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From sets to metric spaces to topological spaces

Nicholas Scoville∗

1 Introduction

Felix Hausdorff (1868-1942) is known as one of the founders of modern topology, due in part to his 1914 systematic textbook *Grundzüge der Mengenlehre* (Fundamentals of Set Theory) [1]. This was one of the first book length treatments of topology, and remains a classic to this day. One of the significant contributions that Hausdorff made in this book was to clearly lay out for the reader the differences and similarities between sets, metric spaces, and topological spaces. It is easily seen how metric and topological spaces are built upon sets as a foundation, while also clearly seeing what is “added” to sets in order to obtain metric and topological spaces. We will follow Hausdorff as he builds topology “from the ground up” with sets as his starting point.

2 Set theory

Part of what makes Hausdorff’s work so interesting is that he does an excellent job of explaining how sets act as a foundation for other mathematical systems. He begins by discussing how set theory is the basis for varied and diverse branches of mathematics.

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Set theory has celebrated its loveliest triumphs in the application to point sets in space, and
in the clarification and sharpening of foundational geometric concepts, which will be conceded
even by those who have a skeptical attitude towards abstract set theory.

First we will be clear about the position of point set theory in the system of general set theory.
One can treat a set purely as a system of its elements, without considering relationships between
these elements.

Exercise 2.1. What does Hausdorff mean when he calls set theory “a system of its ele-
ments without considering relationships between these elements?” What are some purely set
theoretic notions that he might have in mind?

Hausdorff is interested in starting with the basics of set theory, which we are all familiar
with, and adding additional structure. He gives an example below.

Secondly however one can consider relations between the elements. . . This concerns itself, for any
two elements, with one of the three relations \( a \preccurlyeq \succcurlyeq b \), and we could interpret that as a function
\( f(a, b) \) of (ordered) pairs of the set being given, which however, can only take on three values
(only two when restricted to pairs of different elements). For partially ordered sets a fourth
relation or a fourth possible function value was also added.

Let’s investigate this further. Today we would add the adjective “totally” in front of
Hausdorff’s phrase “ordered set,” so when Hausdorff writes “ordered set,” we understand
this as “totally ordered set.” We will contrast this with a partially ordered set. A set \( X \) is
partially ordered if it has a reflexive, antisymmetric, transitive relation \( \leq \). The set is ordered
(or totally ordered) if it has a partial order \( \leq \) such that for every \( a, b \in X \), either \( a \leq b \) or
\( b \leq a \).

Exercise 2.2. a) Explain an ordered set is in your own words.
b) Give an example of an ordered set.

c) What are these “three values” that Hausdorff mentions above? In the case of a partially ordered set, what is the “fourth value”?

Next Hausdorff begins to generalize the particular phenomena that we saw above into a more general framework. This is a very common thing for mathematicians to do; that is, once a concrete, particular notion proves useful, to generalize it or abstract away the particulars in order to develop a more general, comprehensive framework.

Now there is nothing to prevent a generalization of this idea, and we can imagine that an arbitrary function of pairs of a set is defined; that is, to each pair \((a, b)\) of elements of a set \(M\), a specific element \(n = f(a, b)\) of a second set \(N\) is assigned.

For simplicity, let’s call Hausdorff’s above definition a 2-association on the set \(M\).

**Exercise 2.3.** Using modern notation, write down a definition of what it means to have a 2-association on the set \(M\). Explain how this is a generalization of a totally ordered set. Equivalently, explain how a totally ordered set is just a special case of the above.

In yet further generalizations we can take into consideration a function of a triple of elements, a sequence of elements, complexes of elements, subsets and the like of \(M\).

**Exercise 2.4.** Define using modern notation what it would mean to have an \(n\)-association on the set \(M\).

The definition you give in Exercise 2.4 generalizes Hausdorff’s “triple of elements” idea. His next three suggestions concern functions on a set with a certain structure or that satisfy a certain property. Let us make the following definition.
**Definition 2.1.** Let $M$ be a set and $P$ some property, $M_P$ the set of all collections of elements in $M$ that satisfy property $P$. We call any function $f: M_P \to N$ a **property $P$ function**.

**Exercise 2.5.** Let $M = N = \mathbb{N}^{\geq 2}$ and let $P$ be the property “is a prime.” Observe that $M_P$ is the set of all subsets of prime numbers. Find a property $P$ function (there are many, many options).

### 3 Metric Spaces

After having attempted to generalize to the $n^{th}$ degree in Definition 2.1, Hausdorff reels us back in, reminding us that sometimes we can be so general that very little can be said. Instead, Hausdorff finds the “golden mean” between too general and too specific.

A quite generally worded theory of this nature would of course cause considerable complications, and deliver few positive results. But among the special examples that occupy a heightened interest belongs, apart from the theory of ordered sets, especially the theory of point sets in space, in fact here the foundational relationship is again a function of pairs of elements, namely the distance between two points: a function which however now is capable of infinitely many values.

**Exercise 3.1.** Given a set $M$, give the 2-association on $M$ which associates a distance to pairs of elements in $M$.

As you have probably guessed, a distance isn’t simply any function that associates a real number to a pair of points. For example, it seems like we should not allow negative distances. It is also probably the case that we would want the distance from point $A$ to point $B$ to be the same distance as from point $B$ to point $A$. Now we consider the conditions for axioms that Hausdorff will require of a distance.
By a metric space we understand a set $E$, in which to each two elements (points) $x, y$ a non-negative real number is assigned, their distance $xy \geq 0$; in fact we demand moreover the validity of the following

**Distance axioms:**

(α) **(Symmetry axiom).** Always $yx = xy$.

(β) **(Coincidence axiom).** $xy = 0$ if and only if $x = y$.

(γ) **(Triangle axiom).** Always $xy + yz \geq xz$.

We denote specially as the Euclidean $n$-dimensional number space $E_n$ the set of complexes of real numbers

$$x = (x_1, x_2, \ldots, x_n),$$

in which the distance is defined by

$$xy = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \geq 0,$$

and as Euclidean $n$-dimensional space a metric space.

Exercise 3.2. Prove that in Euclidean space, (α) and (β) are always satisfied.

Exercise 3.3. Let $X = \{a, b, c, d\}$. Define a 2-association on $X$ which is a metric structure. That is, find a function on $X$ which associates to each pair of point in $X$ some real number that satisfies Hausdorff’s distance axioms. Since $X$ is a finite set, we call $X$ along with its associated metric a finite metric space.

In a metric space $E$ we understand by a neighborhood $U_x$ of the point $x$ the set of points $y$, whose distance from $x$ is less then a specific positive number $\rho$ ($xy < \rho$). Such a neighborhood depends on the center point $x$ and on the radius $\rho$; a point $x$ has, when one varies the radius, infinitely many neighborhoods, which however as sets need not in general all be different.
Exercise 3.4. Use your finite metric space in Exercise 3.3 to compute all the neighborhoods of the point \( a \). Even though there are infinitely many radii you may pick, only finitely many neighborhoods should differ and hence, you have provided an example of neighborhoods which are not all different.

4 Topological Spaces

Now that Hausdorff has a metric space (i.e. a set with the 2-association satisfying some properties), he will take away some of those properties to arrive at precisely the structure of a topological space in the following way. As you read, be aware that he is searching for axioms, trying to find the best way to, as we have made note of before, abstract away the particulars and identify the essential elements of what makes a topological space a topological space.

One can make this system of neighborhoods be the foundation of the whole theory, with elimination of the concept of distance. . . Thereby we will change our standpoint, as announced earlier, that we will abstain from distances, with whose help we defined neighborhoods, and accordingly place the mentioned properties as axioms in the lead.

By a topological space we understand a set \( E \), in which to the elements (points) \( x \) certain subsets \( U_x \) are assigned, which we call neighborhoods of \( x \), in fact subject to the following

**Neighborhood axioms:**

(A) To each point \( x \) there corresponds at least one neighborhood \( U_x \); each neighborhood \( U_x \) contains the point \( x \).

(B) If \( U_x, V_x \) are two neighborhoods of the same point \( x \), then there is a neighborhood \( W_x \), which is a subset of both \( W_x \subseteq \mathcal{D}(U_x, V_x) \).

\(^{1}\text{The script \( \mathcal{D} \) stands for the German word “durchschnitt,” meaning average or intersection.}\)
(C) If the point \( y \) lies in \( U_x \), then there is a neighborhood \( U_y \), which is a subset of \( U_x \) \( (U_y \subseteq U_x) \).

This definition of a topological space looks quite different than the one you have been exposed to. Nevertheless, we will work through a proof to show that the Neighborhood axioms are equivalent with the open set axioms. Recall that the open set axioms are given by

**Open set axioms** A set \( E \) along with a collection of subsets of \( E \) called open sets satisfy the following conditions:

1. The space \( E \) and the null set \( \emptyset \) are open.
2. The intersection of two open sets is open.
3. The union of any number of open sets is open.

We first need to define what we mean by “open” in terms of neighborhoods. Call a subset \( U \) of \( E \) open in \( E \) if \( U \) can be written as the union of neighborhoods.

**Exercise 4.1.** Prove if a collection of sets of \( E \) satisfies the neighborhood axioms, then the induced open sets (defined above) satisfy the open set axioms.

**Exercise 4.2.** Now assume that \( E \) has a collection of open sets. Show that if we define a neighborhood of a point to be any open set containing that point, then this collection of neighborhoods satisfies the Neighborhood axioms.

It is interesting to note that Hausdorff had a fourth neighborhood axiom

(D) For two different points \( x, y \), there are two neighborhoods \( U_x, U_y \) without any points in common \( (\emptyset (U_x, U_y) = 0) \).

Today this property is referred to as, appropriately enough, the Hausdorff property.

**Exercise 4.3.** Give an example of a topological space which does not satisfy the Hausdorff property (Hint: Try a finite topological space which is not a metric space.)
References

Notes to the Instructor

This project is intended for students in a topology course after they have had some familiarity and experience with a topological space. In particular, this is essential to understand the thrust of Exercise 4.1 and more generally, all of section 4 as the student is asked in this section to prove that Hausdorff’s neighborhood axioms are equivalent to the axioms that the student is familiar with. Students will start with sets and begin to add structure to these sets with a (hopefully) familiar example of an ordered set in Exercise 2.2. From there the student will be led through an abstraction of an ordered set to what we are calling, for lack of a better term, a 2-association in the remaining exercises in this section. In particular, expect a mess from your students in Exercise 2.3. This of course is by design, as coming up with “the right” definition or the right way to think about something is very challenging. After students have come up with answers to this question, discussing their answers, along with “the right” answer, can be a good opportunity for an in-class discussion. In section 3, a metric space is then seen as simply a special case of the more general abstraction that was developed in section 2. It is then Hausdorff’s observation about neighborhoods being defined in a metric space (quote starting “In a metric space $E$, we understand ...”) as the key transition from the metric space to the topological space. Hausdorff accomplishes this by taking the neighborhoods, defined from a metric space, and simply decreeing that these “neighborhoods” satisfy certain properties. The culmination of the project is then in Exercises 4.1 and 4.2 where the student works through a proof that these new axioms of Hausdorff are equivalent to the ones that they are familiar with. As an interesting historical side note, the project concludes by noting that Hausdorff included a fourth axiom in his neighborhood axioms, one which has been dropped as an axiom from the modern point of view. This is the property of being Hausdorff.

One or days class periods devoted to this project should suffice. The first half of the project, especially section 2, is more exploratory and hence, students may benefit from working in groups and discussing these questions among the group and then as a class. Another option is to have students read the first two sections for homework and come to
class prepared to discuss the project and ask questions. The material quickly becomes abstract and challenging, which is by design. This is to encourage students to ask questions and bounce ideas off of each other. The material and exercises in section 4 is much more “standard” in terms of presentation and what the student is expected to do, but also more challenging. In all, Exercises 4.1 and 4.2 have 6 different claims to prove. Hence, some can be assigned as homework, some in groups, and some presented in class by the instructor. It will be a rare student who is able to successfully work through both these exercises completely on his own without either seeing at least 1 done by the professor or seeing how to start such exercises. Part of the difficulty for students is knowing what you get to assume and what you need to show. So it is worth reminding the student that in Exercise 4.1 they get to assume that, for example, given any point $x$, there exists a neighborhood of $x$ (not to mention axioms (B) and (C)). But after working through these exercises, students should have a deeper appreciation for axiomatic systems and the position of topology in relation to other kinds of systems.

The Latex source file is available for modification from the author upon request (nscoville@ursinus.edu).

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