# Riemann's Development of the Cauchy-Riemann Equations

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## 1 Introduction

This project investigates the pair of differential equations often called the Cauchy-Riemann Equations (CRE), and the properties of functions that satisfy these equations. The CRE were first investigated by Jean-Baptiste le Rond d'Alembert (1717–1783) and Leonhard Euler (1707–1783) in 1752 while studying problems in fluid motion. Later, in 1814, Augustin-Louis Cauchy (1789–1857) referenced Euler's work on the CRE and discussed them in the context of evaluating improper integrals in a paper that he presented to the French Academy of Sciences [Cauchy, 1814]. None of these three mathematicians discussed the CRE in connection with geometrical interpretations of complex numbers in the complex plane. In contrast, Bernhard Riemann (1826–1866) brought the geometry of the complex plane into consideration with his discussion of the CRE and differentiable complex functions in his highly influential 1851 Inaugural dissertation[1] entitled *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse (Foundations for a General Theory of Functions of a Complex Variable)*, [Riemann, 1851].

We begin by carefully reading the following passage from the first two sections of Riemann's Inaugural dissertation. Riemann was mostly interested in differentiable functions, and he used the term *function* to mean *differentiable function* in the excerpt below. Also keep in mind that mathematicians of Riemann's time frequently used the differential concept when thinking about derivative.[2] A differential $dw$ or $dz$ was considered an infinitely small quantity, and the derivative $\frac{dw}{dz}$, when defined, was considered equivalent to the ratio of the differentials $dw$ and $dz$.

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[1] An Inaugural dissertation is the first of two doctoral theses that are required within the German university system to obtain a professorship; the second of these theses is known as the Habilitation thesis. Riemann's 1854 Habilitation thesis, entitled "Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe" ("On the Representability of a Function by a Trigonometric Series"), is also famous among mathematicians for its introduction of what is now called the Riemann integral for real-valued functions.

[2] *Historical note*: The concept of differential in Riemann's time had changed from how it was first used by Leibniz and other 17th and 18th century mathematicians, or by Euler and his 18th century colleagues.

Let

$$x + iy, \quad x + iy + dx + idy \tag{1}$$

be two values of the quantity $z$ with an infinitely small difference, and let

$$u + iv, \quad u + iv + du + idv \tag{2}$$

be the corresponding values of $w$.

...

A complex variable $w$ is said to be a [differentiable] function of another complex variable $z$, if $w$ varies with $z$ in such a way that the value of the derivative $\frac{dw}{dz}$ is independent[‡] of the value of the differential $dz$.

Both quantities $z$ and $w$ will be treated as variables that can take every complex value. It is significantly easier to visualize variation over a connected two-dimensional domain, if we link it to a spatial viewpoint.

We represent each value $x + iy$ of the quantity $z$ by a point $O$ of the plane $A$ having rectangular coordinates $x, y$; and every value $u + iv$ of the quantity $w$ by a point $Q$ of the plane $B$, having rectangular coordinates $u, v$. Dependence of $w$ on $z$ is then represented by the dependence of the position of $Q$ on the position of $O$.

Riemann defined $u = u(x, y)$ and $v = v(x, y)$ as real functions of real variables $x$ and $y$ in this passage, and throughout this project.

**Task 1** As an example, let $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Find $u$ and $v$ when $x = 1, y = 3$. Plot these particular points $x + iy = 1 + 3i$ and $u + iv$ on two complex planes $A$ and $B$.

**Task 2** Sketch the two points listed by Riemann in (1) on complex plane $A$ and sketch the two points listed by Riemann in (2) on complex plane $B$, for arbitrary $x, y, u, v$. For the purposes of the sketches, treat the differentials as finite values, tiny in comparison with $x, y, u, v$.

**Task 3** Use this passage from Riemann and your sketches in (2) to explain why $dz = dx + idy$ and $dw = du + idv$. Hint: Think about how we add complex numbers geometrically.

Modern mathematicians don't assume that a complex function must be differentiable, as we shall see in an example below. However, Riemann was mostly interested in differentiable functions.

[‡] *Riemann's footnote*: This assertion is obviously justified in all the cases where one can obtain from the expression of $w$ in terms of $z$, using the rules of differentiation, an expression for $\frac{dw}{dz}$ in terms of $z$. The rigorous general validity of the assertion is left aside for now.

**Task 4** Recall from the multivariable calculus chain rule that the differential of real-valued function $u(x,y)$ is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

(a) State a general formula for $dv$ using the chain rule.

(b) Consider the complex function $w = z^2$ where $z = x + iy$. Write this $w$ in the form $u(x,y) + iv(x,y)$.

(c) Compare your $u(x,y)$ and $v(x,y)$ formulas to those in Task 1. Then confirm directly that $(1+3i)^2 = -8+6i$, as you found in Task 1.

(d) Find $du$ and $dv$ in terms of $x, y, dx, dy$ for $w = z^2$, and then find $dw$.

**Task 5** Consider the complex function $w = \bar{z} = x - iy$ where $z = x + iy$.

(a) Write this $w$ in the form $u(x,y) + iv(x,y)$.

(b) Find $du$ and $dv$ in terms of $x, y, dx, dy$ for $w = \bar{z}$, and then find $dw$.

For the purposes of this project, we will accept Riemann's assertion that a function is differentiable when "the derivative $\frac{dw}{dz}$ is independent of the value of the differential $dz$." The next two tasks examine what this idea means.

**Task 6** To illustrate why $dw/dz$ being independent of $dz$ makes sense for a differentiable function $w$, algebraically simplify

$$\frac{dw}{dz} = \frac{du + idv}{dx + idy}$$

for the function $w = z^2$ discussed in Task 4. You should find that your answer is independent of $dx$ and $dy$. According to Riemann, what does this tell you about the function $w = z^2$? Hint: First regroup the expression $du + idv$ into the sum of two terms, one of which has factor $dx$ and one of which has factor $idy$ (using fact $-1 = i \cdot i$).

Now that we have explored an example of a differentiable function, let's examine a useful function that turns out *not* to be differentiable.

**Task 7** Consider the function $w = \overline{z}$.

(a) Write $\frac{dw}{dz}$ in terms of $dx$ and $dy$ using your answer from Task 5.

(b) Simplify your answer to part (a) when $dy = 0$ and $dz = dx$.

(c) Simplify your answer to part (b) when $dx = 0$ and $dz = idy$.

(d) What can you conclude from your work? Justify your answers using Riemann's notion of derivative.

Riemann stated some very important and remarkable properties of (differentiable) complex functions in the following passage from Section 4 of his Inaugural dissertation. Remember that he used the term *function* to mean *differentiable function*.

If we write the differential quotient $\frac{du+idv}{dx+idy}$ in the form

$$\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}i\right)dx+\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}i\right)dyi}{dx+dyi} \tag{3}$$

it is plain that it will have the same value for any two values of $dx$ and $dy$, exactly when

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}. \tag{4}$$

Hence this condition is necessary and sufficient for $w=u+vi$ to be a [differentiable] function of $z=x+iy$. For the individual terms of the function, we deduce the following:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \tag{5}$$

This equation is the basis for the investigation of the properties of the individual terms of the function.

If you read this passage carefully, you may have noticed that Riemann left out some details and packed a lot of information into just a few sentences. We will explore all this in the next set of tasks.

**Task 8** In your own words, explain what Riemann meant by "Hence this condition is necessary and sufficient for $w=u+vi$ to be a function of $z=x+iy$."

**Task 9** Riemann began the passage by rewriting $\frac{dw}{dz}=\frac{du+idv}{dx+idy}$ in the form (3). Use the chain rule and some algebra to verify this equality. Hint: The algebra should remind you of the regrouping and factoring from Task 6.

Riemann made two separate claims after rewriting $\frac{dw}{dz}=\frac{du+idv}{dx+idy}$ in the form (3). First, if the equations (4) are true, then $w$ is a differentiable function of $z$. To verify this claim, do the next task.

**Task 10** Assume equations (4) are valid, and use them and algebra to simplify the form (3) of $\frac{dw}{dz}$. Then explain why this expression is independent of $dz$ and "will have the same value for any two values of $dx$ and $dy$." Thus, as Riemann explained in his first passage, this means that $w$ is a differentiable function of $z$.

The second claim Riemann made after writing equations (4) was that these equations (4) *must* be true if $w$ is differentiable. That is, if the derivative $\frac{dw}{dz}$ in form (3) has the same value for any pair of $dx, dy$ values, then the equations (4) are valid. To see why this is valid, do the following task.

**Task 11** Choose $dz=dx$ (with $dy=0$) and then choose $dz=idy$ (with $dx=0$). Simplify the derivative $\frac{dw}{dz}$ in each case, using form (3). What can you conclude?

In the next three tasks, we apply the equations (4) to some example functions.

**Task 12** Use equations (4) to show that $w = z^3$ is differentiable at any complex value $z = x+iy$.

As we saw in Task 10, we can find formulas for the derivative of $w$ using the equations (4):

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{6}$$

**Task 13** Use one of these formulas in (6) to show that the derivative of $w = z^3$ is $3z^2$.

**Task 14** Let $z = x + iy$ and define $w = z^2 + iy^2 + (x-1)^2$.

(a) Use the CRE to determine the values of $z$ for which $\frac{dw}{dz}$ exists.

(b) Sketch the set of points from part (a) on the complex plane.

(c) Label the point $3 + 2i$ on your sketch, and find the value of $\frac{dw}{dz}$ at this point.

(d) If we move just a tiny bit away from the point $3 + 2i$ in a random direction to another point $P$, will $\frac{dw}{dz}$ be defined at this point $P$? Explain.

(e) Using your basic understanding of continuity from calculus, do you think that the derivative function $\frac{dw}{dz}$ is continuous (as a function of $z$) at the point $3 + 2i$? Explain.

(f) Let $g(z) = \frac{dw}{dz}$. Find a formula for $g(z)$ using a formula from (6). Then find the values of $z$ for which $\frac{dw}{dz}$ exists.

(g) Reflect on your answers to (e) and (f).

We have seen the equations (4) are very important for differentiable complex functions. They are nowadays usually called the **Cauchy-Riemann equations** (CRE). As Riemann stated in the passage above, the real-valued functions $u$ and $v$ also satisfy the same differential equation (5), called **Laplace's equation**, which is very important in physics. A function that satisfies Laplace's equation is called **harmonic** and has many interesting properties.

**Task 15** Let $c, k$ be constants, $f(x,y) = e^{cx}\sin(cy)$, $w(x,y) = 5y^2 - 5x^2$, and $p(x,y) = 3x^2 + 3y^2$.

(a) Show that $f$ is harmonic.

(b) Show that $w$ is harmonic.

(c) Show that $f + w$ and $kf$ are harmonic functions.

(d) Show that $p$ is not harmonic.

**Task 16** Suppose that $g$ and $h$ are harmonic functions and $k$ is a constant.

(a) Show that $g + h$ is harmonic.

(b) Show that $kg$ is harmonic.

**Task 17** Suppose that $w = u(x,y) + iv(x,y)$ is a differentiable function of $z = x + iy$. Use the CRE (4) to show that $u$ and $v$ satisfy Laplace's equation (5). That is, the real and imaginary parts of a differentiable complex function must each be harmonic! Hint: Use a multivariate calculus fact about mixed partial derivatives.

## 2 Some Modern Reflections on Differentiability

We next address the modern definition of the derivative and give a modern theorem connecting differentiability and the CRE (4).

While Riemann used differentials to gain insight into the differentiability of a function $w = f(z)$, in modern texts we use limits for our definition: we say $f$ has a **derivative at** $z_0$ provided the following limit exists, where $\Delta z = \Delta x + i\Delta y$:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{7}$$

This definition is particularly useful when the function $f$ is difficult to put into the form $u(x, y) + iv(x, y)$.

**Task 18** In Tasks 12 and 13 we showed that $z^3$ is differentiable with derivative $3z^2$ using the CRE and formula (6).

(a) Use this modern definition to show that $f(z) = z^3$ is differentiable at any value $z$ with $f'(z) = 3z^2$.

(b) Use this modern definition and the binomial theorem to show that $f(z) = z^n$ is differentiable at any value $z$ when $n$ is a positive integer. Give a formula for $f'(z)$.

In our second excerpt from Riemann, where he gave the CRE, he stated that the differential quotient

$$\frac{dw}{dz} = \frac{du + idv}{dx + idy}$$

has the same value for any two values of $dx$ and $dy$ exactly when $w$ is differentiable. In Task 11 we set $dz = dx$ and then $dz = idy$ to derive the CRE. In the next task, you will explore this using the modern definition.

**Task 19** Suppose $f'(z_0)$ exists and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ exist at $z_0$.

(a) Use the modern limit definition (7) to show that

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Hint: Use Riemann's idea of "any two values of $dx$ and $dy$"[3] with $\Delta z = \Delta x$, and the limit definition of partial derivative. That is, take the limit as $\Delta z = \Delta x$ approaches zero but $y$ stays constant.

[3]In modern terminology, Riemann's idea translates to saying the limit is independent of the path we take as $\Delta z$ approaches zero.

**Task 19 – continued**

(b) Use the modern limit definition (7) to show that

$$f'(z_0) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Hint: Again use Riemann's idea of "any two values of $dx$ and $dy$," but with a different choice of $\Delta z$ for the limit.

(c) Combine parts (a) and (b) to complete a modern derivation of the CRE.

In Riemann's footnote to his first passage, he indicated that a complex function is differentiable in all the cases where one can obtain from the expression of $w$ in terms of $z$, using the rules of differentiation, an expression for $\frac{dw}{dz}$ in terms of $z$. We saw this above in Task 18 where we derived $(z^n)' = nz^{n-1}$ for positive integer $n$. We can derive a number of standard derivative rules for complex functions using the modern limit definition that should remind you of introductory calculus rules.

**Task 20** Suppose that $f$ and $g$ are differentiable at $z_0$. Use the modern definition of derivative to prove that $f+g$ is differentiable at $z_0$. Give the derivative formulas for this function in terms of $f'$ and $g'$.

More complex function derivative rules can be developed in a similar manner, with proofs similar to those from introductory real function calculus. However, there are some important and fascinating differences between differentiability for real-valued and complex-valued functions, as the Cauchy-Riemann equations phenomenon suggests. Riemann made other important discoveries in this field in his Inaugural dissertation. This story is developed further in a course on complex variables.

## References


A.-L. Cauchy. Mémoire sur les intégrales définies, lu á l'institut le 22 auôt 1814 (Memoir on definite integrals, read to the institute on 22 august 1814). In *Oeuvres complètes Ser. 1*, volume 1, pages 319–506. Gauthier-Villars, Paris, 1814. Volume 1 of Cauchy's *Oeuvres complètes* was published in 1882.

B. Riemann. *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse (Foundations for a General Theory of Functions of a Complex Variable)*. 1851. Inauguraldissertation. English translation by Roger Baker, Charles Christenson and Henry Orde in *Collected Papers of Bernhard Riemann*, Kendrick Press, 2004, pages 1–39.

## Notes to Instructors

### PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed to be used in a course on complex variables. It could also be used in a demanding upper division course on the history of mathematics. Riemann's differential approach works nicely at an intuitive level and motivates modern proofs. On the other hand, a rigorous analysis of what Riemann meant by differentiability ("the value of the derivative $\frac{dw}{dz}$ is independent of the value of the differential $dz$." ) is difficult and deliberately downplayed.

More specifically, the content goals of this project are to:

1. Derive the Cauchy-Riemann equations (CRE) and examine their equivalence with differentiability, using Riemann's differential approach.
2. Apply the CRE to example functions.
3. Introduce and explore harmonic functions.
4. Examine the modern definition of differentiability.
5. Use the modern definition of differentiability and Riemann's ideas to give a modern derivation of the CRE.

### Student Prerequisites

The PSP is written with very few assumptions about student background beyond an introductory calculus course sequence and some basic familiarity with complex numbers. Some comfort with partial derivatives is important.

### PSP Design and Task Commentary

This is roughly a one or two week project. For a complex variables course, the PSP is designed to be used early in the course, largely in place of text section(s) introducing differentiability.

The main goal of Tasks 1–3 is to help students see spatially/geometrically what is going on with the four points in (1) and (2). In particular, these tasks give some geometric motivation for the definitions $dz = dx + idy$ and $dw = du + idv$. This is important in the next excerpt where Riemann started with a differential quotient $\frac{du+idv}{dx+idy}$ and students need to recognize this as $dw/dz$. Moreover, Riemann talked about "any two values of $dx$ and $dy$," and it is valuable for students to be able to visualize these as horizontal and vertical increments in the complex plane. This will be helpful in Tasks 11 and 19.

Tasks 8, 10, 11, 16, 17 and 20 may be challenging for students with little mathematical sophistication.

Task 14 explores a function that is differentiable only on a line in the complex plane, so it is not analytic anywhere. The first part of the task provides practice using the CRE, and the remaining parts explore the consequences of being differentiable on such a "thin" set.

Harmonic functions are very important in a complex variables course, but Tasks 15–17 could be omitted in a history of mathematics course.

### Suggestions for Classroom Implementation

Advanced reading of the project and some task work before each class is ideal but not necessary. See the sample schedule below for ideas.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

### Sample Implementation Schedule (based on a 50-minute class period)

Full implementation of the project can be accomplished in 4 class days, as outlined below.

Students read through the first Riemann passage and do Tasks 1–4 before class. After a class discussion of these first four tasks, students work through Task 6 in groups and then read the second excerpt from Riemann, followed by doing Task 8, with some class discussion, during the first class. Tasks 5, 7 and 9 are assigned for homework.

During the second class, groups work through Tasks 10–12 with some class discussion. Task 13 and 14 are assigned for homework.

During the third class period, students read about Laplace's equation and harmonic functions, and do Tasks 15 and 17. For homework, students do Task 16, read the modern definition of differentiability and do Task 18 (a).

During the fourth class, students finish the PSP.

### Connections to other Primary Source Projects

The following primary source-based projects are also freely available for use in teaching courses in complex variables. The number of class periods required for full implementation is given in parentheses. Classroom-ready pdf versions of each can be obtained (along with their LaTeX code) from their authors or downloaded from `https://digitalcommons.ursinus.edu/triumphs_complex/`.

- *Euler's Square Root Laws for Negative Numbers* by David Ruch (1–2 days)
- *The Logarithm of –1* by Dominic Klyve (2 days)
- *An Introduction to Algebra and Geometry in the Complex Plane* by Nicholas A. Scoville and Diana White (5 days)
- *Argand's Development of the Complex Plane* by Nicholas A. Scoville and Diana White (5 days)
- *Gauss and Cauchy on Complex Integration* by David Ruch (3 days)

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