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The Trigonometric Functions Through Their Origins: Ptolemy Finds High Noon in Chords of Circles

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Trigonometry is concerned with the measurements of angles about a central point (or of arcs of circles centered at that point) and quantities, geometrical and otherwise, which depend on the sizes of such angles (or the lengths of the corresponding arcs). It is one of those subjects that has become a standard part of the toolbox of every scientist and applied mathematician. Today an introduction to trigonometry is normally part of the mathematical preparation for the study of calculus and other forms of mathematical analysis, as the trigonometric functions make common appearances in applications of mathematics to the sciences, wherever the mathematical description of cyclical phenomena is needed. This project is one of a series of curricular units that tell some of the story of where and how the central ideas of this subject first emerged, in an attempt to provide context for the study of this important mathematical theory. Readers who work through the entire collection of units will encounter six milestones in the history of the development of trigonometry. In this unit, we encounter a brief selection from Claudius Ptolemy's *Almagest* (second century, CE), in which the author shows how a table of chords can be used to monitor the motion of the Sun in the daytime sky for the purpose of telling the time of day.

1 An (Extremely) Brief History of Time

We in the 21st century generally take for granted the omnipresence of synchronized clocks in our phones that monitor the rhythms of our lives. Before the widespread use of cellphones in the 1990s, watches and clocks were the standard time-telling devices, but they had to be synchronized manually by their users. The adoption of time zones to coordinate standards for synchronization of time around the world emerged at the end of the nineteenth century, a need that only arose when railroads sought to coordinate their schedules at different and widely separated locations. And before the early 1800s, clockmaking technology had yet to develop mechanisms for keeping time with a degree of regulation that would allow reliable synchronization of time between people in different places to occur in the first place. For thousands of years before this, the telling of time was an entirely local affair, of relevance only to small groups of people and restricted to a particular place on the planet.

In most circumstances, a general estimation of how high the Sun was in the sky, or which constellations were currently visible, was enough to guide people through their affairs. In rare cases, however, priests or natural philosophers desired more control over the timing of events. For the ancient Greeks, the paths of the Sun and stars were obviously circular, traced out against the great

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dome of the heavens, and their motions were reliably periodic. So the geometry of the circle was key to their conception of timekeeping.

As an example of how the geometry of the circle played such an important role for Greek astronomers, consider the problem of designing an effective sundial. The mathematical astronomer Claudius Ptolemy, who lived in Alexandria in Egypt as a Roman subject, addressed this problem in Book II of his famous *Almagest* (Toomer, 1998), an excerpt of which we will read below. Ptolemy was the author of many scientific treatises that survive to the present, the most important of which was this compendium of his astronomical theories.[1] In the *Almagest*, Ptolemy illustrated with geometric justification his geocentric model for the movement of the Sun, Moon and planets about the Earth.[2] In the passage we read from the *Almagest*, we will find that Ptolemy predicted with reasonable accuracy the lengths of noonday shadows on special days of the year, shadows cast by a *gnomon*, a simple stick placed vertically in the ground.[3]

Figure 1: A simple gnomon sundial.[4]

[1]The name *Almagest* was not the title given to the work by its author. Ptolemy gave it a much more prosaic title, *The Mathematical Collection*, but as it was much studied by astronomers in subsequent centuries, it became known more simply as *The Great Collection*. Centuries later, Arabic scholars translated it into their own language with this same title as *Kitāb al-majistī*, and even later, European scholars produced Latin translations, transliterating the title as *Almagest*. It is instructive to consider how it is that Ptolemy's work survives today; it is in no small part due to the spectacular success of the reputation it earned among astronomers who suceeded Ptolemy. In an age before the invention of printing, Ptolemy's work was copied by hand, again and again, while the works of earlier scholars were eclipsed or superseded by Ptolemy's and were set aside. Eventually, the older manuscripts were forgotten, decayed and were lost, except for those mentions in other books by those who had read them in ages past. Furthermore, Ptolemy had the good fortune to have his works preserved at the famous Museum Library in Alexandria, the modern-day source of a vast amount of scientific literature of the ancient world.

[2]See the project "Hipparchus' Table of Chords" at https://blogs.ursinus.edu/triumphs/ for more on ancient Greek astronomy, and specifically, Ptolemy's geocentric model of the universe.

[3]*Gnomon* is a Greek word that might best be translated '*indicator*' (literally, "*the one who knows*"); it refers to an object whose shadow is used to tell the time of day.

[4]Photo by Rich Luhr available at https://www.flickr.com/photos/airstreamlife/3419834481/, and reproduced under Creative Commons license CC BY-NC-ND 2.0.

2 Ptolemy's Model of the Universe

In the geocentric model advocated by Ptolemy, the celestial sphere rotates daily about the Earth, carrying the stars in their constellations around the sky. The Sun, the Moon and the planets[5] had their own sometimes complicated motions in the sky. In general, the Sun lagged a bit behind the stars, slowly drifting eastward day by day, but returning to the same spot in the celestial sphere in one year; this is why different constellations are visible in the nighttime sky in different seasons. The Moon moved against the stars a bit more quickly, making a trek around the celestial sphere once a month, and the other planets had their more idiosyncratic motions.

Of course, in the more modern view, it is the Earth rotating on its axis once a day within the vastness of the cosmos that causes the appearance of the rotation of the heavens, including the daily motion of the Sun and Moon. The axis of rotation of the celestial sphere is therefore the same imaginary line which forms the axis of the rotating Earth, and it pierces the celestial sphere at the North and South celestial poles. (See the diagram below.) For people living in the Northern Hemisphere on the Earth, as did Ptolemy, only the North pole is visible, and it is located very near a bright star in Ursa Minor (the Little Dipper) called, aptly enough, Polaris. Hence, all the stars in the sky (including the Sun) appear to rotate about Polaris. In addition, the celestial equator is that part of the sky directly above places on Earth that lie on its equator.

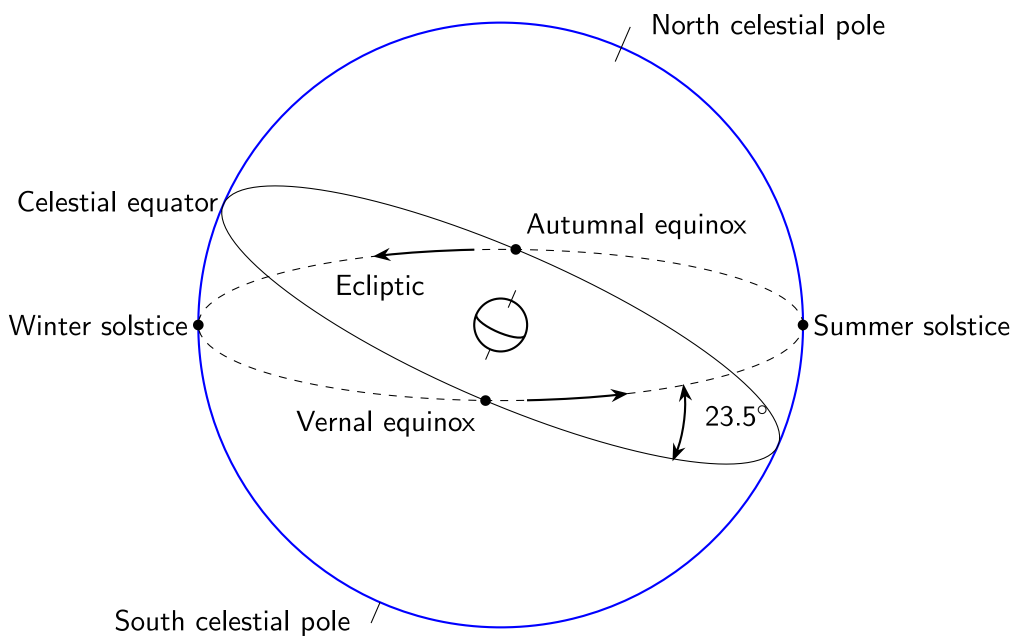


Figure 2: A geocentric view of the Earth, Sun and stars on the celestial sphere.

But modern science tells us that the Earth also revolves around the Sun once a year, and from the perspective of Earth-based observers, this explains why it appears that the Sun moves around the celestial sphere once a year. The path that the Sun travels in its annual circuit across the background stars is called the *ecliptic* circle. The ecliptic is different from the celestial equator, and is tilted with

[5] The ancients actually considered the Sun and Moon to be planets as well because, like Mercury, Venus, Mars, Jupiter and Saturn, the other visible planets, they moved against the background stars, so could not be attached to the celestial sphere. The Greek word *planetes* means *wanderers*.

respect to it by an amount precisely equal to the tilt of Earth's rotational axis relative to the axis of the orbit of the Earth around the Sun.

Ptolemy knew that for any observer like himself who lived north of the Earth's equator, the Sun rode highest in the sky at noon on the day of the summer solstice (around June 21 in our calendar) and lowest on the day of the winter solstice (around December 21).[6] In addition, midway between the solstices are the equinoxes, one on the first day of spring (around March 20) and one on the first day of fall (around September 22). On these days, the amount of daylight is the same as the amount of night[7] – twelve equal hours each, and the Sun rises and sets at the same points on the horizon on both days. Consequently, shadows cast by the Sun at noon are shortest on the summer solstice in the Northern Hemisphere, longest on the winter solstice, and take the same middle length on both days of equinox.

Task 1 The diagram in Figure 2 displays a *geocentric* Earth-Sun-stars system, with the Earth at the center of the celestial sphere and the Sun in motion relative to this sphere. If we consider the Earth to be fixed in space, as did the ancients, then the celestial sphere and the Sun revolve around the Earth once a day along the axis that passes through the celestial poles. As previously noted, the Sun's rotation is a tiny bit slower than the sphere's, in such a way that it drifts against the background stars, returning to the same position relative to the sphere a year (or a bit more than 365 days) later. In this task, you will draw a new diagram to represent these same phenomena in a *heliocentric* system, with the Sun at the center of the celestial sphere and the Earth in motion around the Sun, exhibiting a similar year-long revolution.

(a) Redraw a picture of Earth like the one in Figure 2, showing its equator (as a circle around its middle) and its rotation axis (as tiny line segments sprouting outward from the North and South Poles). Since you are an Earth-based observer in the Northern Hemisphere, mark a point on the front surface of your Earth to identify your position there. Now assume that it is noon on March 20, the day of the vernal equinox. Do you see where the Sun is positioned at this time in Figure 2 shining directly above your position on Earth? Draw a small disk in front of the Earth in your new diagram to represent the position of the Sun at this moment and label it "Sun". (Of course, nothing here will be to proper scale: the Sun is actually more than 300,000 times larger than the Earth!) Label the position of the Earth in your diagram with the timestamp "Vernal equinox."

(b) In order to build a diagram that depicts a heliocentric system, we will consider the Sun to be fixed in space, and depict the Earth in its year-long motion around the Sun. Think about where Earth must be located relative to the Sun on the day of the summer solstice. Then add a second Earth to your diagram to represent

[6] The word *solstice* comes from fusion of the Latin words for 'sun', *sol*, and 'to stand still', *sistere*; the solstices refer to those days when the sun stops moving northward or southward in the sky, and appears to turn around to begin moving in the opposite direction. Modern science tells us that the sun is high in the sky, allowing it to remain in the sky in the northern hemisphere for more than 12 hours a day, when the tilt of the earth's axis in the north is directed toward the star; the sun rides low in the sky and remains up less than 12 hours a day when the earth's axis is tilted away from the Sun exactly half a year later.

[7] The word *equinox* is likewise the merging of Latin words for 'equal', *aequus*, and night, *nox*.

this position and label it "Summer solstice." Repeat a third and fourth time with new copies of the Earth labeled for the days of the "Autumnal equinox" and "Winter solstice." Finally, mark the full orbit of the Earth around the Sun with a dashed curve through these four positions.

(c) If your diagram is properly constructed, then Earth's North Pole should be oriented closest toward the Sun on one of these four days and farthest away on another. Which is which? How does this explain seasonal climate on Earth (that summers are hotter while winters are colder in the Northern Hemisphere, and the opposite in the Southern Hemisphere)?

In *Almagest* I.14,[8] Ptolemy determined the angle of separation between the celestial equator and the ecliptic circle at the two equinoxes, called the *obliquity*; he found the measure of this angle to be $[23;\,51,\,20]^\circ$, that is, $23^\circ\ 51'\ 20''$ in degrees, minutes and seconds. (We use the notation $[a;\,b,\,c]$ to represent numbers in sexagesimal, or base 60, format; this is a short form for $a + b\cdot 60^{-1} + c\cdot 60^{-2}$. Ptolemy and his fellow Greek astronomers inherited this sexagesimal system of numeration from their Babylonian astronomer predecessors.[9]) Ptolemy's value is larger than the current value of 23.5° indicated in the figure above.[10]

Now the particular height of the Sun at noon on any given day of the year also depends on where on the earth the observer stands. If the observer lives further north, then Polaris, which is very near the North Pole of the celestial sphere, lies higher in the sky than for someone who lives further south, closer to the earth's equator. Recall that the celestial equator sits exactly above the terrestrial equator, just as Polaris is situated directly above the terrestrial North Pole. Someone at the earth's North Pole will see Polaris directly overhead, and the observer's horizon will coincide with the celestial equator; someone living at the equator will find the celestial North Pole on the horizon and the celestial equator will cross that person's *zenith*, the point in the sky directly overhead.

Ptolemy and his intended audience, however, lived in middle latitudes, between the Pole and the equator. An observer's *meridian* circle, the circle at the surface of the Earth whose center is the Earth's center and which passes through the observer's position and the North and South Poles, was used to define the observer's geographical *latitude*, namely, the angle of arc along the meridian from the equator to the observer's position. Exactly where the observer stood on the Earth was instrumental in getting the mathematical information right for the computations he was intending to perform.

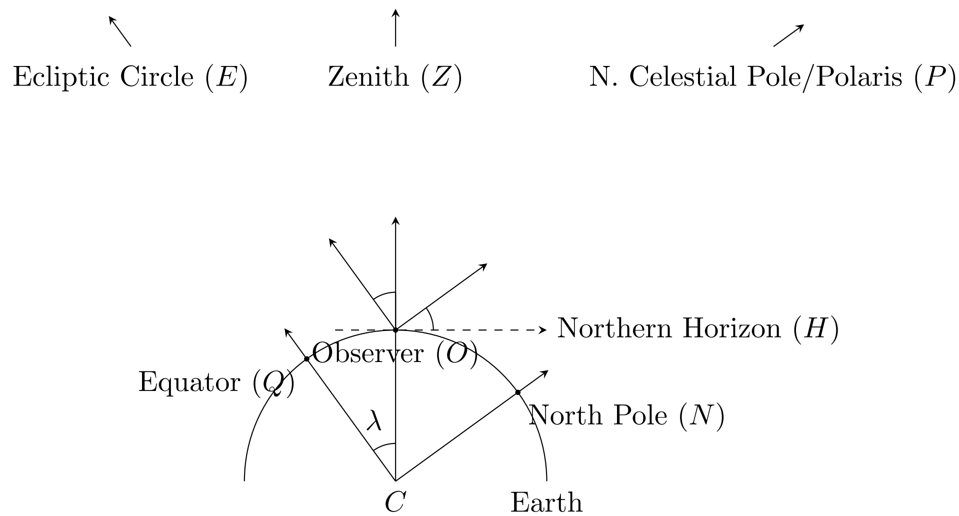
Task 2 The diagram below is a labeled sketch of your vantage when you look up into the sky, assuming that you live in the Northern Hemisphere. It depicts where you are, O, on your meridian circle, a circle which includes the North Pole, N, off to your north and the point on the equator, Q, to your south. The surface of the Earth at your feet you perceive as flat (the dotted line), even though your meridian is part of a circle that runs around the Earth. Your zenith, Z, is way above you on the celestial sphere (not

[8] This refers to Section 14 in Book I of the *Almagest*.

[9] See the project "Babylonian Astronomy and Sexagesimal Numeration" at https://blogs.ursinus.edu/triumphs/ for more on sexagesimal numeration and its role in ancient Babylonian astronomy.

[10] In fact, the Earth's obliquity fluctuates over time, since the axis of the Earth's rotation wobbles like a spinning top, in such a way that the obliquity oscillates between about 22° and 24.5° over a period of some 40,000 years!

pictured here) as it is essentially infinitely far away. In the same way, the northern celestial pole, P, is infinitely far above the North Pole, N, and the ecliptic circle rides above the Earth's equator, at E. In particular, the line through C and Q will pass through E, as will the parallel line drawn at O. Similarly, P lies on the line through C and N as well as on the parallel drawn at O.



Your latitude λ (this symbol is the Greek letter *lambda*) is the measure of the angle $\angle QCO$ made at the center of the Earth C from the equator Q to where you stand O along your meridian circle. It is considered positive if you live north of the equator and negative if you live south of it.

(a) Suppose you live in Tulsa, Oklahoma, where your latitude is $\lambda = 36°$. What is the measure of your angle of sight $\angle EOZ$ between a star positioned on the ecliptic circle E above your meridian and your zenith Z directly overhead? Explain how you arrived at your answer.

(b) What is the angle of elevation $\angle HOP$ of Polaris off of your horizon? [Hint: first consider the measure of $\angle POE$.]

(c) Repeat parts (a) and (b), but for your actual location; you'll have to look up (online) what your latitude is.

(d) What general relationship can you state between the latitude of a Northern Hemisphere observer on the surface of the Earth and the angle of elevation of Polaris off their horizon?

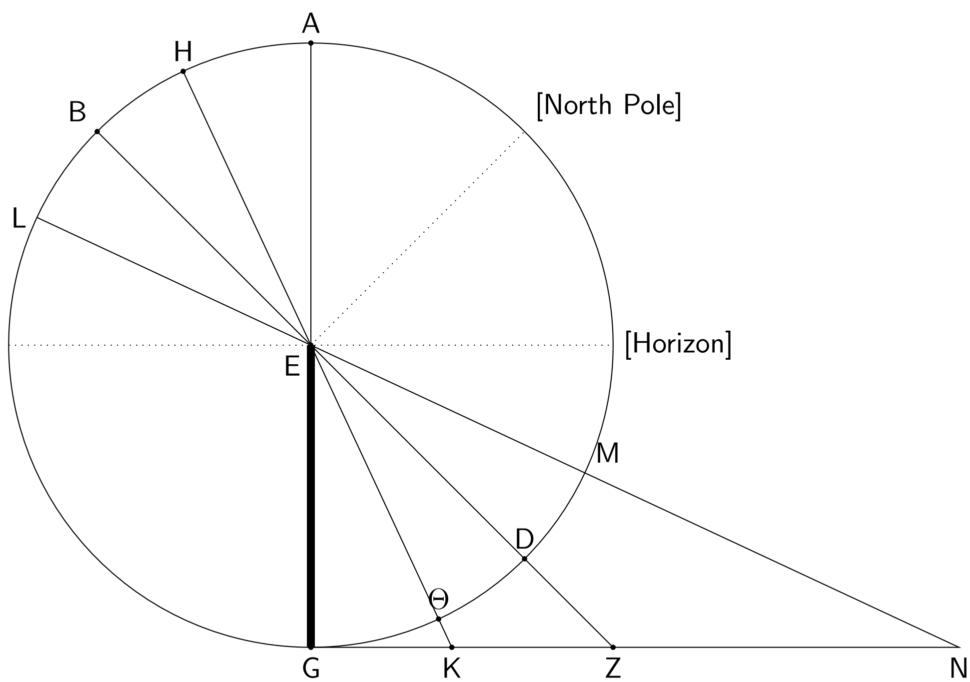
An important and subtle point must be made before we go much further. As we alluded in the very first sentence of this project, we use degrees (or the more accurate degree-minute-second sexagesimal system) to measure the sizes of both angles and arcs along circles. In the first case, we measure angles around a central point, where 360° constitutes one full rotation about the center; while in the second case, we measure arcs along a circle, where 360° constitutes the entire circumference of the circle. Angle measure corresponds to the amount of turn around a point, while arc measure determines a length along a circle. It is good to be aware of this ambiguity, and you can look for it in Ptolemy's writing below.[11]

3 Finding the height of the Sun at your feet

In the passage from *Almagest* II.5 which we will study, Ptolemy considered an observer on the 36th parallel (just like our imaginary Oklahoman in Task 2). This was the latitude of the island of Rhodes in the Aegean Sea, which was a well-known center for astronomical activity at the time.

Ptolemy began by constructing his geometric model of the shadow-casting gnomon and the Sun in its position in the sky, and he arranged a diagram (shown below) to represent the entire assemblage.[12]

The required ratios of shadow to gnomon can be found quite simply once one is given the arc between the solstices and the arc between the horizon and the pole; this can be shown as follows.



Let the meridian circle be ABGD,[13] on centre E.[14] Let A be taken as the zenith, and draw diameter AEG. At right angles to this, in the plane of the meridian, draw GKZN; clearly, this

[11]For instance, we have defined the latitude of a position on the Earth to be the *angle* of arc along the meridian through that position up (or down) from the equator, and in the diagram in the previous Task, notice that λ marks that angle at the center C of the Earth. However, λ also measures the *length* of the arc from Q to O, provided we set the length of the circumference of the Earth to be 360°. On globes and on many maps, latitudes are marked so that they measure the *distances* along meridian circles (one degree of arc along a meridian is roughly 111 km, or about 69 miles).

[12]To help the reader interpret the elements of this diagram, additional dotted lines and labels have been added to Ptolemy's diagram to mark the positions of the celestial North Pole and the Horizon.

[13]The *meridian* of an observer is that imaginary circle on the celestial sphere which passes through the observer's zenith and the points on the horizon at due north and due south; this is the circle depicted in Ptolemy's diagram. Note also that the points are labeled using the Greek alphabet: Alpha, Beta, Gamma, Delta, Epsilon, Zeta, Eta (H), Theta (Θ), Iota (not used as a label), Kappa, Lambda, Mu, Nu, etc.

[14]The celestial sphere is so vast that any point on the Earth can be considered the center of the sphere; Ptolemy

will be parallel to the intersection of horizon and meridian. Now, since the whole earth has, to the senses, the ratio of a point and centre to the sphere of the sun, so that the centre E can be considered as the tip of the gnomon, let us imagine GE to be the gnomon, and line GKZN to be the line on which the tip of the shadow falls at noon. Draw through E the equinoctial noon ray and the [two] solsticial noon rays: let BEDZ represent the equinoctial ray, HEΘK the summer solsticial ray, and LEMN the winter solsticial ray. Thus GK will be the shadow at the summer solstice, GZ the equinoctial shadow, and GN the shadow at the winter solstice.

Then, since arc GD, which is equal to the elevation of the north pole from the horizon, is 36°... at the latitude in question, and both arc ΘD and arc DM are [23; 51, 20]°, by subtraction arc GΘ = [12; 8, 40]°, and by addition arc GM = [59; 51, 20]°.

Therefore the corresponding angles

$$\angle\,\mathsf{KEG} = [12; 8, 40]^\circ,$$
$$\angle\,\mathsf{ZEG} = 36^\circ,$$
$$\angle\,\mathsf{NEG} = [59; 51, 20]^\circ$$

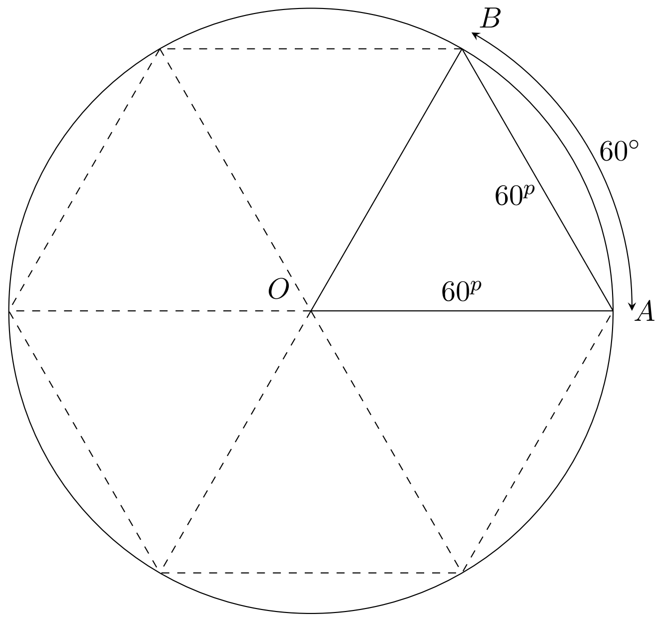
Task 3 What does Ptolemy mean in the first sentence by "the arc between the solstices" and "the arc between the horizon and the pole"? How are these related respectively to the obliquity of the earth and the latitude of the observer at Rhodes?

Task 4 Verify Ptolemy's calculations of the sizes of arcs GΘ and GM, or equivalently, of the angles ∠KEG and ∠NEG. These correspond to the positions of the Sun at noon on the summer and winter solstice days; identify which is which.

Now that Ptolemy had measured the angles ∠KEG, ∠ZEG and ∠NEG, he needed a way to connect those angles to the lengths of the corresponding shadows, GK, GZ and GN. In *Almagest* I.10–11, Ptolemy had artfully applied a number of geometric theorems to produce an extensive table of measurements that he used to solve this important problem.

Constructing his table involved working with a circle of radius measuring 1 unit, whose circumference was divided into 360° (see the diagram below).

chooses this point to be the tip of the gnomon so that the Sun's rays pass through this point and land at the ground at the tip of the gnomon's shadow.



By inscribing a regular hexagon within the circle, it was apparent that the line segment connecting the endpoints A and B of an arc of $60°$ along the circle, what we call the *chord* subtending that arc, itself also measured 1 unit of length. By continuing to employ sexagesimal numeration, he achieved far more accuracy in his calculations by replacing the unit length with an equivalent 60 *parts* of that unit, depicted in the diagram below as 60^p. In other words, the chord length spanning a $60°$ of the circle measured 60^p. We will denote this by writing $\text{Crd arc } AB = \text{Crd } 60° = 60^p$.

This was the simplest entry of Ptolemy's table of chords to calculate. Another straightforward chord computation was that for the arc of measure $180°$.

Task 5 Verify that $\text{Crd } 180° = 120^p$.

As a final example of one of Ptolemy chord calculations, consider Crd $90°$.

Task 6 Draw a circle centered at O of radius 1 unit $= 60^p$, and mark points A and B on the circumference so that they span an arc measuring $90°$, a quarter of the full circle. Then draw in the sides of triangle AOB. Show that you can apply the Pythagorean Theorem to determine that $\text{Crd } 90° = \sqrt{2}$ units. In his Table of Chords, Ptolemy recorded this value as $\text{Crd } 90° = [84;51,10]^p$. Convert this sexagesimal number $[84;51,10]^p$ into its decimal value to show that it is very close to the value of $\sqrt{2}$. How good an approximation is Ptolemy's value?

By these and many more clever geometrical propositions, Ptolemy was able to generate a very detailed table of chord lengths for all arcs from $0°$ to $180°$, in steps of $\frac{1}{2}^\circ$, with chord measures worked out to two sexagesimal places (as in the previous Task). This Table of Chords was an updated and much more extensive table than one produced some 300 years earlier by the Greek astronomer and mathematician Hipparchus of Rhodes.[15] Tables such as these produced by Hipparchus and Ptolemy were the chief advancement in the development of trigonometry that we attribute to the Greeks today: by showing a correspondence between lengths of circular arcs (or equivalently, the angles that determine these arcs) and the lengths of the chordal line segments that span them, he established a method for matching arcs and angles with linear distances. This was the tool that Ptolemy needed to complete his analysis of the lengths of shadows of the Sun, as we will see below in the continuation of the excerpt we are reading from his *Almagest*.

But before we attempt to make sense of this last bit of text, let us first highlight a fundamental fact from the geometry of circles which will also come in handy:

> *If an angle $\angle PQR$ is inscribed in a circle with center O (that is, all three points P, Q, R lie on the circle), then the measure of $\angle PQR$ equals half the measure of the arc PR (or, equivalently, of the central angle $\angle POR$).*

This fact is recorded in Euclid's *Elements* in the following form:

> *Elements*, **Proposition III.20**[16]: In a circle, the angle at the center is double the angle at the circumference when the angles have the same circumference as base.

Because of its appearance in the *Elements*, Ptolemy (and his serious readers) would have been well aware of it.

Task 7 In this Task, we investigate why Euclid's proposition, *Elements* III.20, holds.

(a) Draw a circle with center O and identify three points P, Q, R on the circumference. There are three cases for how these points can be arranged; our first case corresponds to where two of the points, say Q and R, are diametrically opposite one another (so that O lies on one side of the angle $\angle PQR$). Complete such a drawing, then explain why the central angle $\angle POR$ must be twice as large as the inscribed angle $\angle PQR$. (Hint: what kind of triangle is $\triangle OPQ$, and what then must be the relationships between its interior and exterior angles?)

(b) Now arrange the points P, Q, R around the circle so that O lies within $\angle PQR$. To show why the same result holds here, let Q' be the point diametrically opposite Q on the circle, so that $\angle PQR$ can be divided into two angles $\angle PQQ'$ and $\angle Q'QR$. Apply the case from part (a) above to each of these smaller angles to complete the justification in this case.

(c) Finally, arrange the points P, Q, R around the circle so that O lies outside $\angle PQR$. Once again, let Q' be the point diametrically opposite Q on the circle. This time, $\angle PQR$ is the difference between angles $\angle PQQ'$ and $\angle Q'QR$. Apply the case from part (a) again to each of the last two angles, and complete the full justification.

[15] See the project "Hipparchus' Table of Chords" at https://blogs.ursinus.edu/triumphs/ for details about Hipparchus' table of chords.

[16] That is, Proposition 20 of Book III.

We are now ready to make sense of the rest of the excerpt from Ptolemy's *Almagest* II.5, in which we will see how he was able to use his chord table and the measurements of the angles that he had already calculated ($\angle\mathsf{KEG} = [12;8,40]^\circ, \angle\mathsf{ZEG} = 36^\circ, \angle\mathsf{NEG} = [59;51,20]^\circ$) to find the lengths of the shadows cast by the gnomon on the days associated with those angles.

Therefore in the circles about right-angled triangles KEG, ZEG, NEG,

arc GK $= [24;17,20]^\circ$ and arc GE $= [155;42,40]^\circ$ (supplement),

arc GZ $= 72^\circ$ and arc GE $= 108^\circ$, similarly [as supplement],

arc GN $= [119;42,40]^\circ$ and arc GE $= [60;17,20]^\circ$ (again as supplement).

Therefore

where Crd arc GK $= [25;14,43]^p$, Crd arc GE $= [117;18,51]^p$,

and where Crd arc GZ $= [70;32,4]^p$, Crd arc GE $= [97;4,56]^p$,

and where Crd arc GN $= [103;46,16]^p$, Crd arc GE $= [60;15,42]^p$.

Therefore, where the gnomon GE has 60^p, in the same units

the summer [solsticial] shadow, GK $\approx [12;55]^p$,

the equinoctial shadow, GZ $\approx [43;36]^p$,

the winter [solsticial] shadow, GN $\approx [103;20]^p$.

Task 8 In the opening sentence of the last excerpt, Ptolemy proposed to take the three right-angled triangles KEG, ZEG, and NEG, out of the diagram and in each case, inscribe them within a circle.

(a) Let us first think about the general case of a right-angled triangle inscribed in a circle. Assume that PQR is a right-angled triangle with right angle at Q. Use Proposition III.20 from Euclid's *Elements* about the measure of inscribed angles to explain why the center O of the circle in which this triangle is inscribed must be the midpoint of the hypotenuse $\overline{PR}$. In other words, why must the hypotenuse of the triangle be a diameter of the circle?

(b) Now suppose that triangle KEG is inscribed in a circle; explain why arc GK $=$ $[24;17,20]^\circ$ and arc GE $= [155;42,40]^\circ$ as Ptolemy indicated. Similarly verify the corresponding measures of the arcs that Ptolemy reported for the circles that circumscribe the triangles ZEG and NEG.

In the middle sentence of the last excerpt, Ptolemy consulted his own table of chords which he presented in *Almagest* I.11. Recall that Ptolemy chose his reference circle to have a radius of unit length, subdivided sexagesimally into 60 parts. He then determined the lengths of chords in the same unit of parts of the radius' length. In the text, when he wrote "where Crd arc GK $= [25;14,43]^p$," he was saying that the chord spanning the arc GK measures [25; 14, 43] parts in units where the radius of the circle measures 60 parts.

Task 9 If we inscribe the right triangle KEG in a circle of radius one unit = 60 parts, this will match the setup for Ptolemy's table of chords. Assuming that Ptolemy has correctly determined that $\mathsf{Crd\ arc\ GK} = [25;\ 14,\ 43]^p$, use the Pythagorean Theorem to verify that $\mathsf{Crd\ arc\ GE} = [117;\ 18,\ 51]^p$. Similarly, verify the values of $\mathsf{Crd\ arc\ GE}$ from the text that correspond to inscribing each of the other two right triangles, ZEG and NEG, in a circle of radius one unit = 60 parts.

Task 10 Finally, in the last sentence of the excerpt, Ptolemy resized the dimensions of the three triangles: whereas he had originally used units in which the radius of the circumscribing circle is 60 parts (from his table of chords), he now wanted to recalibrate so that the vertical side of the triangle, the gnomon, segment $\overline{\mathsf{GE}}$, has 60 parts. This simply required the use of a proportion to make the final calculations. For instance, $\overline{\mathsf{GK}}$ and $\overline{\mathsf{GE}}$, the sides of right triangle KEG, have the same ratio to each other when inscribed in a circle of radius 60 parts (as when we use Ptolemy's table of chords) as they would if the measuring scale were reset so that the gnomon $\overline{\mathsf{GE}}$ now equaled 60 parts, as in the original diagram:

$$\frac{[25; 14, 43]^p}{[117; 18, 51]^p} = \frac{\mathsf{GK}}{\mathsf{GE}} = \frac{?}{60^p}.$$

Solving the proportion for the unknown gives the length of the summer solsticial shadow that $\overline{\mathsf{GK}}$ represents. Verify that we get the same value for the summer solsticial shadow that Ptolemy obtained. Then do the similar calculations to verify Ptolemy's calculations for the lengths of the equinoctial shadow $\overline{\mathsf{GZ}}$ and the winter solsticial shadow $\overline{\mathsf{GN}}$.

4 Conclusion

Claudius Ptolemy's *Almagest* II.5 presented an extremely practical application of trigonometry, the use of a table of chords to help determine the lengths of the Sun's shadows on selected days of the year. One can easily imagine using this information to design a sundial for use by a community to note the time of day throughout the year. On the ground next to a fixed gnomon (segment $\overline{\mathsf{EG}}$ in Ptolemy's diagram (p. 7), one could mark the positions of points K, Z and N—and others besides—to identify the day of the year when the Sun's shadow at noon will reach those spots. Repeat this for other hours of the day on different days of the year, and voilá, a working clock for the town square! What's more, Ptolemy made many other uses of his table of chords in the *Almagest* to compute solutions to other astronomical problems.

The successes of Greek astronomers like Ptolemy who applied geometrical methods to solve astronomical problems spread across the world. In particular, Syrian and Persian scholars in the Middle East learned, preserved and sometimes added to this knowledge, making use of it in their own parts of the world. Eventually, this science made its way to the Indian subcontinent. In our next episode, we will see how trigonometry was able to take hold, and thrive, in India in a predominantly oral culture.

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Notes to Instructors

This project is the third in a collection of six curricular units drawn from a Primary Source Project (PSP) titled *A Genetic Context for Understanding the Trigonometric Functions*. The full project is designed to serve students as an introduction to the study of trigonometry by providing a context for the basic ideas contained in the subject and hinting at its long history and ancient pedigree among the mathematical sciences. Each of the individual units in that PSP looks at one of the following specific aspects of the development of the mathematical science of trigonometry:

- the emergence of sexagesimal numeration in ancient Babylonian culture, developed in the service of a nascent science of astronomy;
- a modern reconstruction (as laid out in (Van Brummelen, 2009)) of a lost table of chords known to have been compiled by the Greek mathematician-astronomer Hipparchus of Rhodes (second century, BCE);
- a brief selection from Claudius Ptolemy's *Almagest* (second century, CE) (Toomer, 1998), in which the author (Ptolemy) showed how a table of chords can be used to monitor the motion of the Sun in the daytime sky for the purpose of telling the time of day;
- a few lines of Vedic verse by the Hindu scholar Varāhamihira (sixth century, CE) (Neugebauer and Pingree, 1970/1972), containing the "recipe" for a table of sines as well as some of the methods used for its construction;
- passages from *The Exhaustive Treatise on Shadows* (Kennedy, 1976), written in Arabic in the year 1021 by Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī, which include precursors to the modern trigonometric tangent, cotangent, secant, and cosecant;
- excerpts from Regiomontanus' *On Triangles* (1464) (Hughes, 1967), the first systematic work on trigonometry published in the West.

This collection of units is not meant to substitute for a full course in trigonometry, as many standard topics are not treated here. Rather, it is the author's intent to show students that trigonometry is a subject worthy of study by virtue of the compelling importance of the problems it was invented to address in basic astronomy in the ancient world. Each unit may be incorporated, either individually or in various combinations, into a standard course in College Algebra with Trigonometry, a stand-alone Trigonometry course, or a Precalculus course. These lessons have also been used in courses on the history of mathematics and as part of a capstone experience for pre-service secondary mathematics teachers.

In this unit, students consider an application of Ptolemy's table of chord lengths in a circle to determine the lengths of shadows of the Sun at noon on four days of the year, the summer and winter solstices and the spring and fall equinoxes. A circle provides a model of the observer's sky, which is combined with the right triangle formed by a gnomon (the shadow caster) and its shadow. Having inscribed this right triangle in another circle, Ptolemy invoked *Elements* III.20 to realize the sides of the triangle as chords of the circle. This simple geometry is the framework in which Ptolemy employed his Table of Chords to determine the dimensions of the segments in this geometric model. A final proportion determined the length of the shadow at the feet of the observer.

Student Prerequisites

Very little mathematical machinery is needed to work on this PSP. But it would help for students to have been exposed to sexagesimal numeration and to a reasonably good high school geometry course, especially with regard to the geometry of the circle and the measure of arcs and angles. A simple introduction to sexagesimal numeration is provided by the first mini-PSP in this series, "Babylonian Astronomy and Sexagesimal Numeration", and a presentation of the geometry of the circle and the measure of arcs and angles that serve as the foundation of trigonometry will be obtained by working the second mini-PSP in this series, "Hipparchus' Table of Chords".

Suggestions for Classroom Implementation

This project can be implemented in one 75-minute or two 50-minute class periods. For a detailed schedule of how to structure the periods, see the next subsection below. Students are not likely to come with much knowledge of observational astronomy and the variation in the positions of the Sun over the course of a year. These are described in the first pages of this project, but the careful instructor may consider enlisting the help of colleagues who teach introductory astronomy for resources and classroom models that could assist students in making sense of the geometry of the Sun-Earth system. For instance, the author brought a standard globe of the Earth to class to illustrate the concept of latitude, and also made use of a model of the celestial sphere (a transparent globe with constellations, the ecliptic circle, and celestial coordinates printed on its surface) borrowed from the Physics department to show the geometry of these astronomical elements for Earth-based observers.

The most important student tasks, Task 8 and 9, require the arithmetic of angles in their standard degree-minute-second sexagesimal forms. This will challenge students who have not worked much with this kind of arithmetic. A similar issue arises in working the proportion in Task 10.

Sample Implementation Schedule (based on two 50-minute class periods)

In preparation for the first class period, students should be asked to read through Section 2 of the project and try working the first two Tasks. Inform students that it is not expected that everything they read will make sense to them immediately, and that the classroom discussion should help to clear up many of these issues.

Begin the first class with a discussion of Tasks 1 and 2; the first of these is a reimagining of the diagram on p. 3 in its heliocentric form, while the second helps the students to discover that one's terrestrial latitude is also manifested as the angle of the elevation of Polaris in one's sky, a fact that is also noted by Ptolemy in the source text.

The bulk of the class period should be given over to making sense of the first excerpt from *Almagest* II.5 (your class might profit from having students read it aloud slowly to the class), making frequent comparison of the text with Ptolemy's diagram within the source. Once students are comfortable with a sense of what Ptolemy has presented in the first piece of text, have students work in small groups on Tasks 3 and 4. If time remains, they can also work on Tasks 5 and 6, which supplement the introduction to Ptolemy's Table of Chords on pp. 8–10.

Send students away from the first period with the assignment to write up solutions to Tasks 3 and 4 (and perhaps also Tasks 5 and 6) as homework for the next period. In addition, assign students to read ahead, up to the presentation of the second excerpt from the Almagest on p. 3, and to write up their solutions to Task 7.

At the beginning of the second class, a few minutes can be used to check how students dealt with the Tasks they worked on for homework. Then turn to the brief second excerpt following Task 7. Students should work in small groups on solutions for Tasks 8, 9 and 10. This work should be written up formally as part of a final homework assignment.

This implementation schedule can be easily modified for use in a single 75-minute period by assigning the write-up of Tasks 1, 2, 5, 6 and 7 along with the initial reading (described above) before the class meets. The post-class homework assignment will have students write up their solutions to the other five Tasks.

LaTeX code of this entire PSP is available from the author by request to facilitate implementation. The PSP itself may also be modified by instructors as desired to better suit the goals for their course.

Recommendations for Further Reading

Instructors who want to learn more about the history of trigonometry are recommended to consult Glen Van Brummelen's masterful *The Mathematics of the Heavens and the Earth: The early history of trigonometry* (Van Brummelen, 2009), from which much of this work took inspiration.

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