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From Sets to Metric Spaces to Topological Spaces

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1 Introduction

Felix Hausdorff (1868–1942) is known as one of the founders of modern topology, due in part to his systematic treatment of topological spaces in the influential textbook *Grundzüge der Mengenlehre* (*Fundamentals of Set Theory*), [Hausdorff, 1914]. This text, one of the first book-length treatments of set theory, was instrumental in establishing set theory as a foundation for the study of other mathematical fields — a viewpoint that is very familiar to us today, but was still gaining ground in the early 20th century.[1] After devoting the first seven chapters to general set theory, Hausdorff devoted the remainder of the text to a treatment of "point-set theory," as the study of topology was called at that time. One of the significant contributions he made in this treatment was to clearly lay out for the reader the differences and similarities between sets, metric spaces, and topological spaces. It is easy to see how metric and topological spaces are built upon sets as a foundation, while also clearly seeing what is "added" to sets in order to obtain metric and topological spaces. We will follow Hausdorff as he built topology "from the ground up" with sets as his starting point.

2 Set theory

Part of what makes Hausdorff's work so interesting is that he did an excellent job of explaining how sets act as a foundation for other mathematical systems. He began by discussing how set theory is the basis for varied and diverse branches of mathematics.[2]

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[1] Hausdorff fittingly dedicated his 1914 text to Georg Cantor (1845–1918), whom he described as "the creator of the theory of aggregates (sets)." Cantor's earliest work in set theory appeared in a series of papers published between 1874 and 1884, in which several early topological ideas also appeared in connection with his work in real analysis. The author has written several projects in topology based at least in part on Cantor's work, which appear as Chapters 15–18 in this collection.

[2] All translations of Hausdorff excerpts in this project were prepared by David Pengelley, New Mexico State University (retired), 2017.

> Set theory has celebrated its loveliest triumphs in the application to point sets in space, and in the clarification and sharpening of foundational geometric concepts, which will be conceded even by those who have a skeptical attitude towards abstract set theory.
>
> First we will be clear about the position of point-set theory in the system of general set theory. One can treat a set purely as a system of its elements, without considering relationships between these elements.
>
> [Hausdorff, 1914, p. 209]

Task 1 What did Hausdorff mean when he described treating a set as "a system of its elements without considering relationships between these elements?" What are some purely set-theoretic notions that he might have had in mind?

Hausdorff was interested in starting with the basics of set theory, with which we are now all familiar, and adding additional structure. He gave the example below.

> Secondly, however, one can consider relations between the elements . . . . This concerns itself, for any two elements, with one of the three relations $a \overset{<}{\underset{>}{=}} b$, and we could interpret that as a function $f(a, b)$ of (ordered) pairs of the set being given, which however, can only take on three values (only two when restricted to pairs of different elements). For partially ordered sets a fourth relation or a fourth possible function value was also added.
>
> [Hausdorff, 1914, p. 209]

Let's investigate this further. Today we would add the adjective "totally" in front of Hausdorff's phrase "ordered set," so when Hausdorff wrote "ordered set," we understand this as "totally ordered set." We will contrast this with a partially ordered set. Recall that a set X is **partially ordered** if it has a reflexive, antisymmetric, transitive relation $\leq$. The set is **ordered** (or **totally ordered**) if it has a partial order $\leq$ such that for every $a, b \in X$, either $a \leq b$ or $b \leq a$.

Task 2

(a) Explain what a totally ordered set is in your own words.

(b) Give an example of a totally ordered set. Also give an example of a partially ordered set which is not totally ordered.

(c) For a totally ordered set, what are these "three values" that Hausdorff mentioned above? In the case of a partially ordered set, what is the "fourth value"?

Next Hausdorff began to generalize the particular phenomena that we saw above into a more general framework. This is a very common thing for mathematicians to do; that is, once a concrete, particular notion proves useful, mathematicians tend to abstract away its particulars in order to develop a more general, comprehensive framework.

Now there is nothing to prevent a generalization of this idea, and we can imagine that an arbitrary function of pairs of a set is defined; that is, to each pair (a, b) of elements of a set M, a specific element $n = f(a, b)$ of a second set N is assigned.

[Hausdorff, 1914, p. 210]

For simplicity, let's call Hausdorff's above definition a **2-association on the set** M.

Task 3 Using modern notation, write down a definition of what it means to have a 2-association on the set M. Explain how this is a generalization of a totally ordered set. Equivalently, explain how a totally ordered set is just a special case of the above.

In yet further generalizations we can take into consideration a function of a triple of elements, a sequence of elements, complexes of elements, subsets and the like of M.

[Hausdorff, 1914, p. 210]

Task 4 Define using modern notation what it would mean to have an n-association on the set M.

The definition you gave in Task 4 generalizes Hausdorff's "triple of elements" idea. His next three suggestions (i.e.,"a sequence of elements, complexes of elements, subsets") concern functions on a set with a certain structure or that satisfy a certain property. Let us make the following definition.

Let M, N be sets, P some property, and M_P the set of all collections of elements in M that satisfy property P. We call any function $f: M_P \to N$ a **property** P **function**.

Task 5 Let $M = N = \mathbb{N}^{\geq 2}$ and let P be the property "is a prime." Observe that M_P is the set of all subsets of prime numbers. Find a property P function. (There are many, many options.)

3 Metric Spaces

After having attempted to generalize as much as possible in the definition of property P, Hausdorff reeled us back in by reminding us that sometimes we can be so general that very little of interest can be said. Instead, Hausdorff suggested an example that serves as a "golden mean" between too general and too specific.

A quite generally worded theory of this nature would of course cause considerable complications, and deliver few positive results. But among the special examples that occupy a heightened interest belongs, apart from the theory of a [totally] ordered sets, especially the theory of point-sets in space, in fact here the foundational relationship is again a function of pairs of elements, namely the distance between two points: a function which however now is capable of infinitely many values.

[Hausdorff, 1914, p. 210]

Task 6 Given a set M, give the 2-association on M that associates a distance to pairs of elements in M.

As you have probably guessed, a distance isn't simply any function that associates a real number to a pair of points. For example, it seems like we should not allow negative distances. It is also probably the case that we would want the distance from point A to point B to be the same distance as from point B to point A. Here is the complete list of conditions that Hausdorff required of a distance.

By a metric space we understand a set E, in which to each two elements (points) x, y a non-negative real number is assigned, their distance $\overline{xy} \geq 0$; in fact we demand moreover the validity of the following.

Distance axioms:

(α) **(Symmetry axiom).** Always $\overline{yx} = \overline{xy}$.

(β) **(Coincidence axiom).** $\overline{xy} = 0$ if and only if $x = y$.

(γ) **(Triangle axiom).** Always $\overline{xy} + \overline{yz} \geq \overline{xz}$.

We denote specially as the Euclidean n-dimensional number space E_n the set of complexes of real numbers

$$x = (x_1, x_2, \ldots, x_n),$$

in which the distance is defined by

$$\overline{xy} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \geq 0, \ldots$$

[Hausdorff, 1914, pp. 211-212]

Task 7 Prove that in Euclidean space E_n, the axioms (α) and (β) are always satisfied.

Task 8 Let $X = \{a, b, c, d\}$. Define a 2-association on X which is a metric structure. That is, find a function on X which associates to each pair of point in X some non-negative real number and satisfies Hausdorff's three distance axioms. Since X is a finite set, we call X along with its associated metric a **finite metric space**.

In a metric space E we understand by a neighborhood U_x of the point x the set of points y, whose distance from x is less then a specific positive number ρ ($\overline{xy} < \rho$). Such a neighborhood depends on the center point x and on the radius ρ; a point x has, when one varies the radius, infinitely many neighborhoods, which however as sets need not in general all be different.

[Hausdorff, 1914, p. 212]

Task 9 Use your finite metric space in Task 8 to compute all the neighborhoods of the point a. Even though there are infinitely many radii you may pick, only finitely many neighborhoods should differ; hence, the example you have provided an example should consist of neighborhoods which are not all different.

4 Topological Spaces

Now that Hausdorff had a definition for a metric space (i.e. a set together with a 2-association satisfying some properties), he took away the 2-association itself and instead focused on the properties of "neighborhoods" to arrive at a definition of the structure of a general topological space. As you read that definition below, be aware that Hausdorff was *searching* for axioms, by trying to find a way (as we have noted before) to abstract away the particulars and identify the essential elements of what makes a topological space a topological space.

One can make this system of neighborhoods be the foundation of the whole theory, with elimination of the concept of distance ... . Thereby we will change our standpoint, as announced earlier, that we will abstain from distances, with whose help we defined neighborhoods, and accordingly place the mentioned properties as axioms in the lead.

By a topological space we understand a set E, in which to the elements (points) x certain subsets U_x are assigned, which we call neighborhoods of x, in fact subject to the following

Neighborhood axioms:

(A) To each point x there corresponds at least one neighborhood U_x; each neighborhood U_x contains the point x.

(B) If U_x, V_x are two neighborhoods of the same point x, then there is a neighborhood W_x, which is a subset of both ($W_x \subseteq \mathfrak{D}(U_x, V_x)$).[3]

(C) If the point y lies in U_x, then there is a neighborhood U_y, which is a subset of U_x ($U_y \subseteq U_x$).

[Hausdorff, 1914, p. 213]

[3]The script D here stands for the German word "durchschnitt," meaning average or intersection.

Task 10 Determine which (if any) of the Neighborhood axioms are satisfied by the example you came up with in Task 8.

Hausdorff's definition of a topological space may look quite different than the one to which you have been exposed. Nevertheless, we will work through a proof to show that the neighborhood axioms are equivalent with the open set axioms that have become standard in today's definition of a topological space. We first recall that definition:

> By a topological space we understand a set E along with a collection of subsets of E, called open sets, that satisfy the following conditions:
>
> **Open set axioms**
>
> 1. The space E and the null set $\emptyset$ are open.
> 2. The intersection of two open sets is open.
> 3. The union of any number of open sets is open.

We also need to define what we mean by "open" in terms of neighborhoods. Call a subset U of E **open in** E if U can be written as the union of neighborhoods.

Task 11 Prove that if a collection of sets of E satisfies the neighborhood axioms, then the induced open sets (defined above) satisfy the open set axioms.

Task 12 Now assume that E has a collection of open sets. Show that if we define a neighborhood of a point to be any open set containing that point, then this collection of neighborhoods satisfies the neighborhood axioms.

5 Conclusion

It is interesting to note that Hausdorff also had a fourth neighborhood axiom:

> **(D) For two different points x, y, there are two neighborhoods U_x, U_y without any points in common ($\mathfrak{D}\,(U_x, U_y) = 0$).**
>
> [Hausdorff, 1914, p. 214]

Today this property is referred to as, appropriately enough, the **Hausdorff property**. Although Hausdorff himself proposed in his 1914 textbook that this property was part and parcel of what it meant to be a topological space, it has since come to define only one specific type of topological space. Our final two tasks take a closer look at this, providing yet another example of the ways in which today's mathematicians continually seek to generalize the specific.

Task 13 Prove that every metric space satisfies the Hausdorff property.

Task 14 Give an example of a topological space which does not satisfy the Hausdorff property. (Hint: Try a finite topological space which is not a metric space.)

References

F. Hausdorff. *Grundzüge der Mengenlehre (Fundamentals of Set Theory)*. Von Veit, Leipzig, 1914.

Notes to Instructors

Primary Source Project Content: Topics and Goals

This Primary Source Project (PSP) develops the axioms for a topology by starting with a metric space and abstracting away to a topological space. In that sense, the viewpoint is similar to other standard approaches to developing point-set topology (e.g., C. J. Patty's *Foundations of Topology*, Jones and Bartlett Publishers, 2009). What differs is the means by which we move from metric spaces to topological spaces. The project accomplishes this in part by investigating the less well-known set of neighborhood axioms for a topological space. These axioms begin with the undefined concept a neighborhood of a point and lay out properties that a neighborhood must satisfy. It is shown in Tasks 11 and 12 that these axioms are equivalent to the open set axioms. The goals of this project are then two-fold. The first goal is to develop the open set axioms within the more familiar setting of a metric space. The second goal is to expose the axiomatic foundation of topology, both for the sake of seeing an axiomatic system and to see that it has multiple equivalent starting points. This latter goal is not usually emphasized in a topology course. Of course, one usually sees that the open set axioms are equivalent to the closed set axioms, but this hardly seems like a completely different axiomatic system.

Student Prerequisites

This project is intended for students in a topology course after they have had some familiarity and experience with a topological space. In particular, this is essential to understand the thrust of Task 11 and more generally, all of Section 4, as students are asked in that section to prove that Hausdorff's neighborhood axioms are equivalent to the axioms with which they are already familiar.

PSP Design and Task Commentary

Following a brief introduction, Section 2 of the PSP starts with sets and begins to add structure to these sets with a (hopefully) familiar example of an ordered set in Task 2. From there, the remaining tasks in this section lead students through an abstraction of an ordered set to what we are calling, for lack of a better term, a "2-association." In particular, expect a mess from your students in Task 3. This is by design, as coming up with "the right" definition or the right way to think about something is very challenging. After students have come up with answers to this question, discussing their answers, along with "the right" answer, can be a good opportunity for an in-class discussion.

In Section 3, a metric space is presented simply as a special case of the more general abstraction that was developed in Section 2. Hausdorff's statement of the distance axioms for a metric space are examined in Task 7 and 8, the latter of which asks students to define a metric on a four-element set in order to obtain a finite metric space. The final task in this section (Task 9) then has students use their example from Task 8 to justify Hausdorff's claim that "a point x has, when one varies the radius, infinitely many neighborhoods, which however as sets need not in general all be different." The goal here is to help students move away from relying on Euclidean n-space as their sole example of what a metric space must look like.

Hausdorff's observations about the definition of a neighborhood in a metric space (excerpt starting "In a metric space E, we understand ... " at the close of Section 3) also provide the key transition from the metric space to the topological space that is developed in Section 4. Hausdorff accomplished this transition by taking the concept of neighborhoods, as defined in a metric space, and simply decreeing certain properties satisfied by that concept as the axioms for a topological space. The culmination of the project is then in Tasks 11 and 12 of Section 4, where students work through a proof that Hausdorff's newly-created neighborhood axioms are equivalent to the ones with which they are familiar. After working through these tasks, students should have a deeper appreciation for axiomatic systems and the position of topology in relation to other kinds of systems.

As an interesting historical side note, the project concludes (in Section 5) by noting that Hausdorff included a fourth axiom in his neighborhood axioms, one which has been dropped as an axiom from the modern point of view. This is the property of being Hausdorff, which is briefly examined via specific examples in Tasks 13 and 14.

Suggestions for Classroom Implementation and Sample Schedule (based on a 50-minute) class period

The following outline provides a schedule for implementing this project in 3 class days.

Have students read Section 1 and the beginning of Section 2 (through Task 1) and work on Task 1 as advance preparation for the first day of class. Class can then begin by having students share their ideas in either small groups or with the class as a whole. The purpose of this is to get all the students on the same page as far as thinking about properties of purely sets. You can then pose to the class the question of what kinds of properties do things have that imply more than simply a set. This should lead to a whole-class discussion about what is needed in order to have these properties (e.g., to add elements you need a group structure, to take the closure you need a topology, to take distance you need a metric). This discussion sets the stage for students to work through the rest of section 2 in small groups for about 20–25 minutes. Section 2 is more exploratory and students will benefit from discussing these questions in small groups and then as a whole class. After sufficient time has passed, spend 10 minutes debriefing with the class as a whole. With any remaining time, students can begin work on Section 3. Have students write up the tasks from Section 2 for homework. Also have them read the first excerpt of Section 3 (at least through Task 6), and prepare their preliminary answers to Task 6 as advance preparation for the second day of class.

When class begins on Day 2, have students work together on Tasks 6–8 in small groups. The material in Section 3 quickly becomes abstract and challenging, which is by design. This is to encourage students to ask questions and bounce ideas off of each other. It would then be a good idea to either work on Task 8 as a class, or have groups share their responses. The remainder of the class should be devoted to finishing Section 3. Have students write up the tasks from Section 3 for homework. Also have them read Section 4 as advance preparation for the third day of class, and to begin thinking about how they might complete Tasks 10, 11, and 12.

Day 3 can then be devoted to Section 4 of the project, and as much of Section 5 as is reasonable to complete in-class. The material and tasks in Section 4 are much more "standard" in terms of its presentation and what students are expected to do, but also more challenging. In all, Tasks 11 and 12 have 6 different claims to prove. It will be a rare student who is able to successfully work through these claims completely on her own without either seeing at least one done by the professor or seeing how to start such tasks. Hence, you may wish to present at least one of these claims in-class yourself, assign some parts for completion in small groups, and assign the remainder as individual homework. Part of the difficulty for students is knowing what they get to assume and what they need to show. It is accordingly worth reminding them that in Task 11, they get to assume that, for example, given any point x, there exists a neighborhood of x by axiom (A) [not to mention the assumptions given by axioms (B) and (C)]. One strategy that can work well for this is to let students work freely in groups and stop them after 10 minutes or so to remind them of what they get to assume for Task

7. Alternatively, you can walk around the room and see where students tend to get stuck in these two tasks before giving a hint to the whole class. After the hint, have students return to group work and finish the tasks for homework. In addition to any parts of Tasks 11 and 12 that are assigned for individual completion, have students write up the tasks from Section 5 for homework.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Connections to other Primary Source Projects

The following additional primary source-based projects by the author are also freely available for use in teaching courses in point-set topology. The first two projects listed are full-length PSPs that require 10 and 5 class periods respectively to complete. All others are designed for completion in 2 class periods.

- *Nearness without Distance*
- *Connectedness: Its Evolution and Applications*
- *Topology from Analysis* (Also suitable for use in Introductory Analysis courses.)
- *The Cantor set before Cantor* (Also suitable for use in Introductory Analysis courses.)
- *Connecting Connectedness*
- *The Closure Operation as the Foundation of Topology*
- *A Compact Introduction to a Generalized Extreme Value Theorem*

Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_topology`. They can also be obtained (along with their LaTeX code) from the author.

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