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A Compact Introduction to a Generalized Extreme Value Theorem

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1 Introduction

Compactness is a concept that is often introduced in a first course in analysis or topology, and it often gives students great difficulty. It can be very difficult to figure out what the standard definition is saying and even more of a mystery that possessed someone to write down such a definition in the first place. In this short project we will look to when compactness was first defined (albeit slightly different from today's definition) and see what use it had and more importantly, the role it plays in mathematics. This was in an extremely short paper by French mathematician Maurice Fréchet (1878-1973). Fréchet was working at a time when topology was beginning to develop into its own branch of mathematics, due largely in part to his work. We will follow his paper carefully.

2 Fréchet

Fréchet begins his 1904 paper [1] as follows:

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We know how important it would be, in a great number of problems, namely if a quantity U dependent on some elements (points, functions, etc.) actually reached a minimum in the considered range ... The question is solved in the particular case when U is a continuous function of x (or more independent variables). Weierstrass indeed proves that any continuous function in a limited range attains a maximum at least once. There would be great interest in extending this proposition in order to answer the more general problem that we recalled. It is this extension that is the subject of the present note.

The goal of this project is the same as Fréchet's stated goal; that is, to extend the result of Weierstrass to accommodate things like functions, surfaces, etc. But first, let us recall what this result of Weierstrass is. In modern terms, the result that Weierstrass proved is the familiar "extreme value theorem" that you learned in calculus. With that in mind, answer the following:

Exercise 2.1. What must Fréchet mean by a "limited range"? Use a calculus textbook to recall both the idea behind the extreme value theorem and the details if you don't remember.

Exercise 2.2. For reference, state the extreme value theorem using modern language and notation.

Exercise 2.3. When Fréchet talks about "extending" the extreme value theorem, what quantities in the statement is he hoping to replace with what?

Suppose we are given a certain collection $\mathcal{C}$ of elements of any kind (numbers, surfaces, etc.), in which we know how to distinguish the distinct elements. We say that U is a *uniform function (or functional operator) on a set* **E** of elements of $\mathcal{C}$, if an element A of **E** corresponds to a well defined number $U(A)$.

Let us investigate Fréchet's definition, especially in light of his initial comment about Weierstrass' Theorem.

Exercise 2.4.

- Let $\mathcal{C} = \mathbf{R}$ and $U(A) := A^2$. Show that this is a functional operator. Determine if U reaches a minimum for
  - $\mathbf{E} = \mathbf{R}$
  - $(0, 4)$
  - $[0, 4]$

  What is your result a special case of?

- Let $a < b$ be real numbers and $\mathcal{C}$ be all continuous functions on the closed interval $[a, b]$. For each $f \in \mathcal{C}$, define $U(f) := \max_{x\in[a,b]}\{f(x)\}$. Give an example of a set $\mathbf{E} \subseteq \mathcal{C}$ on which U does attain its maximum and a set for which it does not attain its maximum. Can you do this with the further specification that $\mathbf{E}$ is infinite?

Notice the second example illustrates a functional operator that is unlike a typical function that we study in calculus. In this context, we may not know what it would mean for a functional operator to be continuous. Part of Fréchet's need here is to define continuity in a more abstract setting. In order to do this, Fréchet needs the concept of limit, which he now discusses.

To arrive at the concept of continuity of such a function, suppose we have acquired a definition which gives a precise meaning to this phrase: *the infinite sequence* $A_1, A_2, \ldots, A_n, \ldots$ *of elements of* $\mathcal{C}$ *has a limit* B. It suffices that this definition, indeed any definition, satisfy the following two conditions : 1^o If the sequence $A_1, A_2, \ldots, A_n, \ldots$ has a limit, each sequence $A_{p_1}, A_{p_2}, \ldots,$ formed by elements of increasing index from the first sequence also has a limit which is the same;

2^o If none of the elements $A_1, A_2, \ldots$ of the sequence are distinct from A, this sequence has a limit which is A.

Notice that Fréchet does not actually give a definition of limit but rather, requires two properties that he thinks any good definition of limit should have. In order to make this more concrete, we'll illustrate his conditions in $\mathbf{R}$. Recall that in $\mathbf{R}$, a sequence $a_1, a_2, \ldots, a_n, \ldots$ of real numbers has a limit a if for every $\epsilon > 0$, there exists M such that for every $m \geq M$, we have $|a - a_m| < \epsilon$.

Exercise 2.5. Prove that this definition satisfies Fréchet's condition 2^o. What well-known theorem from analysis shows that the limit in $\mathbf{R}$ satisfies Fréchet's condition 1^o?

Next, Fréchet uses the idea of a limit of elements to define a limit for sets.

This being so, we call a *limit element* of the set **E** an element A which is the limit of some sequence of distinct elements taken in **E**. A set **E** is *closed* if it gives rise to no limit element or if it contains its limit elements.

Exercise 2.6. According to Fréchet's definition, are all finite sets closed? Prove or give a counter example.

Exercise 2.7. Use Fréchet's definition to show that for real numbers $a < b$, $[a, b]$ is closed in $\mathbf{R}$.

We can now say that a functional operator U on a closed set E is continuous in E if the numbers $U(A_n)$ always tend to $U(A)$ when each sequence of elements of **E** : $A_1, \ldots, A_n, \ldots$, has a limit A, regardless of the limit element A of **E**.

Is this really the right definition of continuity? To see if this definition makes sense, let's once again look at the special case of $\mathbf{R}$. Recall that in calculus, a function $f \colon \mathbf{R} \to \mathbf{R}$ is continuous at the point $a \in \mathbf{R}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|a - x| < \delta$, we have that $|f(a) - f(x)| < \epsilon$.

Exercise 2.8. Prove that in $\mathbf{R}$, this is equivalent to the standard ϵ-δ definition.

Finally, we will call a set E *[countably]*[1] *compact* if there always exists at least one common element in each infinite sequence of sets $E_1, E_2, \ldots, E_n, \ldots$ contained in E , when they (possessing at least one element each) are closed and each set is contained in the previous.[2]

Let's investigate what Fréchet might have in mind here.

Exercise 2.9.

- Rewrite Fréchet's definition of countably compact using modern notation.
- Which sets in $\mathbf{R}$ can you show to not be countably compact? Which sets seem to be countably compact? (To actually show any set is countably compact is difficult.)

Given the previous definitions, we immediately arrive at the generalization.

Theorem 2.1. Each functional operator U which is continuous on a countably compact and closed set E has at least one limit superior.

[1]While Fréchet used the term "compact" without the "countably" adjective, the modern use of the word compact is slightly different than Fréchet's use, which today we would call "countably compact." Hence in order to avoid confusion, we add the adjective "countably" in each instance of Fréchet's use of "compact."

[2]It is interesting to note that today, this property is known as the closed nested interval property, which is equivalent to the completeness of the real numbers.

We will prove Theorem 2.1 below. For now, some preliminaries.

Exercise 2.10. Explain how a functional operator having a limit superior is related to the functional operator attaining its maximum.

Exercise 2.11. Give an example of a set which is countably compact but not closed.

The preceding theorem plays an important role in the notion of a countably compact set. It is necessary to study the properties of such a set. This is achieved more easily through the following proposition: *A necessary and sufficient condition for a set E to be countably compact is for each set E_i formed from infinitely many elements contained in E to give rise to at least one limit element.*

Exercise 2.12. Prove Fréchet's necessary and sufficient condition for countable compactness.

Exercise 2.13. Now prove Theorem 2.1. You may find the above necessary and sufficient condition helpful.

Fréchet further explains the way countably compact sets behave.

The definition also shows that the countably compact sets enjoy similar properties to those of sets of limit points of a space. In particular, each set formed by a finite number of distinct elements is countably compact, each set formed by a finite number of countably compact sets is itself countably compact... This link is explained when we note that, taking as elements the points of a line for example, and adopting the ordinary definition of the limit of a sequence of points, we find that *each set of limit points of a line is a countably compact set.*

Exercise 2.14. Let us formalize the claims and proofs that Fréchet has just outlined.

1. Prove that a finite set is always countably compact.

2. Prove that the set of limit points of a set is countably compact.

We give Fréchet the last word, as he describes how the main result has the extreme value theorem as a special case.

An interval (where the endpoints are included) is a countably compact and closed set. Thus we discover the particular case of Weierstrass that we recalled.

3 Conclusions

In order to generalize the extreme value theorem when the domain is not just a set of points, Fréchet needed the set of points to also satisfy some properties involving limits. For Fréchet, this turned out to be closed and what he called compact. However, as mentioned above, his definition of compact is a bit different than the one we use today. Today, we use compact to mean a set which is both closed and countably compact in Fréchet's sense. While many students often struggle with the definition of compactness, we have hopefully seen that it has humble and even natural origins in the desire to generalize the important extreme value theorem.

References

[1] Fréchet, M. "Généralisation d'un théoréme de Weierstrass," *Comptes rendus de l'académie des sciences*, 139 (1904), 848- 850.

Notes to the Instructor

This project has two main goals. The first is to generalize the extreme value theorem by showing that it holds in structures more general than $\mathbf{R}$. This has the additional effect of exposing students to the fact that well-known results may be special cases of a more general phenomena. In that sense, it can get the student to begin to ask questions of a mathematical research nature, e.g., was working in $\mathbf{R}$ necessary? Is it necessary that the function be continuous? etc. One could even introduce this project by reminding the students of the extreme value theorem and asking them 1) why does this work 2) why are the given hypotheses necessary and 3) can the hypotheses be generalized? These questions are also teased out in some of the exercises (see below) which are fairly essential to a successful implementation of the project.

Second, and possibly more important, is the goal of introducing compactness in a more motivated or organic way. To tie this goal in with the previous, consider the following: This is a mini-project, meant to be completed in a day or two of class. However, there are several exercises and depending on the skills and abilities of the class, students could spend a long time on a single exercise before they "get it." Here I discuss the exercises that are most important for drawing out the main concepts in this project, and I also mention those which can improve the students' understanding of countable compactness, but which are not essential. Exercise 2.4 is key for students to be able to see why the hypotheses work in the statement of the EVT and why changing them will not necessarily yield the conclusion on the EVT. The instructor might find that the students benefit from working in small groups on this problem, followed by a class discussion where slowly, the importance of the hypotheses in the EVT are drawn out for the whole class to see. Again, understanding this is essential to appreciate why Fréchet is defining this "more general" set of hypotheses, which is precisely countable compactness. This connection can be further investigated by the student through Exercise 2.9. In fact, it would be recommended to have the students work on this problem in groups, share their answers, and have the instructor (if a student has not already done so) make the connection with Exercise 2.4. The hope is that the students will see how this definition is indeed generalizing the hypotheses for the EVT and hence, give the student a better understanding of why someone would write down the definition of compactness in the first place.

The Latex source file is available for modification from the author upon request (nscoville@ursinus.edu).

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