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Nearness Without Distance

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Nearness without Distance

Nicholas A Scoville*

December 21, 2022

Topology is often described as a mathematical discipline which does not have a notion of distance, but does have a notion of nearness. At first glance, this seems like a distinction without a difference. How can things be close to each other without any concept of distance? In this project, we will see how three authors began to study phenomena that had a notion of a distance. We hope to show how their studies led to defining structures which have a notion of nearness but no distance, or, as we call them today, topological spaces.

We begin with an 1872 paper by German mathematician Georg Cantor (1845–1918), best known today as the father of modern set theory. Cantor was studying Fourier series, a way of expanding or approximating a given function with a series of trigonometric functions. While his study of such functions was restricted to the real number line, Cantor ultimately needed definitions and concepts which didn't utilize any notion of distance. This is the first way to begin thinking about nearness without distance. In the case of the real line, the distance is there and exists, but none of the concepts that Cantor used actually utilized any notion of distance. Once we have a concept that doesn't utilize all the structure we have, a natural question to ask is “how little structure can I assume while still having the concept make sense?” In other words, how bad does it have to get before it breaks? The next work we study looks at this question.

After Cantor, we move to a paper written by Emil Borel (1871–1956) in 1903, over thirty years after Cantor's work. Many concepts in topology at this point had been fairly well established. Yet this “nearness without distance” that Cantor constructed on the reals seemed like it could work for other kinds of things . . . lines, planes, etc. For Borel, as long as one is working with objects which have some way to define a distance between them, then he could make sense of, for example, the set of “limit points” of a collection of hyperplanes.

For a theory of distance, we next turn to excerpts from a textbook by Felix Hausdorff (1868–1942). In this text, Hausdorff developed a theory of distance by specifying properties that all distances must satisfy — properties which we now require a topological space to satisfy, as did Hausdorff himself at the very end of his text. From Cantor and a question of Fourier Series, we thus work through a sequence of pioneering ideas to ultimately end up at the modern day definition of a topological space or, more colloquially, a set with nearness but no distance.

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1 Cantor

We first give a little more general background on Cantor. He was born in Saint Petersburg, Russia in 1845. At the age of eleven, he moved to Germany. Cantor completed his education at Darmstadt, Zürich, Berlin, and Göttingen, before becoming a professor at University of Halle in 1869. He was the first mathematician to understand the mathematical meaning of “sizes of infinity,” which he defined in terms of a one-to-one correspondence. One of his notable mathematical discoveries involved using this concept to compare the infinite set of rational numbers with the infinite set of natural numbers, with the result that the infinite cardinalities of these two sets are actually equal.

1.1 Infinite Series

We now begin our survey of Cantor’s paper “Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen” (“On the Extension of a Theorem from the Theory of Trigonometric Series”), [Cantor, 1872].

In this paper, Cantor was studying the Fourier series representation of a function f , which has the form

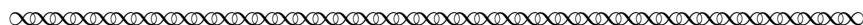
$$f(x) = \frac{1}{2}b_0 + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx)$$

where b_0, a_n, b_n are coefficients. Why would someone study such a strange series? You are probably already familiar with one kind of series. In a calculus course covering sequences and series, you were introduced to power series; that is, the idea that a function f may sometimes be written as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where a_n is some coefficient for each n . This transforms what could be a fairly complex function into a polynomial (albeit an infinite one) which allows you to approximate the function. Another reason such a form is desirable is because under reasonable hypotheses, one can integrate and differentiate the series term by term.

As mentioned above, Cantor was interested in studying Fourier series. Fourier series were a fairly new and extremely powerful tool in the 19th century, for which many applications were found in both physics and mathematics. For example, under reasonable hypotheses, one may differentiate a Fourier series term by term, just as one can a power series in calculus. Below is how Cantor explained the particular problem in the study of Fourier series that interested him.¹



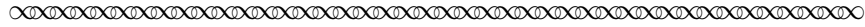
In the following, I will announce a certain extension of the theorem that trigonometric series representations are unique. [I have shown] that two trigonometric series

$$\frac{1}{2}b_0 + \sum (a_n \sin nx + b_n \cos nx) \text{ and } \frac{1}{2}b'_0 + \sum (a'_n \sin nx + b'_n \cos nx)$$

¹All translations of Cantor excerpts in this project, unless otherwise noted, were prepared by the project author, and reviewed by David Pengelley, New Mexico State University (retired), 2017.

which converge for every value of x and have the same sum, agree in their coefficients . . . I have also shown that the theorem holds if we give up either convergence or the agreement of the series sum for a finite number of values of x .

[Cantor, 1872, p. 123]



When Cantor wrote that “I have also shown” above, he was referring to a result of his from one of his earlier papers on Fourier Series, [Cantor, 1870].

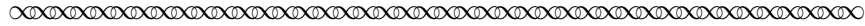
Task 1 What more familiar term do we use today instead of Cantor’s phrase “give up convergence”?

Throughout this section, we will assume f is a function defined on some subset of $[0, 2\pi]$.

Task 2 Let $P \subseteq [0, 2\pi]$ denote the set of points on which f is either undefined or gives up convergence. Give a precise statement of what Cantor has “also shown.”

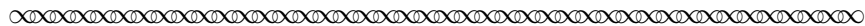
Task 3 Let $n > 0$ be a positive integer. Construct a function f_n which is defined on all of $[0, 2\pi]$ except for n points. Use your result from Task 2 to conclude that f_n , while undefined on n points, still has a unique Fourier series.

Cantor continued:



The extension proposed *here* asserts that the theorem remains valid even when the assumption of the convergence of the series or the agreement of the series sum fails for an *infinite* number of values of x in the interval $[0, 2\pi]$.

[Cantor, 1872, p. 123]



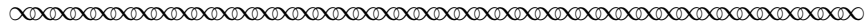
Task 4 Construct a function which is not defined on infinitely many points P for some subset $P \subseteq [0, 2\pi]$. Is there any reason to believe that the function you just defined has a unique Fourier series?

The example you just constructed may or may not have a unique Fourier Series. But as we can see, a function like the one you just defined is what Cantor had in mind. Certainly not all infinite sets P where the function is not defined will lend themselves well to such an extension. Interestingly, in order to carefully construct such sets P , Cantor first took somewhat of a detour into his own theory of constructing real numbers.

1.2 Real Numbers

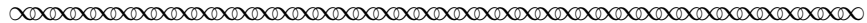
We tend to take the real numbers for granted, but upon some serious reflection, it can be difficult to say what exactly we mean by a real number. For example, what does it mean to add two real numbers? Mathematicians were still wrestling with a precise and rigorous answer to these and related questions in 1872. One definition of the real numbers, that of using a Dedekind cut, was due to Richard Dedekind (1831–1916). Another way to construct the real numbers involves the use of Cauchy sequences. In order to modify the proof in the paper mentioned above, [Cantor, 1870], Cantor amazingly developed his own theory of the real numbers based on this idea. Our purpose in examining his theory of real numbers is to demonstrate the emergence of point-set topology, since the conceptual distinction that Cantor made in his construction was essential to understanding this emergence.

Why would Cantor need a new theory of the real numbers in the first place? Wasn't such a theory existent at the time? According to Cantor's biographer Joseph Dauben, "Cantor was clearly aware of the shortcomings of previous attempts to devise a theory of irrational numbers" [Dauben, 1971, p. 204]. Indeed, Cantor wrote that²



There would be a logical mistake [in that approach to constructing the reals] because the sum $\sum a_v$ is defined by setting it equal to the already defined number b . I believe that this mistake, which was first avoided by Herr Weierstrass, was committed almost universally in earlier times and went unnoticed because it belongs to the rare cases where a real mistake causes no actual harm or error in calculation.³

[Cantor, 1883a, p. 566]



Let us illustrate a variant of Cantor's concern.

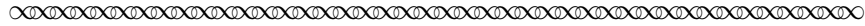
Task 5 Consider the number π .

- (a) How do you write π as a decimal? Use this expansion to write out a sequence of rational numbers that seems like it ought to converge to π ; that is, find a sequence of rationals $\{a_n\}$ such that for every $\epsilon > 0$ there is an integer N such that for all $n \geq N$, $|a_n - \pi| < \epsilon$.
- (b) Use Cantor's comments to explain how the sequence you found above does not actually prove that π exists.

²The translation of this excerpt was prepared by Jim Bisgard, Central Washington University, 2022.

³*Translator's Note:* Cantor wrote the word "Calcül" here. However, following the Orthographische Konferenz of 1901, many occurrences of the letter 'c' were replaced with 'k.' Thus, the corresponding word in "modern" German is "Kalkül" (or rarely "Kalcül"). This word may be translated as "calculus" or "calculation." However, caution should be taken with translating it as "calculus," since (at least for North American English speakers) calculus is typically synonymous with the differential and integral calculus, taught in the last year of high-school or the first year of university. Mathematics has several types of calculus: propositional calculus, Ricci calculus, the calculus of variations, etc., and in these, calculus refers to a set of rules for calculation in that context. Cantor may also have meant "calculus" in the sense that we currently use the word "analysis," since he is writing about the construction of the real numbers.

With this shortcoming in mind, we now turn to Cantor's construction. As we do so, be sure to consider how this avoids the error in the above task.



However, to achieve this end [the extension of the Fourier result] I have chosen to preface a discussion, albeit one consisting mostly of indirect references, that sheds light on certain relations that arise whenever a finite or infinite number of numerical values are given; I am thereby led to certain definitions that are included here only in order to present the theorem under discussion in the clearest possible form.

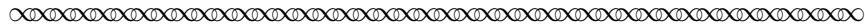
The rational numbers form the groundwork for the following discussion of number values⁴ [Zahlengroße]. I will call this domain [of rational numbers] A (with inclusion of zero).

When I speak of number values in the wider sense, I mean an infinite sequence of rational numbers

$$a_1, a_2, \dots, a_n, \dots \tag{1}$$

which has the property that the difference $a_{n+m} - a_n$ for large n becomes infinitely small, whatever the positive integer m . In other words, that for any (positive, rational) ϵ there exists an integer n_1 so that $|a_{n+m} - a_n| < \epsilon$ when $n \geq n_1$ and when m is any positive integer.

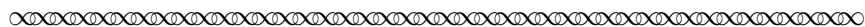
[Cantor, 1872, p. 124]



Cantor refers to any sequence (1) of rational numbers satisfying the above property a fundamental sequence.

Task 6 Give an example of a sequence of rational numbers which is a number value. Give an example of a sequence of numbers which is not a number value. For any rational number q , can you find a sequence which would seem to converge to q ?

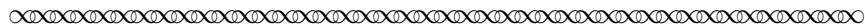
Cantor continued.



This property of sequence (1) I express in the words "*Sequence 1 has a certain limit b .*"

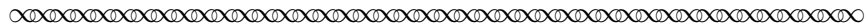
Now these words initially have no other meaning except as an expression for *those* properties of the sequence, and from the fact that we associate with sequence (1) a special character b , it follows that with various sequences, various characters b, b', b'', \dots are formed.

[Cantor, 1872, p. 124]



Immediately after making this definition, Cantor was quick to note that the number value b is simply a formal symbol associated to the fundamental sequence $a_1, a_2, \dots, a_n, \dots$. Notice how this avoids the error of assuming the existence of limits of sequences of real numbers. He simply associated a symbol to any such fundamental sequence. Next Cantor defined a total ordering.

⁴A more literal translation might be "numberness," but this sounds a little funny in English.



If a second sequence

$$a'_1, a'_2, \dots, a'_n, \dots \tag{2}$$

is given, which has a certain limit b' , one finds that the two sequences (1) and (2) always have one of the following 3 relations which exclude each other: either 1. $a_n - a'_n$ is infinitely small as n increases, or 2. from a certain n on, $a_n - a'_n$ always remains larger than a positive (rational) size ϵ , or 3. from a certain n on, $a_n - a'_n$ always remains smaller than a negative (rational) size $-\epsilon$

When the first condition takes place, I set

$$b = b',$$

in the second $b > b'$, in the third $b < b'$.

Similarly we find that sequence (1), which has a limit b , has only one of the following three relationships to a rational number a . Either 1. $a_n - a$ becomes infinitely small as n increases, or 2. beyond a certain n , $a_n - a$ always remains larger than a positive (rational) size ϵ , or 3. beyond a certain n , $a_n - a$ always remains smaller than a negative (rational) size $-\epsilon$.

To demonstrate the existence of this relation, we set

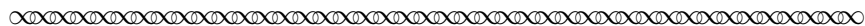
$$b = a, b > a, b < a$$

respectively.

These and the immediately following definitions have the consequence that, if b is a limit of sequence 1, then $b - a_n$ becomes infinitely small with growing n , whereby moreover the designation “ b is the limit of sequence (1)” for b finds a certain justification.

The totality of all number values b are denoted by B .

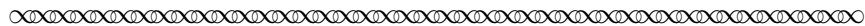
[Cantor, 1872, p. 124]



Task 7 Prove that the symbol “ $=$ ” is an equivalence relation on B .

Given Task 7, we will write B for the set of equivalence classes of number values under the relation $=$. Hence an element $b \in B$ is an equivalence class of fundamental sequences.

Now Cantor was ready to define the operations of addition, subtraction, multiplication and division in B . For conceptual purposes, we may think of this as defining said operations on all real numbers, but technically speaking we are defining how to define these operations on equivalence classes.



If b, b', b'' are three numerical quantities in B , the formulas

$$b \pm b' = b'', bb' = b'', \frac{b}{b'} = b'' \text{ [for } b' \neq 0]$$

serve as an expression that between the corresponding sequences

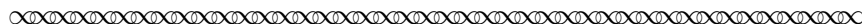
$$\begin{aligned} & a_1, a_2, \dots \\ & a'_1, a'_2, \dots \\ & a''_1, a''_2, \dots \end{aligned}$$

of the numbers b, b', b'' the respective relationships

$$\begin{aligned} \lim(a_n \pm a'_n - a''_n) &= 0 \\ \lim(a_n \cdot a'_n - a''_n) &= 0 \\ \lim\left(\frac{a_n}{a'_n} - a''_n\right) &= 0 \text{ [for } a'_n \neq 0]. \end{aligned}$$

hold.

[Cantor, 1872, p. 125]

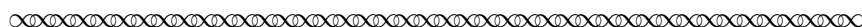


Task 8

Give a justification for why Cantor's definitions of addition, subtraction, multiplication, and division are appropriate.

1.3 Number Line versus Number Values

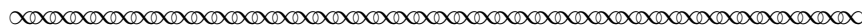
Now that Cantor had constructed a rigorous theory of the reals in which one can add, subtract, etc., he related these number values to the geometric number line (made up of points). Recall from the previous section that Cantor used the symbol A to denote the rational numbers, and that B denotes for us the set of equivalence classes of number values. Cantor wrote:



The points of a straight line are determined conceptually, by assuming a unit of measure and specifying their abscissas, i.e. their distances from a fixed point o of the straight line with $[a] +$ or $-$ sign, depending on whether the point in question lies in the (previously fixed) positive or negative part of the line from o .

If this distance has a rational relationship to the unit, then it is expressed by a number value of a domain A .

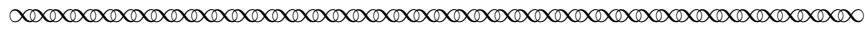
[Cantor, 1872, p. 127]



Task 9

Give a description of what Cantor was explaining in this excerpt using modern language and notation. What is o ?

There are, of course, other points on the number line, points that do not have a rational relationship with a fixed point. This is the "other case" that Cantor next discussed:

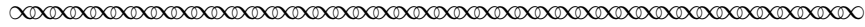


In the other case, if the point is known, for example by a construction, it is always possible to give a sequence

$$a_1, a_2, \dots \tag{3}$$

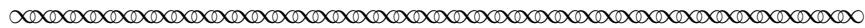
... [so that the] distance from the determined point to the point o is equal to b where b is the corresponding number value of sequence (3).

[Cantor, 1872, p. 127]



Task 10 Express Cantor’s assignment between the geometric line and number values of B in terms of a function.

Task 10 is one half of a bijection that Cantor then created between the geometric number line and set of number values. An assignment from the set of number values B to the geometric number line was taken by Cantor as an axiom. He wrote

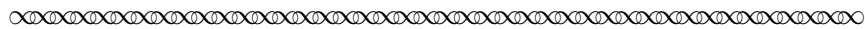


... to make the correspondence with the geometry of the straight line complete is only to add an axiom, which simply consists in [declaring that] to any number value there belongs a certain point of the straight line ...

...

I call this theorem an *axiom* because it is in its nature to not generally be provable.

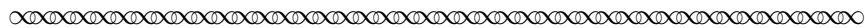
[Cantor, 1872, p. 128]



Task 11 Using the above, show that there is a bijection between the geometric number line and the set of number values B .

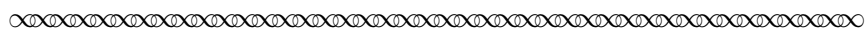
1.4 Topology in Cantor

Recall that at the end of the last section, we saw how Cantor defined a bijection between the points of what we now call the “real number line” and the objects of the set of number values B . Cantor further distinguished his bijection by assigning different names to corresponding objects.

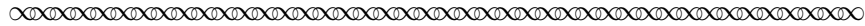


For the sake of brevity I call a given finite or infinite number of numerical values a *set of [numerical] values* and, correspondingly, a given finite or infinite number of points of a line, a *point-set*. Whatever is said of point-sets in what follows may be carried over directly, according to the aforesaid, to sets of values.

[Cantor, 1872, p. 128]



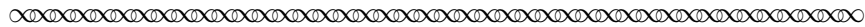
With this distinction in mind, Cantor was now ready to create definitions which describe relationships among point-sets.



If a point-set is given in a finite interval, a second point-set is generally given, and with these generally a third, etc., which are essential for the conception of the nature of the first point-set. . . . To define these derived point-sets, we must begin with the term *limit point*⁵ of a point-set.

By a limit point of a point-set P , I mean a point of the line such that there are infinitely many points of P in every neighborhood of it, and it may happen that it also belongs to the set itself. The neighborhood of a point means here any interval which has the point in its interior.

[Cantor, 1872, pp. 128–129]

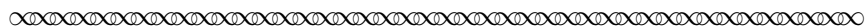


Task 12 Use Cantor’s definitions to give examples of each of the following:

- (a) a set which contains all of its limit points.
- (b) a set which has limit points not in the set.
- (c) a set whose only limit point(s) are not in the set.

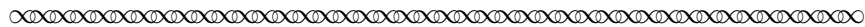
Task 13 What do you think Cantor meant by “in its interior”? Give a definition.

Cantor further commented that



Therefore, it is easy to prove that a [bounded] point-set consisting of an infinite number of points always has at least *one* limit point.

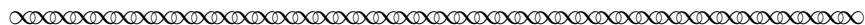
[Cantor, 1872, p. 129]



Task 14 Cantor did not give a definition of “bounded.” Provide such a definition and use it to prove that a bounded point-set consisting of an infinite number of points has at least one limit point.

Cantor next defined these other point-sets “which are essential to understanding the first set.”

⁵Cantor used the term “Grenzpunkt” (“boundary point”) in his 1872 paper; the French translation of that paper (published as [Cantor, 1883b]) used the term “point-set limite.” The term “limit point” has become standard in English.

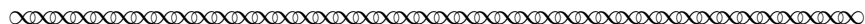


Every point of the line is now in a definite relation to a given set P , either being a limit point of P or not, and thereby along with the point-set P the set of limit points of P is a set which I wish to denote by P' and call the *first derived point-set of P* .

Unless the point-set P' contains only a finite number of points, it also has a derived set P'' , which I call the *second derived point-set of P* . By ν such transitions⁶ one obtains the concept of the ν th derived set $P^{(\nu)}$ of P .

It may happen — and this is the case we are exclusively interested in at present — that after ν transitions the set $P^{(\nu)}$ consists of a finite number of points, and hence has no derived set; in this case we wish to call the original point-set P a set of type ν , so that P', P'', \dots are of types $\nu - 1, \nu - 2, \dots$.

[Cantor, 1872, p. 128]



Task 15 Compute the derived sets for your examples in Task 12.

Task 16 Continue to practice computing the derived set by finding the derived set P' when

- (a) $P = [0, 1]$
- (b) $P = (0, 1]$
- (c) $P = (0, 1)$
- (d) $P = \{.2, .3\}$
- (e) $P = \mathbb{Q} \cap [0, 1]$
- (f) $P = \{\frac{1}{n} : n = 1, 2, 3, \dots\}$
- (g) $P = \{m + \frac{1}{n+1} : m, n \in \mathbb{Z}^+\}$

Note that Cantor himself did not seem to think that a finite set has a derived set, as he stated in the above excerpt. Today, we would say that the derived set of a finite set is the empty set.

Task 17 If P has finitely many elements, show that $P' = \emptyset$.

⁶In his 1872 paper, Cantor restricted the value of ν to finite integers only. At the time he wrote that paper, however, he had already realized that, in the case where $P^{(\nu)}$ is a non-empty set for every finite integer ν , he could extend the notion of “type” for derived sets beyond the finite. To do this, he set $P^\infty = \bigcap_{\nu=1}^\infty P^{(\nu)}$ to obtain a derived set of type ∞ . He then continued the iterative process to obtain $P^{\infty+1} = (P^\infty)'$, $P^{\infty+2} = (P^{\infty+1})'$, and so on. This process could be extended to even higher orders of derived sets, such as $P^{\infty\infty+1}$, $P^{\infty n}$, $P^{\infty\infty}$, $P^{\infty\infty\infty}$, etc. Cantor later substituted the symbol ‘ ω ’ for the ‘ ∞ ’ symbol, to distinguish the actually-infinite ordinals $\omega, \omega + 1, \omega + 2, \dots$ from the concept of potential infinity associated with the ∞ symbol in calculus.

Cantor eventually connected his study of the series of “transfinite ordinals” $\omega, \omega + 1, \omega + 2, \dots$ associated with an ordered iterative process to his use of one-to-one correspondences between two sets as a means to measure their relative sizes, or cardinalities. This led him to introduce an unbounded sequence of “transfinite cardinals,” denoted (by Cantor and today) as $\aleph_0, \aleph_1, \aleph_2, \dots, \aleph_\omega, \aleph_{\omega+1}, \dots$. Here, \aleph_0 is the cardinality of the set of natural numbers \mathbb{N} , and also that of the equally-large set of rational numbers \mathbb{Q} . The problem of determining the cardinality of the set of real numbers \mathbb{R} preoccupied Cantor throughout much of his later life. The conjecture that \mathbb{R} has cardinality \aleph_1 , known as Cantor’s Continuum Hypothesis, continues to be of interest in set theory today. In this way, a question in analysis led not only to the development of point-set topology, but also to today’s modern set theory.

For more information about Cantor’s development of set theory, see [Dauben, 1979].

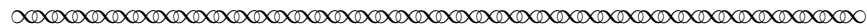
1.5 Cantor's Main Result

Recall from Task 2 that Cantor used P to denote the set of points on the interval $[0, 2\pi]$ on which a function f is either undefined or gives up convergence. His primary goal in [Cantor, 1872] was to prove that, under certain conditions on P , the Fourier series of f is unique. In particular, he was interested in the case where P is an infinite set. In order to state and prove his main result for this case, Cantor first constructed a new function F associated with f . It does not concern us here what this new function F is. The following theorem, stated in modern notation and language, summarizes the essential properties of F that we will need to understand in order to prove Cantor's main result.

Theorem 1 (Cantor)

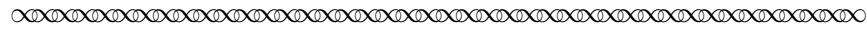
Let f be a function on $[0, 2\pi]$. There exists a function F , based on f , which is continuous on $[0, 2\pi]$. Furthermore, if F is linear on all of $[0, 2\pi]$, then the Fourier series for f is unique.

With this theorem in hand, showing f has a unique Fourier series has been kicked back to showing that the function F is linear. Fortunately, Cantor gave us a practical way to show this.



(A) *If there is an interval (p, q) in which only a finite number of points of the set P lie, then $F(x)$ is linear in this interval.*

[Cantor, 1872, p. 131]



How can we use (A) to show that F is linear even when we give up convergence or are not defined on an infinite set?

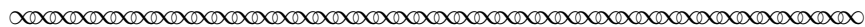
Task 18 Suppose f gives up convergence on $P := \{\frac{1}{n} + 1 : n = 1, 2, 3, \dots\} \cup \{\frac{1}{n} + 2 : n = 1, 2, 3, \dots\}$.

- (a) Use result (A) to show that F is linear on all but a finite number of points of $[0, 2\pi]$. You will need to use the fact that F is the *same* linear function on each interval, a fact that Cantor himself proved in general.
- (b) Argue that the first derived set, P' , is the set of points of $[0, 2\pi]$ for which we can't (yet) conclude that F is linear. Then compute P' .
- (c) Use the fact that F is continuous on all of $[0, 2\pi]$ along with (A) to conclude that F is linear on *all* of $[0, 2\pi]$.

The result you showed in Task 18 is the basic idea behind Cantor's main result. Even though P was infinite, it was relatively easy to apply result (A) to prove that F is linear on all of $[0, 2\pi]$.

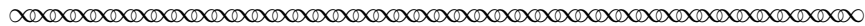
It turns out that F is linear on $[0, 2\pi]$ whenever P is a point-set of the n^{th} kind for some finite integer n . We will prove this by induction on n . For $n = 0$, we have that $(0, 2\pi)$ contains a finite number of points of P , so by (A), F is linear on $(0, 2\pi)$. Because F is continuous on all of $[0, 2\pi]$, it then follows that F is linear on the endpoints 0 and 2π as well. Hence F is linear on all of $[0, 2\pi]$.

Cantor also established the $n = 1$ case; to do so, he first showed that

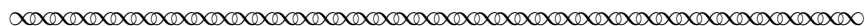


(A') If (p', q') is any interval in which only a finite number of points of the set P' lie, then $F(x)$ is linear in this interval.

[Cantor, 1872, p. 131]

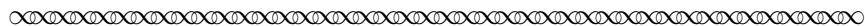


In this case, P' is finite by supposition, so that any subinterval of (p', q') contains at most a finite number of points $x'_0, x'_1, \dots, x'_v \in P'$, where $x'_0 < x'_1 < \dots < x'_v$. We now quote Cantor's argument, in which he used the notation (x'_0, \dots, x'_1) to mean the interval that we would today denote (x'_0, x'_1) .

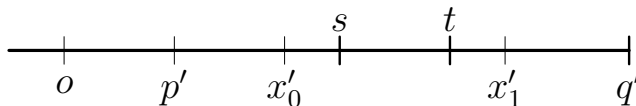


Each of these subintervals generally contains infinitely many points of P so that result (A) does not directly apply; however each interval (s, t) that falls within (x'_0, \dots, x'_1) contains only a finite number of points from P (otherwise other points of the set P' would fall between x'_0 and x'_1), and the function is therefore linear on (s, t) because of (A). The endpoints s and t can be made arbitrarily close to the points x'_0 and x'_1 so that the continuous function $F(x)$ is also linear in (x'_0, \dots, x'_1) .

[Cantor, 1872, p. 131]



Cantor illustrated the situation described in this excerpt with the following picture:



He then noted that it follows, from the argument in this last excerpt and the continuity of F , that F is linear over all of (p', q') . (*Do you see why? If not, look back at Task 18.*) In the case where $(p', q') = (0, 2\pi)$, invoking the continuity of F once more allows us to then conclude that F must be linear over all of $[0, 2\pi]$.

Cantor easily adapted this strategy to a proof that, for an arbitrary finite integer n , the function F is linear on $[0, 2\pi]$ even when convergence fails on a point-set P of the n^{th} kind; that is, when $P^{(n)}$ is finite and $P^{(n+1)} = \emptyset$. The key step in this general proof was to show that the following holds for every finite integer n :

(A⁽ⁿ⁾) If $(p^{(n)}, q^{(n)})$ is any interval in which only a finite number of points of the set $P^{(n)}$ lie, then $F(x)$ is linear in this interval.

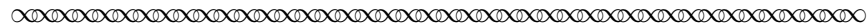
Task 19 Using an argument similar to Cantor's for the $n = 1$ case, assume the inductive hypothesis for $(A^{(n)})$ and prove the inductive step.

In the case where $(p^{(n)}, q^{(n)}) = (0, 2\pi)$, combining $(A^{(n)})$ with Theorem 1 allows us to immediately see that f has a unique trigonometric representation on $[0, 2\pi]$ even when convergence is given up on a point-set P of the n^{th} kind for a finite integer n . The main point of this section, however, is not so much this result, but the mathematics Cantor needed to build the result. Let us summarize what

he did. Cantor needed a structure-preserving bijection between points and sets. This relationship turned out to be that of a limit point, and the set of all limit points was the derived set. Since this is all based on open intervals, we don't really think about distance anymore, although the open set in the case of \mathbb{R} is defined using the distance. From here, one can ask two questions. First, could we make sense of limit point, derived set, etc. if we had a distance on some structure other than the real number line? Second, what if we took as an axiom that we had some notion that allowed us to define limit points, some system that did not begin with a distance? The latter question will be taken up by Hausdorff in section 3, while the former will be the work of Borel in the next section.

2 Borel

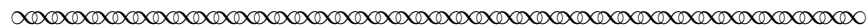
We now turn to a very short but very influential paper written by Émile Borel: “Quelques remarques sur les ensembles de droites ou de plans” (“Some remarks on sets of lines or planes”), [Borel, 1903]. Borel was a French mathematician who made many contributions to both measure theory (analysis) as well as probability. In addition to a successful career as a mathematician, he was active in politics and a member of the French resistance during World War II. In his 1903 paper, Borel took some of Cantor's ideas to the next level. Namely, he abstracted away the need to view point-sets as only “set of points” by considering sets of lines, planes, and other geometrical objects. He set the stage for this abstraction nicely.⁷



The notion of a *set of points* is today quite standard; it seems that we are less accustomed to considering sets whose elements are other geometric elements;[†], however, some of these sets, for example the set of lines in the plane or of planes in space, appear in lots of research and their systematic study, which is moreover easy, is almost as useful as the study of sets of points. Having been led to use such sets in a recent *Memoir*, I would like to indicate here some very simple and very elementary properties, which seem to me to be of a nature that may be able to render service in [the study of] many questions.

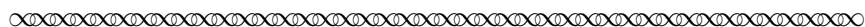
[†]*Borel's footnote:* Of course, I do not mean that such sets have never been considered; but their introduction does not appear to be standard; that is to say that we do not usually speak of them without a preliminary explanation, as we do with sets of points.

[Borel, 1903, p. 272]



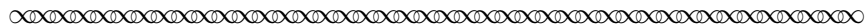
We will follow Borel's new definitions. As we do so, we will be interested not only in how they applied for Borel, but also their interpretation in terms of Cantor's point-sets.

⁷All translations of Borel excerpts in this section were prepared by Janet Heine Barnett, Colorado State University Pueblo, 2022.



First of all, it is important to define what we mean by *lines infinitely close* to a given line; we will adopt the following *geometric* definition: *Given a fixed line D , the variable line D' is said to be infinitely close to D if, any two points A and B being chosen on D , we can for each positive number ϵ find a position of D' such that the distance to [that position of] D' from each of the points A and B is less than ϵ .* It is easy to see that the choice of the points A and B on D can be made arbitrarily; if the definition is verified with a particular choice of these two points, it is verifiable with every choice of these two points, provided that they are distinct.

[Borel, 1903, p. 272]



Borel's phrase "the variable line D' " requires a bit of unpacking. In other words, what is a meaning of this phrase that might make sense?

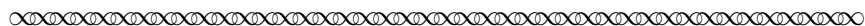
Task 20 Suppose that by "the variable line D' ," Borel simply meant a fixed line. If so, explain what D' could be. Why is this not an interesting definition?

Task 21 Give a definition of "the variable line D' " using modern language that makes Borel's definition interesting. [See Task 22 for a hint.]

Task 22 (a) Show that the collection $\{mx + b : m > 0, b \in \mathbb{R}\}$ is infinitely close to the x -axis.
 (b) Let $r > 0$. Show that the collection $\mathcal{F}_r = \{mx + b : m \geq r, b \in \mathbb{R}\}$ is not infinitely close to the x -axis.

Task 23 Why did Borel specify that two distinct points must be chosen? Give an example to show that specifying only one point is undesirable.

Task 24 Use your definition from Task 21 to adapt Borel's definition of "a fixed line D being infinitely close to variable line D' " to define what it means for a fixed point to be infinitely close to a variable point. Compare your definition with Cantor's definition of a limit point. Show they are equivalent.



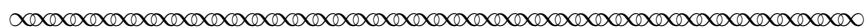
Similarly, *given a fixed plane P , the variable plane P' will be said to be infinitely close to P if, any three non-collinear points A, B, C being chosen on P , we can for each positive number ϵ find a position of P' such that the distance to [that position of] P' from each of the points A, B, C is less than ϵ .*

[Borel, 1903, p. 273]



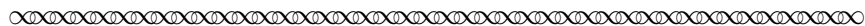
As we are interested in generalizing, it will not be difficult to adapt Borel's above definition to any dimension.

Task 25 Let H be an $(n - 1)$ dimensional hyperplane in n dimensions. Give a definition of what it means for H to be "infinitely close" to the variable hyperplane H' .

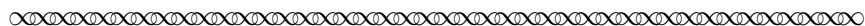


Given a set of lines (or planes), *the derived set* is, by definition, the set of lines (or planes) such that there are lines in the [given] set that are infinitely close to these lines. We will say that a set is *closed* if it contains all the elements of the derived set and that it is *perfect* if it is identical with the derived set.

[Borel, 1903, p. 273]

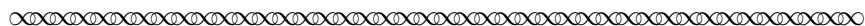


Task 26 Suppose A is a subset of the real line. Apply Borel’s definition of closed to A . In other words, replace “lines” with “points” above and compare with Cantor’s definition. Give examples of several types of closed sets of \mathbb{R} according to Borel’s definition of “closed.” What do some closed subsets of \mathbb{R} look like? Is this consistent with your understanding of closed sets in \mathbb{R} ?



A set is said to be *bounded* when, given some point O , there exists a number A such that the distance from the point O to any line (or plane) in the set is less than A ; obviously, if this property is verified for a point O , then it is verified for all the points in the space (A may vary with the selected point).

[Borel, 1903, p. 273–274]



Task 27 Show that “if this property is verified for a point O , then it is verified for all the points in the space.”

- Task 28**
- (a) Let S_1 be the set of all horizontal lines in the plane with integer y -intercepts. Prove that S_1 is unbounded and has no limit points (i.e., $S'_1 = \emptyset$).
 - (b) Let S_2 be the set of all horizontal lines $y = b$ where $-1 \leq b \leq 1$. Prove that S_2 is bounded and each line in S_2 is a limit point and thus in its derived set.

It should now be clear that Borel was extending Cantor’s definitions for sets of points to sets of lines and planes. But why stop at lines and planes? Why not circles, triangles, or other geometric objects?

Task 29 Let $S_n := \{\vec{x} \in \mathbb{R}^2 : \|\vec{x}\| = \frac{1}{n}\}$ for $n = 1, 2, 3, \dots$ and $S := \bigcup_{n=1}^{\infty} S_n$.

- (a) Draw a picture of S_1, S_2, S_3 , and S_4 .
- (b) Make a conjecture as to what geometric object this set S “ought to be infinitely close to.”
- (c) Give a definition of what it would mean for any $A \subseteq \mathbb{R}^2$ to be infinitely close to a collection \mathcal{A} of sets in \mathbb{R}^2 , and use it to prove your assertion in b).

As you have seen, now we have circles infinitely close to points. In all cases, this requires computing distances between points. As can be inferred from Task 25, this could be cumbersome for high dimensions. Thus it stands to reason that we could benefit from a notion not of distance between *individual points* in the sets, but a distance between the sets themselves.

Yet this is not as easy as it first seems. We may be tempted to define distance as “the minimum (or infimum) value over all pairs of points.” But consider first the two lines $y = 1$ and $y = 1.1$. Now consider $y = |x|$ and the x -axis. Intuitively, which of the two sets of lines ought to be closer to each other? Most likely any good notion of distance should say that $y = 1$ and $y = 1.1$ are very close to each other while $y = |x|$ and x -axis are not close to each other. But if we use the above definition, we would get that there is no distance between $y = |x|$ and the x -axis. (*Verify this!*) Clearly more care is needed.

Finally, in an attempt to further generalize Borel’s work, we wish to leave \mathbb{R}^n and work in any setting with a distance. Our reading of Hausdorff will help guide us through this.

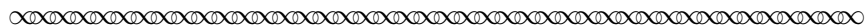
3 Hausdorff

Born in 1868 to Jewish parents, Felix Hausdorff is known as one of the founders of modern topology. Hausdorff studied mathematics and astronomy mainly in the city of Leipzig. He graduated from the University of Leipzig in 1891 and completed his habilitation thesis (a second doctoral degree required to teach at the university level in Germany) there in 1895. He taught at the University of Leipzig until 1910, when he accepted a position at the University of Bonn. Hausdorff continued to teach as a university professor, primarily in Bonn, until he was forced to retire by the Nazi regime in 1935. After being informed that they were to be placed in an internment camp, Hausdorff committed suicide, together with his wife and his wife’s sister, in 1942.

In addition to his contributions to point-set topology, Hausdorff made significant contributions to set theory, measure theory, and functional analysis. He developed, for example, the concepts of Hausdorff spaces, metric spaces and topological spaces. The development of the idea of closeness independent of the ability to be measured also interested Hausdorff.

3.1 Metric Spaces

In this section, we study ideas that Hausdorff first presented in his highly influential work *Grundzüge der Mengenlehre (Fundamentals of Set Theory)* [Hausdorff, 1914]. This was one of the first book-length treatments of topology, and remains a classic to this day. Later, Hausdorff published a significantly revised version of it — essentially an entirely new book — that appeared in two editions (in 1927 and 1935) under the title *Mengenlehre (Set Theory)*. As we read excerpts from these works, Hausdorff will guide us through the different viewpoints by which we might view topology.⁸



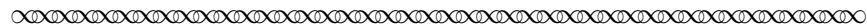
Set theory has celebrated its loveliest triumphs in the application to point-sets in space, and in the clarification and sharpening of foundational geometric concepts, which will be

⁸The translation of this excerpt from [Hausdorff, 1914] was prepared by David Pengelley, New Mexico State University (retired), 2017. All other Hausdorff excerpts in this section are taken (with minor changes by the project author) from the published English translation, [Hausdorff, 1957], of the (revised) German edition [Hausdorff, 1935].

conceded even by those who have a skeptical attitude towards abstract set theory. . . . Based on the concept of distance, one can for instance define the concept of a convergent sequence of points and its limit, and this concept can be chosen fresh as the foundation of point-set theory, with elimination of the concept of distance. Then it would formally concern a set M , in which a function $f(a_1, a_2, \dots, a_n, \dots)$ of sequences of elements is defined, namely to certain sequences (the convergent ones) an element of M itself (the limit) is assigned. . . .

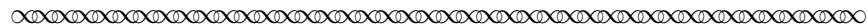
Now a theory of spatial point-sets would, by dint of the numerous properties of ordinary space coming into the picture, naturally carry a very special character, and if one wanted to settle from the start upon this single instance, one would have to develop a new theory for each of the point-sets of a line, a plane, a sphere, etc. Experience has shown that one can avoid this pleonasm and establish a more general theory, which encompasses not only the cases mentioned, but also yet other sets (Riemann surfaces, spaces of finite and infinitely many dimensions, curve and function sets, among others). In fact this gain of generality is not linked with a raised complication, rather exactly conversely, it is linked with a considerable simplification, in which we, at least for the essentials of the theory, only have to make use of very few and simple assumptions (axioms).

[Hausdorff, 1914, pp. 209–210]



Task 30 What do you think Hausdorff was trying to say here? Do you agree with him?

In the above excerpt, Hausdorff discussed using a distance to construct topological spaces. Notice the import of the phrase “one can . . . establish a more general theory, which encompasses not only the cases mentioned, but also yet other sets . . . this gain of generality is not linked with a raised complication, rather exactly conversely, it is linked with a considerable simplification.” This is done, according to Hausdorff, by making use of axioms. We will first investigate his axiomatic theory of distance and, ultimately, the axioms for a topological space to which that theory of distance led Hausdorff.

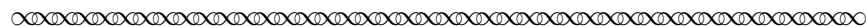


Let E be a set, whose elements we now refer to as *points*. To each pair (x, y) we assign a real number xy , the *distance* between points, that is, a real function $xy = f(x, y)$ defined in (E, E) . We require of this function that the following *distance axioms* or *postulates*, be satisfied:

- (α) $xx = 0$;
- (β) $xy = yx > 0$ for $x \neq y$
- (γ) $xy + yz \geq xz$

The axiom (γ) . . . is called the *triangle inequality* . . . the set E is called a *point-set* or *metric space*.

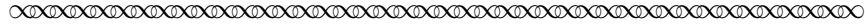
[Hausdorff, 1957, p. 109]



Task 31 Explain Hausdorff’s distance axioms in words. Are they appropriate? Are they what you would expect a good notion of distance to satisfy?

Task 32 Prove that $E := \mathbb{R}^n$ with $xy := \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$ satisfies the distance axioms.

From his distance point of view, Hausdorff next reconstructed concepts with which we are familiar.



If x is a point and δ a positive number, then the set of points γ whose distance from x is $< \delta$ is called a *neighborhood* of x and δ is its radius. It will be denoted by U_x , or more precisely, by $U_x(\delta)$.

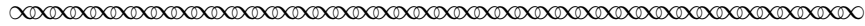
In terms of the concept of a neighborhood the fact that a point is a *limit point* (*point of accumulation*) of the sequence x_n is expressed by saying: Every neighborhood U_x contains almost all (infinitely many) points of the sequence x_n . For, in fact, for every $\delta > 0$ we have ultimately (infinitely often) $x_n < \delta$ or $x_n \in U_x(\delta)$.

Let A be a point-set. If a point x [in A] is such that there exists a neighborhood $U_x \subseteq A$, then it is called an *interior point*, otherwise, it is called a *border point* [hereon “boundary point”] of A . Let A_i be the set of interior points, A_b the set of boundary points. Then we have a decomposition

$$A = A_i \cup A_b$$

of A into disjoint summands. Let us call A_i the *interior* of A , and A_b the *border* [hereon “boundary”] of A . A set that consists entirely of interior points will be called an *open set* ($A_b = 0$); one that consists entirely of boundary points ($A_i = 0$) will be called a *border set* [hereon “boundary set”].

[Hausdorff, 1957, p. 127]



Task 33 Using Hausdorff’s definitions, compute the accumulation points for the following sequences:

- (a) $x_n = \frac{1}{n}$
- (b) $y_n = \frac{(-1)^n}{n}$
- (c) $z_n = (-1)^n + \frac{1}{n}$

Hausdorff next gave examples, computing boundary points and interior point of several sets. How about you do this?

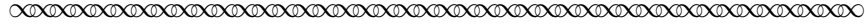
Task 34 Compute the interior and boundary points of the following sets:

- (a) a circular disk including the circumference, i.e. $x_1^2 + x_2^2 \leq 1$.
- (b) a circular disk without the circumference, i.e. $x_1^2 + x_2^2 < 1$.
- (c) the circumference of a circle, i.e. $x_1^2 + x_2^2 = 1$.
- (d) the set of rational points.
- (e) the set of irrational points.

Task 35 Apply Hausdorff's definitions in the special case of Borel's infinitely close set of lines. Are these consistent?

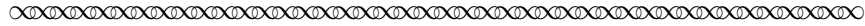
Task 36 Use Hausdorff's definition of "open" to show that the complement of a closed set (in the sense of Borel) is an open set.

In order to make a deeper investigation into these concepts, Hausdorff next introduced continuity. He used the notation $[f > 0]$ to represent the set of all values x in the domain of f such that $f(x) > 0$.

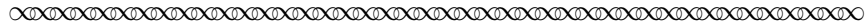


If we assign to each point x of E a unique point $y = f(x)$ in another metric space (or in the same metric space), then the function is said to be *continuous at the point* $x \dots$ if $y = f(x), \eta = f(\xi)$, then $y\eta$ can be made arbitrarily small by making $x\xi$ sufficiently small; that is, for every $\sigma > 0$, it is possible to find some $\varrho > 0$ such that if $x\xi < \varrho$, then $y\eta < \sigma \dots$ If $f(x)$ is real $[f: E \rightarrow \mathbb{R}]$ and continuous, then the set $[f > 0]$ is open.

[Hausdorff, 1957, p. 128]



Task 37 Using his definition of continuity, prove Hausdorff's assertion that if f is real and continuous, then $[f > 0]$ is open.

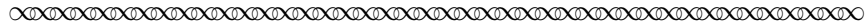


We shall now try to convince ourselves that all the open sets can be obtained by this method. For a point x and a non-empty set B , let us define a *lower distance*

$$\delta(x, B) = \inf_{y \in B} xy,$$

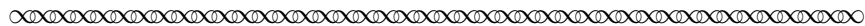
the greatest lower bound of the distances of the point x from the points y of B .

[Hausdorff, 1957, p. 128]



Task 38 Fix a non-empty set B . Prove that $\delta(x, B)$ is a continuous function of x .

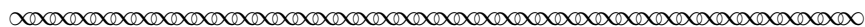
Task 39 Prove that $\delta(x, B) > 0$ if and only if x is an interior point of $A = E - B$.



Thus for the continuous function $f(x) = \delta(x, B)$, the set $[f > 0]$ is identical with A_i and if $A = A_i$ is open, identical with A . In the exceptional case $B = 0, A = E$, we can use the constant function $f(x) = 1$. Therefore we have:

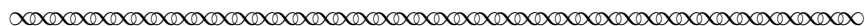
II. For every open set A there exists a function $f(x)$ continuous in E such that the set $[f > 0]$ is identical with A . \dots If $A \subseteq B$ then of course $A_i \subseteq B_i$; A_i is a *monotone* function of A . The set A_i is always open \dots [and] A_i is the greatest open subset of A .

[Hausdorff, 1957, p. 128–129]



Task 40 What does the term “monotone” mean in this context? Why does the term make sense here?

Task 41 State precisely what Hausdorff meant by “ A_i [is to be] the greatest open subset of A .”⁹ Then prove that it is.



If A, B, \dots are any number of sets, not necessarily a finite number, and if

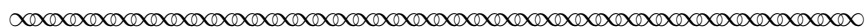
$$S = A \cup B \cup \dots, D = A \cap B \cap \dots$$

are their [union] and intersection, then the property of monotonicity implies in either case that

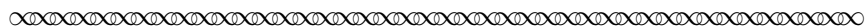
$$S_i \supseteq A_i \cup B_i \cup \dots \text{ [and] } D_i \subseteq A_i \cap B_i \cap \dots$$

For a finite number of sets, however, a stronger result holds

[Hausdorff, 1957, p. 129]



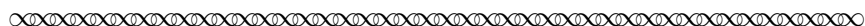
Task 42 The “stronger” result that Hausdorff had in mind concerns a statement that can be deduced when one assumes that $D = A \cap B$. What is this claim? Prove it.



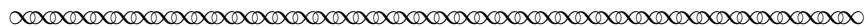
It follows immediately that

III. *The [union] of any number of open sets and the intersection of a finite number are themselves open sets.*

[Hausdorff, 1957, p. 129]

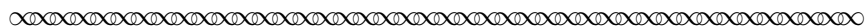


Hausdorff then went on to point out that this is not the case for infinite sets.



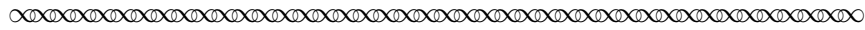
In the Euclidean plane the intersection of concentric open circular discs with radii $\varrho + \frac{1}{n}$ ($n = 1, 2, \dots$) is the disk of radius ϱ inclusive of the circumference.

[Hausdorff, 1957, p. 129]



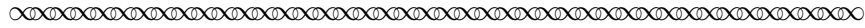
Task 43 Prove that this is a counterexample to III.

⁹It does not mean A_i is better than all the other subsets of A .



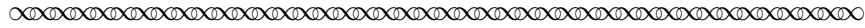
Let A be a point-set in the space E and x a point of E (not necessarily of A). We make the following definition: x is called a β -point of A if every neighborhood U_x contains infinitely many points of A . The set of [all] β points will be denoted by A_β .

[Hausdorff, 1957, p. 130]



Task 44

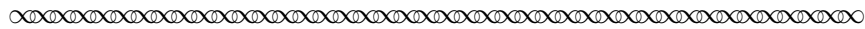
What familiar concept are the “ β points of A ”? What is another name for A_β ?



We say that the set A is *dense-in-itself* if $A \subseteq A_\beta$, *closed* if $A \supseteq A_\beta$, *perfect* if $A = A_\beta$.
... It follows that

II. Open and closed sets are the complements of one another.

[Hausdorff, 1957, pp. 133–134]



Task 45

Prove II.

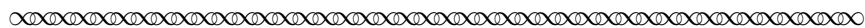
Task 46

Using III and II, prove Hausdorff’s claim that

“I. The [union] of a finite number of closed sets and the intersection of an arbitrary number of closed sets is closed.”

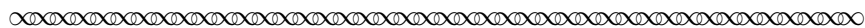
3.2 Topological Spaces

As promised, we now turn to Hausdorff’s theory of topological spaces, driven by his theory of distance.

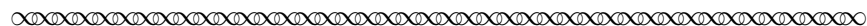


The mathematical discipline concerned with [topological invariance] is called Topology or Analysis Situs. (The latter term, due to Leibniz, was re-introduced by Riemann.) ... This seems like a suitable occasion to touch, in all brevity, on those point-set theories that emphasize the topological point of view from the very beginning and work only with the topologically invariant concepts ... What are primary in the topological space E are the sets that are *closed* (in E) and their complements, the *open* sets The closed or open sets can be taken as our starting point and left undefined, or they can be defined from related concepts (limit point, neighborhood), but always derived in such a way as to keep invariant their topological character; the more detailed nature of the space is then determined by *axioms*.

[Hausdorff, 1957, p. 257]



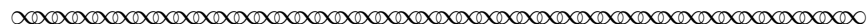
Hausdorff then stated his axioms for a set to be a topological space.



The closed sets must, regardless of anything else, satisfy the following conditions:

- (1) The space E and the null set \emptyset are closed.
- (2) The union of two closed sets is closed.
- (3) The intersection of any number of closed sets is closed.

[Hausdorff, 1957, p. 258]



Task 47 Prove that the closed sets induced by a distance form a topological space.

These are the same conditions that are given today except with “closed” replaced with “open” and “union” replaced with “intersection” (and “intersection” replaced with “union”).

Task 48 Are Hausdorff’s “closed” set axioms equivalent to what we use today with “open”? Why or why not?

4 Conclusion

From 1872–1914, we have seen topology evolve from a specialized concept in analysis to its own branch of mathematics, studied for its own sake and interest. Today, topology is one of the main branches of mathematics, and its mastery is necessary to pursue other areas of math. We hope that this project has shown that definitions and concepts do not fall from the sky or are written down for no reason. Rather, the development of ideas and concepts is organic, starting with a particular problem in a well-established area of mathematics (Cantor and the uniqueness of Fourier series), being generalized to the “next obvious” larger class of objects (Borel’s nearness of lines and planes), and developing into a full blown theory in its own right (Hausdorff’s book on the subject).

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- F. Hausdorff. *Set Theory*. Chelsea Publishing Company, New York, 1957. English translation of [Hausdorff, 1935] by John R. Aumann et al.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is meant to have the student work through multiple problems in order to build up to the modern axioms for a topology. It is meant to be one of the first things that a student of topology sees. Students will be exposed to limit points, the derived set, open sets, closed sets, and continuous functions. The main goals of the project are not only for students to learn this material, but almost just as important, to “bridge the gap” between topology and other mathematics. Students often have a hard time understanding what it is they are doing when they are doing topology, and *why* they are doing what they are doing. The hope is that this project will naturally lead the student into the ideas and concepts in topology so that the “why” is no longer a question.

Student Prerequisites

This project is for a first course in point-set topology, and it is assumed that the student has familiarity with the basic proof techniques (direct proof, proof by contradiction, induction, etc.) as well as equivalence relations and partitions. Furthermore, these are high-level concepts that are being explored and hence a good deal of mathematical maturity and a desire to think deeply and about abstract concepts is also needed.

The source text in this PSP includes excerpts from Cantor’s work on the uniqueness of Fourier series representations of functions. Although no prior knowledge of Fourier series is necessary to complete that section of the project, instructors could also preface its implementation by having students read “Joseph Fourier: The Man Who Broke Calculus,” Chapter 6 of the open source textbook *How We Got From Here to There: A Story of Real Analysis* by Robert Rogers and Eugene C. Boman (available at http://personal.psu.edu/ecb5/ASORA/chapt_interegnum.html). This can provide both a friendly introduction to Fourier series and a seamless transition to Cantor’s work.

PSP Design and Task Commentary

Beginning with Cantor, the idea is that the students can see in Fourier series a kind of math that is familiar to them, even if they have never worked with Fourier series per se. The question of uniqueness and convergence of Fourier series is again, something that they should be able to appreciate. From this natural question, concepts like limit points and derived sets which are purely point-set notions arise by the end of the Cantor section. In this way, it is hoped that the students will appreciate where these more abstract definitions come from. One task that could prove particularly challenging is Task 14. A more “intuitive” proof idea is appropriate for this task, especially for students who have not had a course in analysis. Since the set in question is bounded, it is in some closed set, say $[a, b]$. Now cut $[a, b]$ in half. Then at least one of these halves is infinite, say $[a, a_1 = (a + b)/2]$. Now cut that in half, and at least one of those halves is infinite, etc. There are then two directions in which you can lead the student. One is to try and argue that the intersection of all these sets contains a single point (nested interval theorem). Then you could argue that you can get as close to that single point as you want by going far enough into the sequence of cuts. This works well with Cantor’s definition of “point” because if you construct a sequence by choosing points from the sequence of nested intervals you get a Cauchy sequence, which is a “point” by Cantor’s definition. Task 8 also asks for informal justification as to why Cantor’s definitions are appropriate. While a formal justification would involve many details, here we simply hope that students will observe

that this definition recreates the standard operations in the way that we have all known since grade school.

The next section on Borel is meant to take the ideas of Cantor to which students were exposed and bring them to the next level. This section can be quite challenging, but part of the purpose is indeed for the student to struggle through the ideas put forth by Borel. We begin by attempting to say what it means for a set of lines to converge to a line. His definition is a little bit odd if we understand it from a modern point of view, and so the students are expected to struggle through it until they grasp what Borel must have meant in order to make the definition non-trivial. However, once this mental block is removed, it can be much easier for the student to think about limit points and derived sets abstractly, in a setting where the kinds of objects don't really matter, so long as there are notions of distances. This moves into the last section on Hausdorff.

The section on Hausdorff is taken from his textbook, and it does have a more modern feel to it in terms of the way he proceeds. Still, there is much to be gleaned from the way Hausdorff thinks. He begins with distance axioms, the perfect next step after Borel's generalizations. With an abstract notion of distance also comes an abstract notion of continuity, a notion that students of analysis will recognize. By the time Hausdorff completed his thoughts, he had shown that any metric space satisfies what we would today recognize as the axioms for a topological space.

Suggestions for Classroom Implementation

Implementation of the entire project will take about ten 50-minute class periods in total, using a combination of small-group and whole-class discussions. See the Sample Implementation Schedule below for one possibility, including individual assignments to be made as advance preparatory work. Formal write-up of select tasks can also be required as individual homework.

L^AT_EX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

The following outline provides a schedule for implementing this project in 10 class days.

Day	Preparatory Homework	Classroom Plan
1	Read through Section 1.1.	Students do Tasks 1-4.
2	Do Task 5.	Review student questions on Task 5; do Tasks 6-8.
3	Complete Tasks 6-8; read Section 1.3.	Do Tasks 9-11.
4	Read Section 1.4; do Tasks 12-13.	Do Tasks 14-17.
5	Read Section 1.5.	Do Tasks 18-19.
6	Read Section 2; do Tasks 20-21.	Review Tasks 20-21; do Tasks 22-24.
7	Do Tasks 25-26.	Review Tasks 25-26; do Tasks 27-29.
8	Read Section 3.1; do Tasks 30-32.	Do Tasks 33-36.
9	Do Tasks 37-38.	Review Task 37; do Tasks 39-42.
10	Read Section 3.2; do Tasks 43-45.	Do Tasks 46-48.

For Further Reading

A nice source on Cantor's life and work is Joseph Dauben's *Georg Cantor: His Mathematics and Philosophy of the Infinite*, [Dauben, 1979].

Connections to other Primary Source Projects

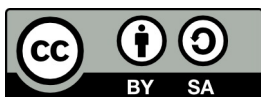
The following additional primary source-based projects by the author are also freely available for use in teaching courses in point-set topology. The first two projects listed are full-length PSPs that require 5 and 3 class periods respectively to complete. All others are designed for completion in 2 class periods.

- *Connectedness: Its Evolution and Applications*
- *From Sets to Metric Spaces to Topological Spaces*
- *Topology from Analysis* (Also suitable for use in Introductory Analysis courses.)
- *The Cantor set before Cantor* (Also suitable for use in Introductory Analysis courses.)
- *Connecting Connectedness*
- *The Closure Operation as the Foundation of Topology*
- *A Compact Introduction to a Generalized Extreme Value Theorem*

Classroom-ready versions of these projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_topology. They can also be obtained (along with their L^AT_EX code) from the author.

Acknowledgments

The development of this student project has been partially supported by the Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Program with funding from the National Science Foundation's Improving Undergraduate STEM Education Program under Grant Nos. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily represent the views of the National Science Foundation.



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