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A Compact Introduction to a Generalized Extreme Value Theorem

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1 Introduction

Compactness is a concept that is often introduced in a first course in analysis or topology, but one which students in those courses often find to be considerably difficult. Not only can it be challenging to figure out what the standard definition is saying, it can be even more of a mystery to figure out what possessed someone to write down such a definition in the first place. In this project, we look back to when compactness was first defined (albeit slightly differently than it is today) to see what use it had then and, more importantly, the role it continues to play in mathematics today. This first definition appeared in an extremely short paper entitled “Génèralisation d'un théorème de Weierstrass” (“Generalization of a theorem from Weierstrass”), [Fréchet, 1904], by French mathematician Maurice Fréchet (1878–1973). Fréchet was working at a time when topology was beginning to develop into its own branch of mathematics, due largely to his own research contributions. We will follow his paper carefully.

2 Fréchet

Fréchet began his paper as follows:[1]

> I. We know how important it will be, in a great number of problems, to know if a quantity U which depends on some elements (points, functions, etc.) actually attains a minimum on the domain under consideration. . . .
>
> This question is resolved in the particular case when U is a simple function of x (or [a function] of several independent variables). Weierstrass has indeed shown that any continuous function on a limited interval attains its maximum at least once. There will be great interest in extending this proposition so as to respond to the more general problem we have recalled [above]. It is this extension that is the subject of the present Note.

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[1]All translations of Fréchet excerpts in this project were prepared by the project author.

The goal of this project is the same as Fréchet's stated goal; that is, to extend the result of Weierstrass to accommodate things like functions, surfaces, etc. But first, let us recall what this result of Weierstrass is. In modern terms, the result that Weierstrass proved is the familiar "Extreme Value Theorem" that you learned in calculus. With that in mind, answer the following:

Task 1 What must Fréchet have meant by a "limited interval"? Use a calculus textbook to recall both the idea behind the Extreme Value Theorem and the details if you don't remember.

Task 2 For reference, state the Extreme Value Theorem using modern language and notation.

Task 3 When Fréchet talked about "extending" the Extreme Value Theorem, what quantities in the statement was he hoping to replace with what?

Here is the first definition in Fréchet's paper.

> II. We assume given some collection $\mathcal{C}$ of arbitrary elements (numbers, surfaces, etc.), in which we know how to distinguish distinct elements. We can say that U is a *uniform function (or functional operator) of a set* **E** of elements of $\mathcal{C}$, if to any element A of **E** corresponds a well-determined number $U(A)$.

Let us investigate Fréchet's definition, especially in light of his initial comment about Weierstrass' Theorem.

Task 4 (a) Let $\mathcal{C} = \mathbf{R}$ and $U(A) := A^2$. Show that this is a functional operator. Determine if U reaches a minimum for each of the following.

(i) $\mathbf{E} = \mathbf{R}$

(ii) $(0, 4)$

(iii) $[0, 4]$

Using (i)-(iii) as evidence, under what conditions does it seem that U reaches a minimum? Formulate a carefully worded conjecture.

(b) Let $a < b$ be real numbers and $\mathcal{C}$ be all continuous functions on the closed interval $[a, b]$. For each $f \in \mathcal{C}$, define $U(f) := \max_{x \in [a,b]}\{f(x)\}$. Give an example of a set $\mathbf{E} \subseteq \mathcal{C}$ on which U does attain its maximum and a set for which it does not attain its maximum. Can you do this with the further specification that $\mathbf{E}$ is infinite?

Notice the second example illustrates a functional operator that is unlike a typical function that we study in calculus. In this context, we may not know what it would mean for a functional operator to be continuous. Part of what Fréchet needed to do here was to define continuity in a more abstract setting. In order to do this, Fréchet needed the concept of limit, which he next discussed.

> To arrive at the concept of continuity of such a function, we will assume we have acquired a definition that gives a precise meaning to this phrase: *the infinite sequence* $A_1, A_2, \ldots, A_n, \ldots$ *of elements of* $\mathcal{C}$ *has a limit* B. It will be enough for us that this definition, whatever it may be, satisfies the following two conditions : 1^o If the sequence $A_1, A_2, \ldots, A_n, \ldots$ has a limit, each sequence $A_{p_1}, A_{p_2}, \ldots,$ formed by elements of increasing index from the first sequence also has a limit which is the same; 2^o If none of the elements $A_1, A_2, \ldots$ of the sequence are distinct from A, this sequence has a limit which is A.

Notice that Fréchet did not actually give a definition of limit, but rather required two properties that he thought any good definition of limit should have. In order to make this more concrete, we'll illustrate his conditions in $\mathbf{R}$. Recall that in $\mathbf{R}$, a sequence $a_1, a_2, \ldots, a_n, \ldots$ of real numbers has a limit a if for every $\epsilon > 0$, there exists M such that for every $m \geq M$, we have $|a - a_m| < \epsilon$.

Task 5 Prove that this definition satisfies Fréchet's condition 2^o. What well-known theorem from analysis shows that the limit in $\mathbf{R}$ satisfies Fréchet's condition 1^o?

Next, Fréchet used the idea of a limit of a sequence of elements to define a limit for sets.

> This being so, we will call a *limit element* of the set **E** an element A which is the limit of some sequence of distinct elements taken in **E**. A set **E** is *closed* if it gives rise to no limit element or if it contains its limit elements.

Task 6 According to Fréchet's definition of a closed set, are all finite sets closed? Prove or give a counterexample.

Task 7 Use Fréchet's definition of a closed set to show that for real numbers $a < b$, the interval $[a, b]$ is closed in $\mathbf{R}$.

Fréchet next defined continuity for functional operators.

> We can now say that a functional operator U on a closed set E is continuous on E if the numbers $U(A_n)$ always tend to $U(A)$ when any sequence of elements of **E** : $A_1, \ldots, A_n, \ldots,$ has a limit A, regardless of the limit element A of **E**.

Is this really a definition of continuity? To see if this definition makes sense, let's once again look at the special case of $\mathbf{R}$. Recall that in calculus, a function $f\colon \mathbf{R} \to \mathbf{R}$ is continuous at the point $a \in \mathbf{R}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|a - x| < \delta$, we have that $|f(a) - f(x)| < \epsilon$.

Task 8 Prove that in **R**, Fréchet's definition of continuity is equivalent to the standard ϵ-δ definition.

We now come to Fréchet's definition of compactness.

Finally, we will call *[countably]*[2] *compact* any set E for which there always exists at least one common element for each infinite sequence of sets E_1, E_2, ..., E_n ..., contained in E, when these (having at least one element each) are closed and each set is contained in the previous one.[3]

Let's investigate what Fréchet might have had in mind here.

Task 9
(a) Rewrite Fréchet's definition of countably compact using modern notation.
(b) Which sets in **R** can you show to not be countably compact? Which sets seem to be countably compact? (To actually show a set is countably compact from the definition is generally difficult.)

We now come to Fréchet's statement of his main theorem.

III. By means of the preceding definitions, we arrive immediately at the generalization :

THEOREM 1. *Each functional operator U which is continuous on a countably compact and closed set E ... has at least one limit superior.*

We will prove Theorem 1 below. For now, some preliminaries.

Task 10 Explain how a functional operator having a limit superior is related to the functional operator attaining its maximum.

Task 11 Give an example of a set which is countably compact but not closed.

[2]While Fréchet used the term "compact" without the "countably" adjective, the modern use of the word compact is slightly different than Fréchet's use, which today we would call "countably compact." Hence in order to avoid confusion, we add the adjective "countably" in each instance of Fréchet's use of "compact."

[3]It is interesting to note that a special case of this property is the closed nested interval property, which is equivalent to the completeness of the real numbers. Fréchet's result generalizes this.

Let's turn now to the proof of Theorem 1.

IV. Since the previous theorem plays an important role in the notion of a countably compact set, it is necessary to study the properties of such a set. This is achieved more easily through the following proposition:

> *The necessary and sufficient condition for a set E to be countably compact is that any set E_i formed from an infinity of distinct elements contained in E gives rise to at least one limit element.*

Task 12 Prove Fréchet's necessary and sufficient condition for countable compactness.

Task 13 Now prove Theorem 1. You may find the above necessary and sufficient condition helpful.

Fréchet continued by further explaining the way countably compact sets behave.

The definition also shows that the countably compact sets have properties similar to those of limited sets of points of a space. In particular, any set formed by a finite number of distinct elements is countably compact, any set formed by a finite number of countably compact sets is itself countably compact, ...

This approximation can be explained by noting that, taking as elements the points of a line for example, and adopting the ordinary definition of the limit of a sequence of points, we find that *any limited set of points of a straight line is a compact set.*

Recall that in Task 1, you considered what Fréchet meant by "limited interval." Here he used the phrase "limited set." Evidently, a limited interval should be a special case of a limited set.

Task 14 Let us formalize and prove the claims that Fréchet has just outlined.

(a) Prove that a finite set is always countably compact.

(b) Prove that a finite union of countably compact sets is countably compact.

(c) Prove that any closed, bounded interval of $\mathbf{R}$ is countably compact.

We give Fréchet the last word, with his description of how the Extreme Value Theorem is a special case of his main theorem.

An interval (where the endpoints are included) is a countably compact and closed set. Thus we discover the particular case of Weierstrass that we recalled.

3 Conclusion

In order to generalize the Extreme Value Theorem when the domain is other then a closed real interval, Fréchet needed the set of points to also satisfy some properties involving limits. For Fréchet, these properties turned out to be 'closed' and what he called 'compact.' However, as mentioned above, his definition of compact was a bit different than the one we use today. While understanding today's definition of compactness can be a challenge, we have hopefully seen that this concept had humble and even natural origins in the desire to generalize the important Extreme Value Theorem.

References

M. Fréchet. Génèralisation d'un théorème de Weierstrass (Generalization of a theorem from Weierstrass). *Comptes Rendus de l'Acadèmie des Sciences*, 139:848–850, 1904.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) has two main goals. The first is to generalize the Extreme Value Theorem (EVT) by showing that it holds in structures more general than $\mathbf{R}$. This has the additional effect of exposing students to the fact that well-known results may be special cases of more general phenomena. In that sense, it can get the student to begin to ask questions of a mathematical research nature, e.g., was working in $\mathbf{R}$ necessary? Is it necessary that the function be continuous? One could even introduce this project by reminding the students of the Extreme Value Theorem and asking them 1) why does this work; 2) why are the given hypotheses necessary; and 3) can the hypotheses be generalized? These questions are also teased out in some of the tasks (see below) which are fairly essential to a successful implementation of the project.

Second, and possibly more important, is the goal of introducing compactness in a more motivated or organic way. To tie this goal in with the previous, consider the following: This is a short project, meant to be completed in a day or two of class. However, there are several exercises, and depending on the skills and abilities of the class, students could spend a long time on a single exercise before they "get it." Here I discuss the exercises that are most important for drawing out the main concepts in this project. Task 4 is key for students to be able to see why the hypotheses work in the statement of the EVT and why changing them will not necessarily yield the conclusion of the EVT. The instructor might find that the students benefit from working in small groups on this problem, followed by a class discussion where slowly, the importance of the hypotheses in the EVT are drawn out for the whole class to see. Again, understanding this is essential to appreciate why Fréchet defined this "more general" set of hypotheses, which is precisely countable compactness. This connection can be further investigated by the student through Task 9. In fact, it is recommended to have the students work on this problem in groups, share their answers, and have the instructor (if a student has not already done so) make the connection with Task 4. The hope is that the students will see how this definition is indeed generalizing the hypotheses for the EVT and thereby gain a better understanding of why someone would write down the definition of compactness in the first place.

Student Prerequisites

This project is appropriate for a course in either topology or analysis, and as such, students should be familiar with the basics of proof as well as familiarity with results from calculus e.g. the Extreme Value Theorem for Task 4. In theory, these are the minimum prerequisites. However, most students who have had no exposure to topology or analysis might find this project challenging. For example, Tasks 6 and 7 have students work through the definition of a limit element, while Task 8 has the student show that a certain definition of continuity is equivalent to the standard ϵ-δ definition. Thus, while there are no other formal prerequisites besides mathematical proof, more exposure to concepts in topology or analysis beyond the basics of proof is recommended. The first three chapters in Rudin's *Principles of Mathematical Analysis* or Chapter 2 of Munkres's *Topology; A first course*, for example, would be good exposure for the students before attempting this project.

PSP Design and Task Commentary

See the PSP content section above for some comments on the PSP design and tasks, as well as the sample implementation schedule below for some further commentary on project tasks.

Suggestions for Classroom Implementation and Sample Schedule (based on 75-minute class periods)

As mentioned in the PSP content section above, there are at least a couple of options for implementing this project, depending on the background and ability of students, instructor interests, and time allotted. The following implementation schedule attempts to complete the entire project in two 75-minute class periods.

Day 1

- **Preassignment:** Assign students to read the Introduction and first excerpt from Fréchet and completion of Tasks 1–3 for day 1. They can possibly write up solutions for these tasks to hand in, or just prepare notes. In either case, students should come to class prepared to discuss answers to tasks 1-3 with the class.
- **In class discussion (15 minutes):** Lead a class discussion around the student answers to Tasks 1–3. It is crucial that students have a good grasp of the statement of the Extreme Value Theorem, as well as the ways in which Fréchet planned to extend it (Tasks 2 and 3). To that end, help the students see out how a modern statement of the Extreme Value Theorem can be teased out of Fréchet's quote. Also be sure that students come to the realization in Task 1 that Fréchet used the phrase "limited interval" to mean what we would call a "closed and bounded interval." This idea is important later in the project (e.g., for understanding the excerpt related to Task 14.
- **Working in groups (20 minutes):** Have students work in groups or individually on Task 4. This will help students see more clearly what the Extreme Value Theorem says ad furthermore, why the hypotheses are necessary.
- **Debrief (10 minutes):** After students have worked on Task 4, regroup as a whole class to discuss student answers. For example, three groups can be chosen to present their answers to parts (a-i), (a-ii), and (a-iii) on the board, while the class as a whole can discuss part (b).
- **Working in groups (20 minutes):** Next have students work on Tasks 5–7. Most likely, 15 minutes will not suffice to finish these three tasks, so as indicated below, these can be assigned for homework along with Task 8 as an optional homework problem.
- **Debrief (10 minutes):** The key points that the instructor wants to stress here are Fréchet's definitions of limit and closed. These can be illustrated by discussing as a class either Task 6 or Task 7, or some other example provided by the instructor. Also be sure to discuss the second part of Task 5 about the "well-known theorem" from analysis[4] if this is a theorem that the students have seen.
- **Homework:** Tasks 6, 7, 8. Have students read the Fréchet quote immediately after Task 8 and think about Task 9.

[4] If a sequence converges, then every subsequence converges to the same limit.

Day 2

- **In class discussion (25 minutes):** Discuss different ideas about Task 9, especially how one might show that certain sets are or are not countably compact. Then move to discuss Theorem 1 as a class. To help facilitate this and illustrate concepts, you can work through Tasks 10 and 11 as a class.
- **Working in groups (25 minutes):** Have students work in groups on Tasks 12–14. Students most likely will not be able to finish all three of these tasks. Have them write up the rest for homework.
- **Debrief (50 minutes):** In the remaining time, discuss ideas students had for Tasks 12–14 and where they are stuck. Try to guide students in the right direction, tying everything together.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Connections to other Primary Source Projects

The following additional primary source-based projects by the author are also freely available for use in teaching courses in point-set topology. The first three projects listed are full-length PSPs that require 10, 5, and 3 class periods respectively to complete. All others are designed for completion in 2 class periods.

- *Nearness without Distance*
- *Connectedness: Its Evolution and Applications*
- *From Sets to Metric Spaces to Topological Spaces*
- *Topology from Analysis* (Also suitable for use in Introductory Analysis courses.)
- *The Cantor set before Cantor* (Also suitable for use in Introductory Analysis courses.)
- *Connecting Connectedness*
- *The Closure Operation as the Foundation of Topology*

Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_topology`. They can also be obtained (along with their LaTeX code) from the author.

Recommendations for Further Reading

Readers who would like to know more about Fréchet's work in topology will find the following of interest.

- Taylor, A. A Study of Maurice Fréchet: I. His Early Work on Point Set Theory and the Theory of Functionals. *Archive for History of Exact Sciences*, 27(3):233–295, 1982.

- Taylor, A. A Study of Maurice Fréchet: II. Mainly about his Work on General Topology, 1909–1928. *Archive for History of Exact Sciences*, 34(4):279–380, 1985.

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