section of the project. We now turn to Abel’s paper, with his first two theorems. As you read them, see if they remind you of convergence tests from your Introductory Calculus course.



**Theorem I** If one denotes a series of positive quantities by  $\rho_0, \rho_1, \rho_2 \dots$ , and the quotient  $\frac{\rho_{m+1}}{\rho_m}$ , for ever increasing values of  $m$ , approaches a limit  $\alpha$  greater than 1, then the series

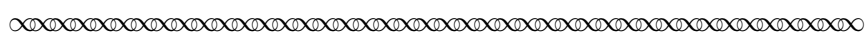
$$c_0\rho_0 + c_1\rho_1 + c_2\rho_2 + \dots + c_m\rho_m + \dots ,$$

where  $c_m$  is a quantity which, for ever increasing values of  $m$ , does not approach zero, will be necessarily divergent.

**Theorem II** If in a series of positive quantities  $\rho_0 + \rho_1 + \rho_2 + \dots + \rho_m + \dots$  the quotient  $\frac{\rho_{m+1}}{\rho_m}$ , for ever increasing values of  $m$ , approaches a limit  $\alpha$  smaller than 1, then the series

$$c_0\rho_0 + c_1\rho_1 + c_2\rho_2 + \dots + c_m\rho_m + \dots , \tag{16}$$

where  $c_0, c_1, c_2$  etc. are quantities that are never greater than one, will be necessarily convergent.<sup>9</sup>



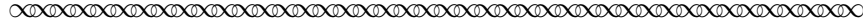
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<sup>9</sup>We have used  $c$  where Abel wrote  $\varepsilon$  in these theorems, in order to reduce confusion in modern  $\varepsilon$  arguments.



**Task 37** Interpret Abel's Theorems I and II in the special case where  $c_m = 1$  for all  $m$ . What name did we give these results in an Introductory Calculus course? Cauchy actually gave these results in the special case where  $c_m = 1$  in his book; Abel generalized them for his needs in Theorem IV later in his 1826 paper.

Abel did not give a proof of his first theorem, but he did supply a proof of his Theorem II, given in the next excerpt from his paper. Read it carefully, because it needs some minor adjustments for a modern level of rigor.



Indeed, by assumption, one may always take  $m$  large enough that  $\rho_{m+1} < \alpha\rho_m$ ,  $\rho_{m+2} < \alpha\rho_{m+1}$ ,  $\dots$ ,  $\rho_{m+n} < \alpha\rho_{m+n-1}$ . It follows from there that  $\rho_{m+k} < \alpha^k\rho_m$ , and consequently

$$\rho_m + \rho_{m+1} + \dots + \rho_{m+n} < \rho_m (1 + \alpha + \alpha^2 + \dots + \alpha^n) < \frac{\rho_m}{1 - \alpha}, \quad (17)$$

therefore, for all the more reason,

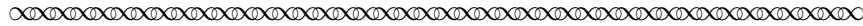
$$c_m\rho_m + c_{m+1}\rho_{m+1} + \dots + c_{m+n}\rho_{m+n} < \frac{\rho_m}{1 - \alpha}.$$

Now, since  $\rho_{m+k} < \alpha^k\rho_m$  and  $\alpha < 1$ , it is clear that  $\rho_m$  and consequently the sum

$$c_m\rho_m + c_{m+1}\rho_{m+1} + \dots + c_{m+n}\rho_{m+n}$$

will approach zero.

The above series [(16)] is therefore convergent.



This proof is largely correct, but has a flaw, as we shall see. Perhaps you have had a similar experience in your own proof writing! Fortunately, we can make an adjustment and save Abel's proof.

**Task 38** In his proof of Theorem II, Abel stated that "one may always take  $m$  large enough that  $\rho_{m+1} < \alpha\rho_m$ ." Use the example  $\rho_n = \frac{n}{2^n}$  to show that this is not always true.

**Task 39** You will correct part of Abel's Theorem II proof in this task.

- Show that if  $\lim \frac{\rho_{m+1}}{\rho_m} < 1$  then we can (i) find a number  $\beta$  such that  $\lim \frac{\rho_{m+1}}{\rho_m} < \beta < 1$ , and (ii) find an integer  $N \in \mathbb{N}$  for which  $m \geq N$  implies that  $\rho_{m+1} < \beta\rho_m$ .
- Use part (a) and Abel's ideas to show that the sequence  $(\rho_m)$  converges to 0.
- Use part (a) and Abel's ideas to prove a statement analogous to (17).

**Task 40** To understand Abel's Theorem II statement completely, we need to remember to interpret the statement where  $c_0$  etc. are quantities that are never greater than one" carefully. To see this, set  $\rho_m = 1/2^m$  and  $c_m = -2^m$  and show that the series  $\sum \rho_m c_m$  diverges.

**Task 41** Based on the task above, let's interpret Abel's Theorem II hypotheses about the  $c_k$  as "the quantities  $|c_k|$  are never greater than one." With this adjustment,

- (a) Write a modern version of Abel's Theorem II.
- (b) Use Abel's proof method and your results from Task 39 to give a modern  $\epsilon$ - $N$  proof of your modern version of Theorem II.

Abel did not give a proof of his first theorem, perhaps thinking it obvious. See if you can verify his claim in the next task.

**Task 42** Give a modern proof of Abel's Theorem I. Here are some suggestions:

- (a) First explain why the sequence  $(\rho_k)$  diverges with  $\lim \rho_k = \infty$ .  
Hint: Think about Abel's argument relating  $\rho_{m+k}$  and  $\rho_m$  in his Theorem II proof.
- (b) Translate the hypothesis about the sequence  $(c_k)$  to a statement about a subsequence  $(|c_{n_k}|)$  of  $(|c_k|)$ .
- (c) Draw a conclusion based on the limiting behavior of a subsequence of  $(|c_k \rho_k|)$  corresponding to  $(|c_{n_k}|)$ .

**Task 43** In the 1700s, Euler derived a power series for  $\ln(x+1)$  :

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (18)$$

Use Abel's theorems and standard sequence theorems to prove this series converges at a given real number  $x$  when  $|x| < 1$  and diverges when  $|x| > 1$ . Be sure to clearly identify the  $c_k, \rho_k$  and  $\alpha$  values.

**Task 44** Generalize Abel's Theorem II to a theorem with hypothesis " $(c_k)$  bounded" in place of " $|c_k|$  are never greater than one." Prove your claim.

**Task 45** Apply the theorems from Tasks 41, 44 and standard sequence theorems to determine convergence or divergence of the series below. Be sure to clearly identify the  $c_k, \rho_k$  and  $\alpha$  values and make sure the theorem hypotheses are met.

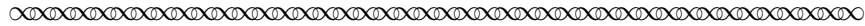
- (a)  $\sum \frac{3k-1}{2^k(k+1)}$
- (b)  $\sum \frac{k+1+(-1)^k 3^{k+1}}{k+1} \frac{3^{k+1}}{2^{k-1}}$
- (c)  $\sum y^k/k!$  for fixed, arbitrary  $y \in \mathbb{R}$ .

**Task 46** Recall from the project introduction that Abel was interested in proving convergence of the generalized binomial series (3)

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \dots$$

for various  $m$  and  $x$  values. For  $m \in \mathbb{R}$  but not necessarily an integer, use Abel's theorems to show that series (3) converges when  $|x| < 1$  and diverges when  $|x| > 1$ .

We next return to Abel for his third theorem, which he needed as a tool for proving his major power series result in Theorem IV.



(Abel Section 2 continued)

**Theorem III** On denoting by  $t_0, t_1, t_2, \dots, t_m, \dots$  a series of any quantities whatever, if  $p_m = t_0 + t_1 + t_2 + \dots + t_m$  is always less than a determined quantity<sup>10</sup>  $B$ , one will have

$$r = c_0 t_0 + c_1 t_1 + c_2 t_2 + \dots + c_m t_m < B c_0$$

where  $c_0, c_1, c_2, \dots$  denote positive decreasing quantities.

Indeed, one has

$$t_0 = p_0, \quad t_1 = p_1 - p_0, \quad t_2 = p_2 - p_1, \quad \text{etc.}$$

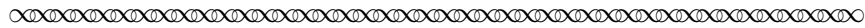
therefore

$$r = c_0 p_0 + c_1 (p_1 - p_0) + c_2 (p_2 - p_1) + \dots + c_m (p_m - p_{m-1}) \tag{19}$$

or rather

$$r = p_0 (c_0 - c_1) + p_1 (c_1 - c_2) + \dots + p_{m-1} (c_{m-1} - c_m) + p_m c_m \tag{20}$$

But  $c_0 - c_1, c_1 - c_2, \dots$  are positive, so the quantity  $r$  will clearly be less than  $B c_0$ .



Let's examine this result and Abel's proof from a modern viewpoint.

**Task 47** Rewrite the theorem statement with appropriate quantifiers, and clarify the phrase "decreasing quantities."

**Task 48** Justify the algebraic rearrangement of terms in  $r$ , between (19) and (20).

**Task 49** Justify Abel's claim in his Theorem III proof that "quantity  $r$  will clearly be less than  $B c_0$ ."

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<sup>10</sup>We have used  $B$  where Abel wrote  $\delta$  in these theorems, in order to reduce confusion in modern  $\epsilon$ - $\delta$  proofs.

**Task 50** It will be helpful to have a version of this theorem with stronger conclusion  $|r| < Bc_0$  instead of  $r < Bc_0$ . This stronger conclusion naturally requires a stronger hypothesis on the partial sums  $p_m$ . State and prove an “absolute value” version of Theorem III with the stronger hypothesis  $|p_k| < B$  for all  $k$  and conclusion  $|r| < Bc_0$ . Abel’s beautiful rearrangement of the terms in  $r$  will still be crucial for your proof!

**Task 51** Consider the partial sums  $s_m$  of series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \quad (21)$$

- (a) Use your “absolute value” version of Theorem III from Task 50 to prove that  $|s_m| < 1.01$  for all  $m$ .
- (b) For this same series (21), find  $m$  so that for any  $n$  we have  $|s_{m+n} - s_m| < 0.001$ , and prove your claim using your “absolute value” version of Theorem III from Task 50.

Hint: Choose  $t_0, c_0, B$  to align with the  $m$ th term of series (21).

The series (21) is an example of an alternating series, which you may recall from your Introductory Calculus course. Here is a useful theorem for guaranteeing convergence of a certain class of alternating series.

**Theorem 2.** *If  $(d_k)$  is a decreasing sequence of positive numbers with  $\lim(d_k) = 0$ , then the alternating series  $\sum (-1)^k d_k$  is convergent.*

**Task 52** Prove Theorem 2 using the Cauchy criterion, your “absolute value” version of Theorem III, and the ideas used in Task 51 (b).

**Task 53** Use Theorem 2 to prove the following series converge, or explain why the theorem cannot be applied to the particular series.

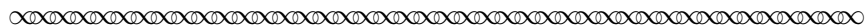
- (a)  $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$
- (b)  $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$
- (c)  $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots$

Like Cauchy, Abel wanted to apply his results on numerical series to variable series, and to address the issue of continuity. We investigate his efforts in the remainder of the project.

### 3.1 Abel’s power series theorem

After developing his first three theorems on numerical series, Abel considered the situation where a function  $f$  was defined by an infinite series in terms of a variable  $\alpha$ . In the theorem below<sup>11</sup>, he stated an important result for power series.

<sup>11</sup>In Theorem IV and its proof, we use  $d$  where Abel wrote  $\delta$ , in order to avoid confusion with  $\epsilon$ - $\delta$  proofs.

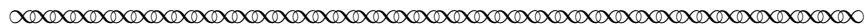


**Definition** A function  $f(x)$  will be said to be a continuous function of  $x$  between the limits  $x = a$  and  $x = b$ , if for any value of  $x$  contained between these limits, the quantity  $f(x - \beta)$ , for ever decreasing values of  $\beta$ , approaches the limit  $f(x)$ .

**Theorem IV** If the series

$$f(\alpha) = v_0 + v_1\alpha + v_2\alpha^2 + \dots + v_m\alpha^m + \dots$$

converges for a certain value  $d$  of  $\alpha$ , it will also converge for every value smaller than  $d$  and, for this kind of series, for ever decreasing values of  $\beta$ , the function  $f(\alpha - \beta)$  will approach the limit  $f(\alpha)$ , assuming that  $\alpha$  is equal to or less than  $d$ .



We begin by examining the statement of Theorem IV. First notice that Abel was making two claims. First, he claimed that the infinite series  $f(\alpha)$  will “converge for every value smaller than  $d$ .” From our previous readings, we suspect that this is not to be taken literally. Let’s take his meaning on  $\alpha$  to be: for every  $\alpha$  value,  $0 \leq \alpha < d$  where  $d$  is positive.

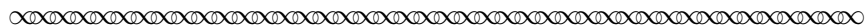
Abel’s second claim was that “the quantity  $f(\alpha - \beta)$ , for ever decreasing values of  $\beta$ , approaches the limit  $f(\alpha)$ ” for  $0 \leq \alpha \leq d$ .

**Task 54** Abel’s definition of continuity is essentially the same as Cauchy’s. To make this definition consistent with our modern definition, what do you think he meant by “for ever decreasing values of  $\beta$ ?” Rewrite his continuity claim with modern terminology. Be careful with the special case  $\alpha = d$ .

**Task 55** Explain how Abel’s Theorem IV is a variant of Cauchy’s Theorem I, page 14. Be sure to compare and contrast both the hypotheses and the conclusions.

**Task 56** As a first application of this theorem, consider the power series (18), which we showed converges for  $x = 1$  in Task 53. For what other  $x$  values does Abel’s Theorem IV guarantee convergence of this power series?

We next read Abel’s proof of his Theorem IV. As you read his proof, think about how you can adjust it for a modern proof. In particular, notice how he used the symbol  $\omega$ .



For brevity, in this memoir we will understand by  $\omega$  a quantity which may be smaller than any given quantity, however small.<sup>12</sup>

**Theorem IV** If the series

$$f(\alpha) = v_0 + v_1\alpha + v_2\alpha^2 + \dots + v_m\alpha^m + \dots$$

converges for a certain value  $d$  of  $\alpha$ , it will also converge for every value smaller than  $d$  and, for this kind of series, for ever decreasing values of  $\beta$ , the function  $f(\alpha - \beta)$  will approach the limit  $f(\alpha)$ , assuming that  $\alpha$  is equal to or less than  $d$ .

Suppose

$$\begin{aligned} v_0 + v_1\alpha + v_2\alpha^2 + \dots + v_{m-1}\alpha^{m-1} &= \varphi(\alpha), \\ v_m\alpha^m + v_{m+1}\alpha^{m+1} + v_{m+2}\alpha^{m+2} + \text{etc.} \dots &= \psi(\alpha), \end{aligned}$$

so

$$\psi(\alpha) = \left(\frac{\alpha}{d}\right)^m v_m d^m + \left(\frac{\alpha}{d}\right)^{m+1} v_{m+1} d^{m+1} + \text{etc.},$$

therefore, from Theorem III,  $\psi(\alpha) < \left(\frac{\alpha}{d}\right)^m p$  where  $p$  denotes the greatest of the quantities  $v_m d^m, v_m d^m + v_{m+1} d^{m+1}, v_m d^m + v_{m+1} d^{m+1} + v_{m+2} d^{m+2}$  etc. Therefore for every value of  $\alpha$ , equal to or less than  $d$ , one may take  $m$  large enough that one will have

$$\psi(\alpha) = \omega.$$

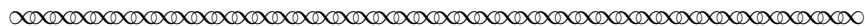
Now  $f(\alpha) = \varphi(\alpha) + \psi(\alpha)$ , so  $f(\alpha) - f(\alpha - \beta) = \varphi(\alpha) - \varphi(\alpha - \beta) + \omega$ . Further,  $\varphi(\alpha)$  is a polynomial in  $\alpha$ , so one may take  $\beta$  small enough that

$$\varphi(\alpha) - \varphi(\alpha - \beta) = \omega;$$

so also one has in the same way

$$f(\alpha) - f(\alpha - \beta) = \omega,$$

which it was required to prove.



**Task 57** What part of this proof is needed for Abel's first claim, that the infinite series  $f(\alpha)$  will converge for  $0 \leq \alpha < d$ ? On what variables does  $\omega$  depend for this part of his proof? How can we translate Abel's phrase "a quantity which may be smaller than any given quantity, however small" for a modern proof?

**Task 58** Abel let  $p$  denote the greatest of an infinite number of quantities. From a modern point of view, how would you critique this?

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<sup>12</sup>Abel put this remark into a footnote.

### 3.2 Modernizing Abel's proof that $f(\alpha)$ converges for $\alpha < d$

Notice that Abel used Theorem III in his Theorem IV proof with an *infinite* sum, the remainder term  $\psi(\alpha)$ , but the  $r$  in Theorem III involves a *finite* sum. Also observe that for a modern proof, we can't use his infinite series "tail"  $\psi(\alpha)$  until we know it converges. Moreover, we don't have a candidate for this series sum, so a modern proof will need to use the Cauchy criterion. For these reasons, let's introduce the notation

$$\begin{aligned}\varphi_m(\alpha) &= v_0 + v_1\alpha + v_2\alpha^2 + \cdots + v_{m-1}\alpha^{m-1} \\ \psi_{m,n}(\alpha) &= v_m\alpha^m + v_{m+1}\alpha^{m+1} + v_{m+2}\alpha^{m+2} + \cdots + v_{m+n-1}\alpha^{m+n-1}.\end{aligned}$$

**Task 59** Show that for arbitrary  $\alpha, m, n$  we have  $\psi_{m,n}(\alpha) = \varphi_{m+n}(\alpha) - \varphi_m(\alpha)$ .

We need to adjust Abel's  $p$  definition for our modern proof with the Cauchy criterion as follows:

$$P_m = \sup \{ |v_m d^m + v_{m+1} d^{m+1} + v_{m+2} d^{m+2} + \cdots + v_{m+n-1} d^{m+n-1}| : n \in \mathbb{N} \}$$

Observe that  $P_m$  depends only on  $d, m$  and the coefficients  $v_k$ .

**Task 60** Let  $\epsilon > 0$ . Prove there exists  $N \in \mathbb{N}$  such that  $P_m < \epsilon/3$  for all  $m \geq N$ , using Task 22 and the Theorem IV hypotheses.

**Task 61** Let  $m, n \in \mathbb{N}$  and  $0 \leq \alpha < d$ . Use Abel's ideas from his Theorem IV proof and your "absolute value" version of Theorem III from Task 50 to show that

$$|\psi_{m,n}(\alpha)| \leq \left(\frac{\alpha}{d}\right)^m P_m.$$

Be sure to make clear which factors in  $\psi_{m,n}(\alpha)$  correspond to which  $c_k$  and  $t_k$  values in Theorem III.

**Task 62** Let  $\alpha$  be fixed with  $0 \leq \alpha < d$ . Use the task results above and the Cauchy criterion to give a modern  $\epsilon - N$  proof that the sequence of partial sums  $\{\varphi_m(\alpha)\}$  converges. We will call the sum  $f(\alpha)$ , in keeping with Abel's name for this convergent infinite series.

Task 62 gives us a modern proof that the infinite series  $f(\alpha)$  in Theorem IV will converge for  $0 \leq \alpha \leq d$ . Now we tackle the second part of Abel's proof regarding the continuity of  $f$ .

### 3.3 Modernizing Abel's proof that $f(\alpha)$ is continuous for $\alpha \leq d$

To prove continuity with modern terminology, given  $\alpha \leq d$  and  $\epsilon > 0$  we need to find a  $\delta > 0$  so that  $|\beta| < \delta$  implies that  $|f(\alpha - \beta) - f(\alpha)| < \epsilon$ . Observe that Abel used the symbol  $\omega$  four times. Recalling his statement "we will understand by  $\omega$  a quantity which may be smaller than any given quantity, however small," we can see that he might not have meant for these four  $\omega$ 's to be literally identical.

**Task 63** On what variables does  $\omega$  depend for the last three times he used it?

Now that we know the infinite series  $f(\alpha)$  converges from the first part of the proof, we can safely use Abel's infinite term remainder  $\psi(\alpha)$ , but we must take care to remember it depends on  $m$  as well as  $\alpha$ . When this is particularly important, we can use  $\psi_m(\alpha)$  in place of  $\psi(\alpha)$  for emphasis.

**Task 64** Let  $\alpha$  be fixed with  $0 \leq \alpha < d$ . Explain why  $\psi_m(\alpha) = \lim_{n \rightarrow \infty} \psi_{mn}(\alpha)$ .

The next task will help us modernize Abel's claim that " $f(\alpha) - f(\alpha - \beta) = \varphi(\alpha) - \varphi(\alpha - \beta) + \omega$ ."

**Task 65** Let  $m$  be arbitrary. Show that

$$|f(\alpha) - f(\alpha - \beta)| \leq |\varphi_m(\alpha) - \varphi_m(\alpha - \beta)| + |\psi_m(\alpha)| + |\psi_m(\alpha - \beta)|.$$

For what  $\beta$  values is this valid?

We need to convert Abel's  $\omega$  statements into appropriate  $\epsilon - \delta$  statements. We need an  $\epsilon$  bound on  $|\psi_m(\alpha)|$  and another bound on  $|\varphi(\alpha) - \varphi(\alpha - \beta)|$ . Less obviously, we need a bound on  $|\psi_m(\alpha - \beta)|$ , which Abel absorbed into one of his  $\omega$ 's in the claim  $f(\alpha) - f(\alpha - \beta) = \varphi(\alpha) - \varphi(\alpha - \beta) + \omega$ .

It turns out that the trickiest of these three bounds is for  $|\psi_m(\alpha - \beta)|$ , because we need the bound to work *for all*  $\beta$  with  $|\beta| < \delta$ , not just a single  $\beta$ . To get this bound, look at your definitions of  $\psi_{m,n}(\alpha)$  and  $P_m$  just before Task 60. Notice  $P_m$  does not depend on  $n$  or  $\alpha$ ! In fact observe that

$$|\psi_{m,n}(\alpha)| \leq \left(\frac{\alpha}{d}\right)^m P_m \tag{22}$$

for all  $n$  and for any  $\alpha$ ,  $0 \leq \alpha \leq d$ .

**Task 66** For a given  $\epsilon > 0$  and  $0 \leq \alpha \leq d$ , find  $N \in \mathbb{N}$  so that  $m \geq N$  and  $0 \leq \alpha - \beta \leq d$  imply that

$$|\psi_m(\alpha - \beta)| < \frac{\epsilon}{3}$$

The bound (22) and Task 60 should be helpful!

**Task 67** Using a modern  $\epsilon - \delta$  argument with your results from the past few tasks, rewrite Abel's proof that the infinite series  $f(\alpha)$  is continuous at each  $\alpha$  for  $0 \leq \alpha \leq d$ .

Hint: Remember that  $\varphi_N(\alpha)$  is a polynomial, hence continuous.

Congratulations, you have now worked your way through a very difficult and important theorem! If you struggled at times, you are not alone. In 1863, the highly accomplished mathematician J. Liouville (1809–1882) published a different proof of this theorem that he attributed to his friend J. Dirichlet (1805–1859) [Liouville, 1863]. Liouville stated in his article that he confessed to Dirichlet his confusion with Abel's proof, and Dirichlet then gave his alternative explanation. Unfortunately, Dirichlet died before he could publish his proof.

Now let's use Abel's results to analyze where a function defined as an infinite series is continuous.

**Task 68** Define a function  $f$  by

$$f(x) = x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \frac{x^5}{\sqrt{5}} - \dots$$

Use Abel's theorems to find  $x$  values at which  $f$  is continuous.

Hint: Recall Task 53.



## 4 Conclusion

We have examined Cauchy's Theorem I and Abel's response to it using the series

$$\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \quad (23)$$

on a neighborhood of  $x = \pi$ . Part of the difficulty for Cauchy was that mathematicians had not yet worked out the issues for an infinite series involving a variable  $x$  converging at a *single*  $x$  value, versus converging *uniformly* for a *set* of  $x$  values. Here is a modern definition of *uniform* convergence for a power series on a set  $A$ .

**Definition 3.** Suppose  $\sum_{k=0}^{\infty} v_k x^k$  is a power series, and define partial sum  $\varphi_m(x) = \sum_{k=0}^{m-1} v_k x^k$  for each  $m \in \mathbb{N}$ . We say the series converges **uniformly** to  $f(x)$  on set  $A \subset \mathbb{R}$  if for each  $\epsilon > 0$  there is a natural number  $K(\epsilon)$  such that if  $n \geq K(\epsilon)$  then

$$|f(x) - \varphi_n(x)| < \epsilon \quad \text{for all } x \in A.$$

Note in particular that  $K(\epsilon)$  is independent of the points  $x$  in  $A$ .

It turns out that Cauchy's Theorem I is valid if we insist on *uniform* convergence of his series functions  $u_n(x)$  on a set  $A$ . Part of the success of Abel's Theorem IV proof is that it actually shows uniform convergence on the set  $[0, d]$ , even if Abel didn't explicitly say so in 1826. Indeed, the terminology for uniform convergence did not exist at the time.

**Task 69** Carefully review your modern proof of the power series convergence on  $[0, d]$  for Abel's Theorem IV, and explain why the convergence is uniform.

**Task 70** Prove that the geometric series converges uniformly to  $\frac{1}{1-x}$  on the set  $A = \left[-\frac{1}{3}, \frac{1}{3}\right]$ .

Hint: Use Cauchy's expression (6) for the finite geometric series.

Often a series will converge "pointwise" at each  $x$  in a set, but will fail to converge uniformly.

**Task 71** In this task you will show that the geometric series does **not** converge uniformly to  $\frac{1}{1-x}$  on  $(-1, 1)$ .

(a) Carefully write out the negation of the uniform convergence definition for the geometric series on  $(-1, 1)$ .

(b) Prove that the convergence of the geometric series to  $\frac{1}{1-x}$  on  $(-1, 1)$  is not uniform.

Hint: Consider the sequence  $\left((1/2)^{1/n}\right)$ , which converges to 1.

As you might guess, the trigonometric series (23) does not converge uniformly to  $x/2$  on set  $(-\pi, \pi)$ . Moreover, it turns out that conditions for continuity and convergence of power series are quite a bit different than conditions for continuity and convergence of a series of trigonometric functions. These challenges would keep mathematicians busy in the decades after Cauchy's *Cours d'analyse* and Abel's 1826 paper.

Regarding the binomial series, Abel went on in his 1826 paper to prove a number of rigorous convergence results for complex numbers  $x$ , which are outside the scope of this project.

Sadly, Abel would only have a few more years to work on mathematics, for he contracted tuberculosis while on his Paris visit, and died in 1829 at the age of 27. Nevertheless, in his short lifetime he did an amazing amount of first class mathematics for which he has been much celebrated.

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## Notes to Instructor

### PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed for a course in Real Analysis. It starts with Abel's concerns about infinite series in general and the amazing Fourier series  $x/2 = \sum (-1)^k \sin(kx)/k$  to give the students a quick hook into the topic of infinite series and immerse them into the problems of the day. Then the project moves through the first section on infinite series in Cauchy's *Cours d'Analyse*, which reads a lot like most current analysis text introductions to infinite series. However, he doesn't quite phrase things in terms of  $\epsilon$ - $N$  language, so there are plenty of challenging details for students to iron out. Cauchy finishes the section with a famous "near miss" Theorem I on the continuity of an infinite series of continuous functions. We return to Abel, who gave a counterexample to this theorem.

In the next section of the project, students work through several of Abel's results, including a slightly generalized ratio test, and then a clever rearrangement theorem which is often referred to as Abel's Lemma in modern books. This theorem is used, via some tasks, to prove the Alternating Series Test from Introductory Calculus. Abel's results are then used to prove his Theorem IV on power series convergence, now often referred to as one of Abel's theorems on infinite series, which is a partial patch to Cauchy's problematic Theorem I.

From this point, it would be natural for a analysis class to move into a discussion of power series and intervals of convergence, and the various convergence tests not covered in this project, such as the root test. The project also motivates the need to pin down the concept of *uniform* convergence — the fundamental source of problems with Cauchy's Theorem I. This topic is discussed briefly in the project conclusion.

The specific content goals of this project are:

1. Develop a modern convergence definition with quantifiers for infinite series based on Cauchy's definition.
2. Analyze convergence for geometric and harmonic series using Cauchy's arguments.
3. Develop and apply modern versions of the "Divergence Test," the "Comparison Test" and a Cauchy condition for convergence based on Cauchy's work.
4. Develop and apply modern versions of the "Ratio Test" and the "Alternating Series Test" based on Abel's work.
5. Develop modern proofs of Abel's theorem on convergence and continuity of power series based on Abel's proofs.

### Student Prerequisites

Students should have already done a rigorous study of sequences, limits, and continuity for real-valued functions. In particular, students should know the Cauchy criterion for sequences. This equivalence between Cauchy and convergent sequences in  $\mathbb{R}$  is treated as obvious by Abel and Cauchy in this project's excerpts from their writing, and the project doesn't dwell too much on this.

## PSP Design and Task Commentary

This project consists of four sections.

### Section 1 Introduction

The discussion and examples here are largely to provoke thought in the students and motivate the need for a systematic approach to series. Euler's series (1), stimulated by the vibrating string controversy, is a lovely gateway into Fourier series. A complete investigation of this topic deserves plenty of time and is not the focus of this project. This series exposes a problem with Cauchy's Theorem I that we meet at the end of Section 2.

### Section 2 Cauchy

Task 9 is important for giving students some historical perspective. There was no universal terminology or notation for the magnitude/absolute value of a real number in the 1820's, and both Abel and Cauchy need to be read with this in mind. There are a number of tasks sprinkled throughout the section establishing basic series properties that are not necessary for the flow of Cauchy's arguments, but which are useful elsewhere and provide good homework practice.

Many students struggle with the distinction between series and sequences, and Cauchy's initial expression of series as a sequence  $u_0, u_1, u_2, \dots, u_n, u_{n+1}, \dots$  in his first excerpt may confuse some students. The discussion and task immediately after this excerpt are designed to clear up any confusion about this series naming convention of Cauchy's. On the other hand, instructors may wish to point out that this convention allows Cauchy to talk about divergent series without following into the trap of writing

$$u_0 + u_1 + u_2 + \dots$$

Cauchy does start writing the sum of *convergent* series this way in the fourth excerpt.

Task 35 is difficult, but some wrestling with it should give students some insight into Cauchy's difficulty with uniform versus pointwise behavior. The task also motivates Abel's program, investigated in Section 3 of the PSP. However, the task is not intended to start a *comprehensive* treatment of convergence of function series, unless the instructor chooses to end the PSP after Section 2. For more on teaching with Cauchy's Theorem I and his proof attempt, see Fred Rickey's excellent article "Cauchy's Wrong Proof," available at <http://fredrickey.info/hm/CalcNotes/CauchyWrgPr.pdf>.

### Section 3 Abel

Abel did not need Theorems I and II to prove his Theorems III, IV, so their proofs can be treated lightly if desired. On the other hand, as suggested in Task 37, they provide a "backdoor" approach to the important Ratio Test. While Cauchy listed this theorem in Section 6.2 of his *Cours d'Analyse*, he did not give its proof there.

As noted in the discussion before inequality (22), the quantity  $p$  does not depend on  $n$  or  $\alpha$ , and in some sense this inequality captures the uniformity of convergence. Indeed, elements of Abel's proof of Theorem IV are very similar to the modern proof that a sequence of continuous functions, converging uniformly to a function  $f$ , result in a continuous limit function  $f$ . Abel tried to stretch his proof to the case where the  $v_m$  are continuous functions in a following Theorem V, but his proof fails due to the uniform convergence issue and lack of conditions on the  $v_m$  functions.

The rest of Abel’s proof for the Binomial Theorem involves copious algebraic manipulation of complex numbers, and is not addressed in this project.

## Section 4 Conclusion

A modern definition of uniform convergence is given and students do some tasks using the definition. They are asked to reflect on Cauchy’s Theorem I and Abel’s Theorem IV with regard to this uniform convergence concept, providing some synthesis of the ideas from the project. This can provide the instructor a launch into a full treatment of function convergence after the PSP.

## Suggestions for Classroom Implementation

The complete PSP is roughly a four week project, while the first two sections can be done in around two weeks, under the following methodology (basically David Pengelley’s “A, B, C” method described on his website <https://web.nmsu.edu/~davidp/>):

1. Students do some advanced reading and light preparatory tasks before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the tasks or your grading load will get out of hand! Some instructor have students write questions or summaries based on the reading.
2. Class time is largely dedicated to students working in groups on the project - reading the material and working tasks. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a “participation” grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the tasks they missed. This is usually a good incentive not to miss class!
3. Some tasks are assigned for students to do and write up outside of class. Careful grading of these tasks is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

L<sup>A</sup>T<sub>E</sub>Xcode of this entire PSP is available from the author by request to facilitate preparation of ‘in-class task sheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

## Sample Implementation Schedule (based on a 50-minute class period)

The complete PSP is roughly a four week project, while the first two sections can be done in around two weeks (ending with Day 6 below).

Students read through the introductory material and first excerpt and do Tasks 1–3 before the first class. After discussing their results at the beginning of Class 1, students read the second Abel excerpt, do Task 4, read the first Cauchy excerpt and work on Tasks 6, 7. Task 5 may be assigned for homework.

As preparation for Class 2, students do Tasks 8, 9. After discussing their results at the beginning of Class 2, students work on and discuss Tasks 10, 11, 13a. Tasks 12, 13b may be assigned for homework.

As preparation for Class 3, students read the second Cauchy excerpt on page 2 and do Task 14. After discussing their results at the beginning of Class 3, students work on and discuss Tasks 15, 16, 17, 18a. Task 18b,c can be assigned for homework.

As preparation for Class 4, students do Tasks 19,20. After discussing their results at the beginning of Class 4, students work on and discuss Tasks 21, 23. Then they read the third Cauchy excerpt and work on Tasks 25, 26, 27. Tasks 22, 24, 27 can be assigned for homework.

As preparation for Class 5, students read the third Cauchy excerpt and do Task 28. After discussing their results at the beginning of Class 5, students work on and discuss Tasks 29, 30, 31, 32, 33a. Task 33b,c can be assigned for homework.

As preparation for Class 6, students read the fourth Cauchy excerpt and do Task 34. After discussing their results at the beginning of Class 6, students work on and discuss Task 35 (this is hard and probably needs a brief class discussion led by the professor, but resist the temptation to give a full proof and discussion of uniform vs. pointwise behavior). Then students read the Abel excerpt on “exceptions” to Cauchy’s theorem, and work on Task 36. Task 36 can be assigned for homework.

As preparation for Class 7, students read the first Abel excerpt in Section 3 and do Task 37. After discussing their results at the beginning of Class 7, students read the next Abel excerpt from his Theorem II proof, and work on Tasks 38, 39, 40, 41. Task 41 can be assigned for homework.

As preparation for Class 8, students do Task 42. After discussing their results at the beginning of Class 8, students work on Tasks 43,44,45a. Task 45bc,46 can be assigned for homework.

As preparation for Class 9, students read the Abel excerpt on his Theorem III and do Task 47. After discussing their results at the beginning of Class 9, students work on Tasks 48, 49, 50, 51.

As preparation for Class 10, students do Task 52. After discussing their results at the beginning of Class 10, students do Task 53a and read the next Abel excerpt on his Theorem IV, and work on Tasks 54, 55, 56. Task 53b,c can be assigned for homework.

As preparation for Class 11, students read the Abel excerpt on his Theorem IV proof and do Tasks 57, 58. After discussing their results at the beginning of Class 11, students work on Tasks 59, 60, 61. Task 62 can be assigned for homework.

As preparation for Class 12, students do Tasks 63, 64. After discussing their results at the beginning of Class 12, students work on Tasks 65, 66, 67. Task 68 can be assigned for homework.

As preparation for Class 13, students read the first part of the PSP Conclusion, and do Task 69. After discussing their results at the beginning of Class 13, students finish the PSP.

## **Connections to other Primary Source Projects**

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be completed in at most 2 class periods are designated with an asterisk (\*). Classroom-ready versions of the last two projects listed can be downloaded from [https://digitalcommons.ursinus.edu/triumphs\\\_topology](https://digitalcommons.ursinus.edu/triumphs\_topology); all other listed projects are available at [https://digitalcommons.ursinus.edu/triumphs\\\_analysis](https://digitalcommons.ursinus.edu/triumphs\_analysis).

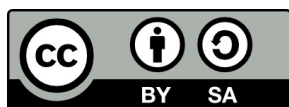
- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis\** (Janet Heine Barnett)
- *Investigations into Bolzano's Bounded Set Theorem* (David Ruch)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)  
Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit\** (David Ruch)  
A second version of this project suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Henri Lebesgue and the Development of the Integral Concept\** (Janet Heine Barnett)
- *Euler's Rediscovery of  $e^*$*  (David Ruch; sequence convergence, series & sequence expressions for  $e$ )
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor\** (Nicholas A. Scoville)  
Also suitable for use in a course on topology.
- *Topology from Analysis\** (Nicholas A. Scoville)  
Also suitable for use in a course on topology.

## Recommendations for Further Reading

The articles in [Jahnke, 2003] and [Bottazzini, 1986] give some perspective on other 19th century works in analysis.

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