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The Trigonometric Functions Through Their Origins: Hipparchus' Table of Chords

Danny Otero

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Daniel E. Otero*

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Trigonometry is concerned with the measurements of angles about a central point (or of arcs of circles centered at that point) and quantities, geometrical and otherwise, which depend on the sizes of such angles (or the lengths of the corresponding arcs). It is one of those subjects that has become a standard part of the toolbox of every scientist and applied mathematician. Today an introduction to trigonometry is normally part of the mathematical preparation for the study of calculus and other forms of mathematical analysis, as the trigonometric functions make common appearances in applications of mathematics to the sciences, wherever the mathematical description of cyclical phenomena is needed. This project is one of a series of curricular units that tell some of the story of where and how the central ideas of this subject first emerged, in an attempt to provide context for the study of this important mathematical theory. Readers who work through the entire collection of units will encounter six milestones in the history of the development of trigonometry. In this unit, we examine the second of these six episodes by examining a modern reconstruction of a lost table of chords known to have been compiled by the Greek mathematician-astronomer Hipparchus of Rhodes.

1 Hipparchus, Ptolemy, and Astronomy in Ancient Greece

Hipparchus was an astronomer, geographer, and mathematician who lived during the second century BCE.[1] As is the case with most notable figures of the ancient world who were not military, political, or religious leaders, we know very little about his life, save that he was born in Nicaea, the major city of the kingdom of Bithynia (the region around present-day Istanbul, Turkey), that he died in Rhodes (the largest of the Aegean islands), and that he was an accomplished astronomer. He was among the first to set forth a *heliocentric* planetary theory, one that put the Sun at the center of a system of planets which orbit it.[2] Only one of his writings still survive, and this is a very minor treatise; our interest in him here is due to the fact that he was cited significantly by another much more influential astronomer, Claudius Ptolemy, who lived some 300 years later in the second century

*Department of Mathematics, Xavier University, Cincinnati, OH, 45207-4441; otero@xavier.edu.

[1]Hipparchus was likely a contemporary of one of the most famous athletes of the ancient world, Leonidas, who also hailed from Rhodes. Leonidas was a celebrated runner, who won Olympic crowns in four successive Olympic games during the middle of the second century BCE. In fact, his record of twelve Olympic crowns for victories in Olympic races throughout his athletic career stood unbeaten for over 2000 years until Michael Phelps, the American swimmer, won his thirteenth gold medal at the Rio de Janeiro Olympics in the summer of 2016.

[2]Based on mentions of him by other writers, it appears that Aristarchus of Samos was the first to posit that the Sun was the center of the universe; Aristarchus lived about a century before Hipparchus.

CE in Alexandria in Egypt. Ptolemy's reference to the work of Hipparchus helped to preserve for modern readers some of the earlier astronomer's work.

Ptolemy was the author of many scientific treatises that survive to the present, the most important of which was the *Almagest*,[3] a compendium of his own astronomical theories. In this work Ptolemy illustrated with geometric demonstrations his *geocentric* model for the movement of the Sun, Moon, and planets about the Earth.[4]

What characterized these Greek contributions to the development of astronomy was their incorporation of ideas from new advances in the study of geometry. These theories gave them the means to describe more clearly how it is that heavenly objects move the way they do, descriptions based on objectively deductive principles about the nature of geometrical objects. In this project, readers will be introduced to two important developments in the genesis of trigonometry: the decision to apply the new deductive science of geometry – in particular, the geometry of the circle and sphere – to the study of the dynamics of the heavens; and also, the creation of a two-column table to represent interrelated measurements between arcs in a circle and the chords that span those arcs, an early precursor to our modern trigonometric functions.

2 Circles and Spheres: Geometry in the Service of Astronomy

In the hands of Greek philosophers, mathematics leapt beyond its use as a collection of systems for counting and measuring the world to become a deductive science that collected abstract truths, as well as the means for certifying those truths. They erected systematic theories about points and lines, triangles and circles, cones and spheres,[5] In the following passage drawn from his *Almagest* (I.2[6]), Ptolemy summarized some basic astronomical principles using geometrical terms, making it clear that the most important geometry of interest to astronomers was that of the sphere and the circle.

> ...the heaven is spherical in shape, and moves as a sphere; the earth too is sensibly spherical in shape, when taken as a whole; in position it lies in the middle of the heavens very

[3] The name *Almagest* was not the title given to the work by its author. In Ptolemy's day, it had a much more prosaic title, *The Mathematical Collection*, but as it was much studied by astronomers in subsequent centuries, it became known more simply as *The Great Collection*. Centuries later, Arabic scholars translated it into their own language (with this same title) as *Kitāb al-majistī*, and even later, European scholars produced Latin translations, transliterating the title as *Almagestus*. Eventually, and finally, it was Anglicized as *Almagest*.

[4] It is instructive to consider how it is that Ptolemy's work survives today while Hipparchus' does not. This is in no small part due to the spectacular success of the reputation of the *Almagest* among astronomers who came after Ptolemy. In an age before the invention of printing, Ptolemy's work was copied by hand, again and again, while the works of earlier scholars were eclipsed or superseded by Ptolemy's and were set aside. Eventually, the older manuscripts were forgotten, then decayed and were lost, except for those mentions in other books by those who had read them in ages past. Furthermore, Ptolemy had the good fortune to have his works preserved at the famous Museum Library in Alexandria, the modern-day source of a vast amount of scientific literature of the ancient world.

[5] We must mention in this regard Euclid (ca. 300 BCE) and his *Elements*, the most well-known compendium of such theories in Greek geometry. It is arguably the most influential piece of mathematical writing in all of history. We find many citations to the *Elements* made by Ptolemy elsewhere in the *Almagest*. In fact, in the next unit of this series, we will focus on reading a section of the Almagest that appears after the one we will read below. There we will see how Ptolemy depends heavily on geometrical foundations drawn from the *Elements*. and before long, thinkers like Ptolemy employed these theories to describe – and explain – the movements of heavenly bodies.

[6] This notation indicates that the passage comes from Section 2 of Book I of the *Almagest*.

much like its centre; in size and distance it has the ratio of a point to the sphere of the fixed stars; and it has no motion from place to place.

. . . the ancients . . . saw that the sun, moon, and other stars were carried from east to west along circles which were always parallel to each other, that they began to rise up from below the earth itself, as it were, gradually got up high, then kept on going round in similar fashion and getting lower, until, falling to earth, so to speak, they vanished completely, then, after remaining invisible for some time, again rose afresh and set; and [they saw] that the periods of these [motions], and also the places of rising and setting, were, on the whole, fixed and the same. . .

No other hypothesis but [the sphericity of the heavens] can explain how sundial constructions produce correct results; furthermore, the motion of the heavenly bodies is the most unhampered and free of all motions, and freest motion belongs among plane figures to the circle and among solid shapes to the sphere. . .

Task 1

(a) What is Ptolemy generally trying to convince us of in the excerpt above?

(b) Cite as many pieces of evidence as you can from Ptolemy in support of his claim. Then add one or more of your own justifications, either for or against his claim. (Be sure to clearly distinguish your evidentiary points from those that Ptolemy made.) From what you know about the history of science, do you think it likely that Ptolemy would agree with each of your confirmations or rebuttals? Why, or why not?

(c) How convincing overall do you find the argument that he has made here? (It was wildly successful in convincing many in the ancient world, as the *Almagest* was treated as *the* authority in these matters for centuries following its writing.)

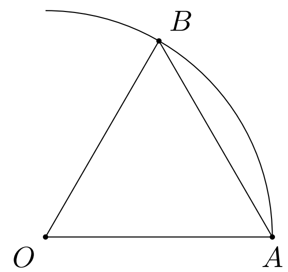
At the beginning of this excerpt, Ptolemy described the vastness of the heavens by stating that "in size and distance [the Earth] has the ratio of a point to the sphere of the fixed stars." That is, wherever one was located on the surface of the Earth, it made sense to consider yourself at the center of the immense celestial sphere. In this way, he also made clear his cosmological view: unlike Hipparchus, who thought that the Sun was at the center of the universe with the Earth and planets circling it, Ptolemy believed not only that the Earth was at the center, but that its location was fixed: "it has no motion from place to place."[7]

At the core of the Greek understanding of the motion of the heavens, then, is knowledge of the geometry of the sphere and of its two-dimensional counterpart, the circle, which describes the paths of the heavenly bodies around the celestial sphere. The application of this geometry to handling

[7]Indeed, except for the rare occurrence of an earthquake, humans perceive no movement of the Earth; the sky *actually does* appear to move around the observer! Modern science only succeeded in convincing the general public of the view that the Earth moves around the Sun after centuries of argument and rhetorical harangue, largely through the work of Copernicus and Galileo in the sixteenth and seventeenth centuries. Later, in the twentieth century, astronomers realized that even this view was incomplete: the galaxies that make up the universe are generally receding from each other over time. That is, no matter where you are in the universe, distances to the galaxies around you are expanding with time. So in some real sense, the universe has no center at all!

problems in astronomy is the real origin of what would eventually become trigonometry. We will explore a little of this geometry in the following two tasks, before taking a look at how Hipparchus made use of it in the next section of this project.

Task 2 On a circle of unit radius ($OA = 1$), mark off a point B so that $AB = 1$. If we join O to B, we will then have constructed triangle AOB.



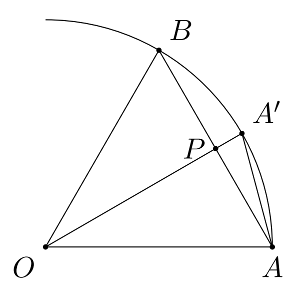
(a) Why is it that triangle AOB is equilateral?

(b) What is the measure of $\angle AOB$? Explain how you know.

(c) Mark off the point C on the circle so that $BC = 1$, with C *on the opposite side of B than A*. By the same reasoning as in parts (a) and (b), triangle BOC is equilateral. If we continue to mark points $D, E, \ldots$, moving counterclockwise around the circle, spaced one unit apart, how many such points will fit before the next point takes us beyond our original starting point at A? Explain how you know what this number is.

(d) What kind of regular polygon $ABCD \ldots A$ is thereby inscribed in the circle by connecting the consecutive points we have identified in the steps above? (A polygon is called *regular* if all its sides have equal measure.)

Task 3 We continue the exploration begun in the previous Task.



(a) Let the line segment OA' bisect $\angle AOB$ and cut the segment $\overline{AB}$ at the point P. Then triangle AOA' is isosceles. Use this to determine the measures of all the angles in triangles AOA', AOP, and APA'.

(b) Use your answers in part (a) and the Pythagorean Theorem to find the lengths of the sides of triangles AOP and APA', and in particular, the length of $\overline{AA'}$.

(c) The segment $\overline{AA'}$ is one side of a regular polygon which can be inscribed in the circle. How many sides does this polygon have?

3 The Table of Chords

Greek geometers had developed techniques to determine the exact lengths of the sides of certain regular polygons, such as the one you constructed in Task 3, but not just any such polygon: for instance, the length of the regular *heptagon* (having seven sides) could only be approximated. Given a circle with center point O and any two radii, $\overline{OA}$ and $\overline{OA'}$, these radii will bound the central angle $\angle AOA'$, corresponding to the arc from A to A'. The line segment $\overline{AA'}$ is called the *chord* corresponding to the arc $\overset{\frown}{AA'}$or the angle $\angle AOA'$.

It was recognized early on that knowledge of the lengths of these chords would be extremely useful for solving astronomical problems. Thus, it is not surprising to learn that Hipparchus was reported to have prepared a table of arcs and their corresponding chords for use by astronomers (and astrologers). Unfortunately, no copy of this work of Hipparchus survives today.[8] However, a number of scholars in recent years have proposed a likely reconstruction of his table, which we present below.[9]

Figure 1: **Hipparchus' Table of Chords (Reconstruction)**

Arcs	**Chords**
[7, 30]	[7, 30]
[15, 0]	[14, 57]
[22, 30]	[22, 21]
[30, 0]	[29, 40]
[37, 30]	[36, 50]
[45, 0]	[43, 51]
[52, 30]	[50, 41]
[60, 0]	[57, 18]
[67, 30]	[63, 40]
[75, 0]	[69, 46]
[82, 30]	[75, 33]
[90, 0]	[81, 2]

Arcs	**Chords**
[97, 30]	[86, 9]
[105, 0]	[90, 55]
[112, 30]	[95, 17]
[120, 0]	[99, 14]
[127, 30]	[102, 46]
[135, 0]	[105, 52]
[142, 30]	[108, 31]
[150, 0]	[110, 41]
[157, 30]	[112, 23]
[165, 0]	[113, 37]
[172, 30]	[114, 21]
[180, 0]	[114, 35]

Task 4 Write at least three observations and at least three questions you have about how to read this as a Table of Chords.

[8] We have independent reports that Hipparchus had indeed prepared such a table, one from Vettius Valens (120 - ca. 175 CE), a contemporary of Ptolemy and author of a respected astrological work, and another from Theon of Alexandria (ca. 335 - 405 CE), in his *Commentary on the Almagest.*

[9] For a deeper discussion of how this table was reconstructed, and a "convincing, but circumstantial" rationale for it, see (Van Brummelen, 2009, 41-46). Since this table is not taken from a historical documented source, we do not present it here in the sans-serif font we reserve for such texts.

The Table is the supposed production of a Greek astronomer working within the tradition of Babylonian science, so we should interpret the numbers in the table as being represented in *sexagesimal* (that is, base 60) numeration. In other words, the table entries are numerals which you may find to be of an unfamiliar type. If you have seen sexagesimal numerals before, subsection 3.1 below should remind you how they work. In subsection 3.2, we will then return to our exploration of the other features of Hipparchus' Table.

3.1 Sexagesimal Numeration

Our familiar decimal (or base 10) numeration makes use of the ten digits $0, 1, 2, \ldots, 9$, to represent any number, the positions of a particular digit within a specific numeral indicating what size of unit it is meant to represent: ones, tens, hundreds, and so on, with larger powers of 10 off to the left to count larger and larger groupings of units. Similarly, tenths, hundredths, thousandths, and even smaller fractions of units (the negative powers of 10) appear after a decimal point to the right of the units digit to measure finer and finer parts of units. For instance, the number we call $2\frac{3}{4}$ is written in decimal form as 2.75 because the 2 represents two units, the 7 seven tenths of these units, and the 5 five additional hundredths (that is, tenths of tenths): $2.75 = 2$ units + 7 tenths + 5 hundredths $= 2 \cdot 10^0 + 7 \cdot 10^{-1} + 5 \cdot 10^{-2} = 2 + \frac{7}{10} + \frac{5}{100} = 2\frac{75}{100} = 2\frac{3}{4}$. In different positions, even the same set of digits would represent a different number: $7.52 = 7 + \frac{5}{10} + \frac{2}{100}$, and $75.2 = 7 \cdot 10 + 5 + \frac{2}{10}$.

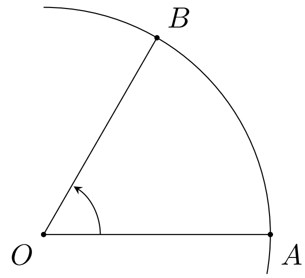
In the same way, sexagesimal numeration makes use of 60 distinct digits, representing the numbers $0, 1, 2, \ldots, 59$. Each digit's position within a numeral identifies the power of 60 associated to that digit. We will separate sexagesimal digits by commas, and, instead of a "sexagesimal point", we will signify the break between units and sixtieths with a semicolon (;). Thus, 100, which is 1 sixty + 40 units, is represented in sexagesimal as $[1, 40]$; likewise, 300 is represented as $[5, 0]$ and $2\frac{3}{4} = [2; 45]$. Note that this last numeral requires only two sexagesimal digits, since 45 is a *single* sexagesimal digit.

Task 5 Represent the three (decimal) numbers 78, 4.0075, and 4800 in sexagesimal numeration.

Now that we have the means to read the numbers in Hipparchus' Table, let's look more closely now at its structure. You may want to revisit your answers to Task 4 to see if you can explain in this new light some of the patterns in the Table that you identified there. Do these patterns appear to be intentional in the design of the Table? The rest of this section will be devoted to an investigation of what this Table means and why it was of use to Greek mathematicians and astronomers.

3.2 Measuring the Arcs

We begin by making a subtle point about the geometry of central angles in a circle and their corresponding arcs: given a circle with center O and two points A and B on the circumference, the two radii $\overline{OA}$ and $\overline{OB}$ determine an angle $\angle AOB$ and an arc $\overset{\frown}{AB}$, which by longstanding convention we read as opening *in the counterclockwise direction*, the angle opening from $\overline{OA}$ to $\overline{OB}$, and the arc from A to B along the circle. In modern mathematics, when we say that the measure of the angle $\angle AOB$ is $57°$, we are measuring the amount of *rotation* about the center O undergone by turning $\overline{OA}$ into $\overline{OB}$. But the ancients used this same system of units to measure instead the *length* of the curved arc $\overset{\frown}{AB}$ along the circle from A to B in the same direction.



With this in mind, the simple observation that the entries in Hipparchus' Table end at the telling value $[180, 0]$ appears to indicate that it is likely meant to represent an arc of 180° (half the full circle).[10] This suggests strongly that the values in the *Arcs* column of the Table of Chords are meant to measure arcs along the circle, or equivalently (as we have just observed), angles made at the center of the circle. We should also expect the column labeled *Chords* to carry the values of the lengths of the chords that span the corresponding arcs; these too are measurements in sexagesimal form. We will adopt a simple notation for this, referring to the Chord for a given Arc with the notation Crd: for instance, we see from the table that $\text{Crd}\,[37, 30] = [36, 50]$.

Task 6 Use the Table to identify...

(a) the value of $\text{Crd}\,[90, 0]$

(b) the value of $\text{Crd}\,[112, 30]$

(c) the Arc whose Chord is $[43, 51]$

(d) the Arc whose Chord is $[108, 31]$

For the remainder of this subsection, we will focus on the structure of the Arcs column of the table; in the next subsection, we'll investigate what's going on in the Chords column.

If the Arcs are being measured in units of degrees, then we should expect all the commas in these entries to properly be semicolons: [7, 30], the first Arc entry in the table, is a sexagesimal representation of the number $7 \cdot 60 + 30 = 450$, *not* $7\frac{1}{2}$; that number would be written as [7; 30]. But perhaps they were not meant to be interpreted in units of degrees, but in units of *arcminutes*, that is, in units of sixtieths of a degree.

Since the Babylonian astronomers who pioneered the measurement of circular arcs (or of central angles that determine those arcs) worked with a sexagesimal numeration system, their chief measurement unit for arcs, the degree, was naturally subdivided into sixtieths. Later European scholars would identify these as the first smaller parts of the degree, or, writing in Latin, *partes primae minutiae* – in English, *minute*. For the same reason, astronomers subdivided the temporal hour into sixtieths, and called these minutes as well. Then, to discriminate minutes of time from minutes of arc, we call the latter *arcminutes*. To further subdivide arcs (or time), one would have to distinguish sixtieths of sixtieths of a degree (or of an hour), requiring the second smaller parts of the degree (or hour), in Latin *partes primae secondae*, or *seconds* (or *arcseconds*) in English. Under this scheme, [7,

[10] Because there is no actual document authored by Hipparchus to consult here, there is no way of knowing if, much less how, the entries in the Arcs column ending in a 0 digit actually displayed this 0 as a zero symbol. For instance, the entry [180, 0] might have been entered as [180,]!

30] would be interpreted as $7 \cdot 60 + 30 = 450$ arcminutes, *which is the same as* $7\frac{1}{2}^{\circ}$. In the end, then, it doesn't much matter whether there was an unintended missing semicolon in the table entries: [7, 30] arcminutes measures *the very same arc* that also measures [7; 30] degrees.[11]

Task 7 Use the Table to express the Arcs in Task 6(c, d) in units of degrees.

You may also have noticed a simple pattern in the entries of the Arcs column: the first entry [7, 30] represents an arc of $7\frac{1}{2}^{\circ}$, the second represents 15°, the third $22\frac{1}{2}^{\circ}$, and so on. Thus, all entries are successive multiples of the first one. What's so special about $7\frac{1}{2}^{\circ}$? The next four Tasks are designed to help you answer this question.

Task 8

(a) The full 360° angle at the center O of a circle can be divided equally into six angles by radii drawn from O out to the circumference. Let A, B, C, D, E, F be the points on the circumference where these six radii strike the circle itself. Connecting the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EF}, \overline{FA}$ produces a regular hexagon inscribed in the circle. These congruent segments can each be understood as the chords of the corresponding congruent arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DE}$, $\overset{\frown}{EF}$, $\overset{\frown}{FA}$. What is the common measure of these arcs? And does Hipparchus' Table list this arc and its corresponding chord? (You might want to compare your work here with what you did for Task 2.)

(b) Now skip the points B, D and F and connect the other points of the regular hexagon to produce the equilateral triangle ACE, inscribed in the same circle. What is the common measure of the arcs $\overset{\frown}{AC}$, $\overset{\frown}{CE}$, $\overset{\frown}{EA}$? Does Hipparchus' Table list the value of this common arc and the corresponding common chords, $\overline{AC}, \overline{CE}, \overline{EA}$?

Task 9 Divide the full 360° angle at the center O of a circle into five equal parts by radii drawn from O out to the circumference. Let A, B, C, D, E be the points on the circumference where these five radii strike the circle. Connecting the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EA}$ produces a regular pentagon inscribed in the circle. These congruent segments are chords of the corresponding congruent arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DE}$, $\overset{\frown}{EA}$. What is the common measure of these arcs? Does it appear as an Arc in Hipparchus' Table?

Task 10

(a) Divide the full 360° angle at the center O of a circle into four equal parts by radii drawn from O out to the circumference. Let A, B, C, D be the points on the circumference where these four radii strike the circle. Connecting the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ produces an inscribed square. These congruent segments are chords of the corresponding congruent arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DA}$. What is the common measure of these arcs? Does it appear as an Arc in Hipparchus' Table?

[11] Remember, the entire table is a modern "educated guess" of an object whose existence we have only some reasonalbe evidence for!

(b) Now halve the arcs between the vertices of the square to get the points A', B', C', D', so that A' is the midpoint of $\widehat{AB}$, B' is the midpoint of $\widehat{BC}$, etc. If we now connect consecutive points around the circle, we produce an inscribed regular octagon. Does the measure of the arc $\widehat{AA'}$ appear in Hipparchus' table?

(c) If this halving process were repeated once more, how many sides would the resulting inscribed regular polygon have? And does the measure of the arc between consecutive vertices of this polygon appear in Hipparchus' Table?

Task 11 (a) Suppose that $ABC \ldots A$ is a regular polygon with n sides inscribed in a circle. For what values of n is the arc $\widehat{AB}$ an entry in Hipparchus' Table? Use your work on Tasks 8-10 to help answer this question.

(b) Of course, every value of n you found in part (a) must be greater than 2. But what is the largest value of n whose arc can be found in the Table? Let's call that value N. Identify a property shared by all the values of n you found in (a) that relates them to N (besides the obvious one that $n < N$)?

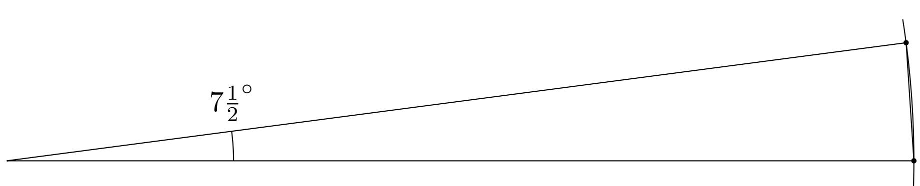
(c) What is the measure of the arc $\widehat{AB}$ in this regular N-gon, as it appears in Hipparchus' Table? What is its measure in degrees? Where $A, B, C, D, \ldots$ are consecutive vertices of the N-gon, what are the measure of the arcs $\widehat{AC}$, $\widehat{AD}$, etc., in degrees?

(d) Based on your answers above, how is the column of Arcs in Hipparchus' Table organized?

3.3 Measuring the Chords

Examining the Chords column for similar patterns, you may also have discovered that these numbers are roughly equal to their Arcs near the beginning of the table but appear to grow less quickly as the size of the Arcs increases. How can we explain this phenomenon?

Our first clue is to consider the first table entry: it reports that the Chord corresponding to an Arc of $7\frac{1}{2}$ degrees is also $[7; 30] = 7\frac{1}{2}$, i.e., $\text{Crd}[7; 30] = [7; 30]$. But if the length of the arc of a circle is a distance, we could imagine straightening this arc's length to obtain a line segment of nearly the same measure. That is, we could measure the chord *in the same units* as the arc. The diagram below shows that the arc and the chord are indeed roughly equal, as they are hard to distinguish when the angle is small, just as Hipparchus' Table confirms.



This gives evidence that the Chord entries were interpreted as lengths in the same units of arcminutes as were used to measure the lengths of the Arcs along the circle.

Finally, you may have observed that the later entries in both columns of Hipparchus' table are not properly rendered in true sexagesimal form. For instance, the final entry of [180, 0] arcminutes in the Arcs column ought to be written in sexagesimal as [3, 0, 0]: the number 180 is not a proper sexagesimal digit, as it is greater than or equal to 60. Ptolemy also did not use a strictly sexagesimal numeration in the table of chords he published in the *Almagest* I.11 (Toomer, 1998).[12] For both Hipparchus and Ptolemy, as for most other Greek astronomers, Arcs were consistently measured in units of degrees, but sexagesimal notation was used *only to reference fractions of a degree.* Quantities that were one degree or larger were recorded in a base-10 system (Pedersen, 2011, pp. 49-52). This is what we are seeing in this version of Hipparchus' Table. From the time of the Greeks until today, astronomers in the Western world continued to make use of this "hybrid" sexagesimal number system of degrees/minutes/seconds whenever they measured arcs or angles.

Task 12 Let's see if we can translate entries of Hipparchus' Table into a more modern form, with angles measured in degrees and chord lengths in units of arcminutes.

(a) Hipparchus' table shows that $\text{Crd}\,[15,0] = [14,57]$ arcminutes. Convert the Arc measure to degrees, and the measure of its Chord into a decimal number of arcminutes (rounding to four places).

(b) The table also shows that $\text{Crd}\,[37,30] = [36,50]$ arcminutes. Convert this Arc measure to degrees, and the measure of its Chord into a decimal number of arcminutes (rounding to four places).

Task 13 (a) Recall the formula you learned in school that relates the circumference C of a circle and its radius r. (If you've forgotten it, look it up.) Given that the circle contains 360°, how many minutes of arc measure the entire circumference of the circle? According to the formula, how long is the radius of the circle in arcminutes?

(b) From Task 2(a), we conclude that $\text{Crd}\,[60,0]$ should be equal to the radius of the underlying circle. Show how Hipparchus' table verifies this.

(c) What then is the length of the diameter of the circle? How does this relate to the tabulated Chord length for an Arc of measure 180°?

Task 14 In Hipparchus' table, Arcs step up in units of $7\frac{1}{2}^\circ$. Arcs greater than 180° would not need to be considered, since their chord values would simply repeat ones already found in the table. Verify this by drawing a picture displaying a circle with central angle whose corresponding arc on the circle has measure $187\frac{1}{2}^\circ$, the next entry that would have appeared in the table if it were to continue past 180°. What entry in the table already gives the corresponding chord value? According to the table, what is the measure of the Chord for Arc $337\frac{1}{2}^\circ$?

[12] As Hipparchus did, Ptolemy's table of chords listed arcs up to 180°, but it was far more extensive: in fact, it listed 360 entries, finding the chord for every arc that was a multiple of $\frac{1}{2}^\circ$, together wiht an aid for getting very close approximations for the chords of any arc measure in between!

Task 15 Translate at least the first eight entries of Hipparchus' Table of Chords from sexagesimal into decimal numbers, using degrees and minutes to measure the Arcs but only arcminutes for the Chords. For instance, the first pair of entries would give an Arc measure of $7°30'$ and a Chord measure in arcminutes of $7 \cdot 60 + 30 = 450'$. Similarly, the second pair of table entries illustrate that Crd $15°0' = 14 \cdot 60 + 57 = 897'$.

4 Conclusion

We have learned that, because the Greeks saw the circle and sphere as appropriate geometric models for the paths of motion of the heavenly bodies – and for the heavens themselves, the geometry of the circle was central[13] to their astronomical theories. The creation of tables of chords, like the reconstructed one of Hipparchus which was explored in this project, allowed them to relate the linear distances between points on a circle (or sphere) to the lengths of their arcs, or the measures of the angles they spanned at the center. In the next unit of this series of projects, we will examine in detail how a table of chords was used to solve a practical astronomical problem, the telling of time by the measure of the sun's shadows.

[13] Pun intended!

References

J. L. Berggren. *Episodes in the Mathematics of Medieval Islam.* Springer, New York, NY, 2003.

Barnabas Hughes, O.F.M. *Regiomontanus: On Triangles.* University of Wisconsin Press, Madison, WI, 1967.

Victor Katz. *A History of Mathematics: an introduction.* Addison Wesley, Reading, MA, second edition, 1998.

Victor Katz, Annette Imhausen, Eleanor Robson, Joseph W. Dauben, Kim Plofker, and J. Lennart Berggren. *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: a Sourcebook.* Princeton University Press, Princeton, NJ, 2007.

E. S. Kennedy. *The Exhaustive Treatise on Shadows, by Abū Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī.* Institute for the History of Arabic Science, Aleppo, Syria, 1976.

Otto Neugebauer and David Pingree. *The Pañcasiddhāntikā of Varāhamihira.* Munksgaard, Copenhagen, Denmark, 1970/1972.

John North. *The Fontana History of Astronomy and Cosmology.* Fontana Press, London, 1994.

Olaf Pedersen. *A Survey of the Almagest, with annotation and new commentary by Alexander Jones.* Springer, New York, NY, 2011.

Kim Plofker. *Mathematics in India.* Princeton University Press, Princeton, NJ, and Oxford, 2008.

E. G. Richards. *Mapping Time: the calendar and its history.* Oxford University Press, Oxford, 1998.

Jeff Suzuki. *Mathematics in Historical Context.* Mathematical Association of America, Washington, DC, 2009.

G. J. Toomer. *Ptolemy's Almagest.* Princeton University Press, Princeton, NJ, 1998. Translated and annotated by G. J. Toomer; with a foreword by Owen Gingerich.

Glen Van Brummelen. *The Mathematics of the Heavens and the Earth: the early history of trigonometry.* Princeton University Press, Princeton, NJ, and Oxford, 2009.

Notes to Instructors

This project is the second in a collection of six curricular units drawn from a Primary Source Project (PSP) titled *A Genetic Context for Understanding the Trigonometric Functions*. The project is designed to serve students as an introduction to the study of trigonometry by providing a context for the basic ideas contained in the subject and hinting at its long history and ancient pedigree among the mathematical sciences. Each of the individual units in that PSP looks at one of the following specific aspects of the development of the mathematical science of trigonometry:

- the emergence of sexagesimal numeration in ancient Babylonian culture, developed in the service of a nascent science of astronomy;
- a modern reconstruction (as laid out in (Van Brummelen, 2009)) of a lost table of chords known to have been compiled by the Greek mathematician-astronomer Hipparchus of Rhodes (second century, BCE);
- a brief selection from Claudius Ptolemy's *Almagest* (second century, CE) (Toomer, 1998), in which the author (Ptolemy) shows how a table of chords can be used to monitor the motion of the Sun in the daytime sky for the purpose of telling the time of day;
- a few lines of Vedic verse by the Hindu scholar Varāhamihira (sixth century, CE) (Neugebauer and Pingree, 1970/1972), containing the "recipe" for a table of sines as well as some of the methods used for its construction;
- passages from *The Exhaustive Treatise on Shadows* (Kennedy, 1976), written in Arabic in the year 1021 by Abū Rayḥ̄ān Muḥammad ibn Aḥmad al-Bīrūnī, which include precursors to the modern trigonometric tangent, cotangent, secant and cosecant;
- excerpts from Regiomontanus' *On Triangles* (1464) (Hughes, 1967), the first systematic work on trigonometry published in the West.

This collection of units is not meant to substitute for a full course in trigonometry, as many standard topics are not treated here. Rather, it is the author's intent to show students that trigonometry is a subject worthy of study by virtue of the compelling importance of the problems it was invented to address in basic astronomy in the ancient world. Each unit may be incorporated, either individually or in various combinations, into a standard course in College Algebra with Trigonometry, a stand-alone Trigonometry course, or a Precalculus course. These lessons have also been used in courses on the history of mathematics and as part of a capstone experience for pre-service secondary mathematics teachers.

In this unit, students are introduced to the basic elements of the geometry of the circle and the measure of its arcs, central angles and chords, whose interrelationships formed the foundation for the early development of trigonometry as a mathematical tool for Greek astronomy. The project opens with a reading of a brief excerpt from Claudius Ptolemy's *Almagest* (second century CE) that provides scientific and philosophical context for why the geometry of the circle was so important to astronomers. This is followed by an investigation of a (modern reconstruction of a) table of chords attributed to Hipparchus of Rhodes (second century, BCE). This work is intended to give students reasons for why and how degree measure works, as well as gently introducing them, through an examination of a table of chords, to the study of trigonometrical functions.

Student Prerequisites

Nearly nothing in the way of special prerequisites is required of students for this particular unit beyond what a typical high school student knows about basic plane geometry. But instructors interested in implementing the current project in their classroom may want to consider also using the first unit of the collection, "Babylonian Astronomy and Sexagesimal Numeration." This can be most effective if the instructor wishes students to work through these materials at times separated by more than just a few days, since the units have been pieced apart here to maximize their independent utility; otherwise, one could also consider making use of the full PSP described above.

The central features of most of the units in the full PSP are the primary source texts and the sequence of tasks that accompany them. In this unit, however, the single primary source text offers mainly context for the mathematical work at hand, which centers on a modern reconstruction of a lost work, Hipparchus' Table of Chords. The goal of the project is to connect the geometry of the circle with the measurement of its chords as propounded by Greeks like Hipparchus and Claudius Ptolemy. This work continued the advancement of mathematical astronomy from the Old Babylonian era, whose legacy is still felt today in the way we measure angles, arcs and time.

Instructors who wish to expose students to more experience with sexagesimal numeration are also encouraged to consider the PSP "Babylonian Numeration" by Dominic Klyve, which is available for free download at `https://blogs.ursinus.edu/triumphs/`.

Suggestions for Classroom Implementation

This particular unit is meant to be completed in two to three 50-minute (or two 75-minute) classroom periods, plus time in advance for students to do some initial reading and time afterwards for them to write up their solutions to the tasks. It should be emphasized that student written work should be far more explicit and detailed in its production than the oral communication in which they engage in the classroom, communication that is often accompanied by the recording of rather telegraphic notes.

Sample Implementation Schedule (based on two 50-minute class periods)

Students should read through the end of section 3.1 before the first class begins. This will expose them to the single included source text and the associated commentary, the Table of Chords, and the first five Tasks. Assign students to also work Tasks 1, 4 and 5 beforehand.

The first 5 minutes of the first period can be devoted to comparing students' responses to Task 1(a) and (c). Now, students are likely to be mystified by Tasks 2 and 3, so the next 20 minutes or so should be set aside to assist them, preferably in groups of 4-6, as they work through the geometric details of the procedures laid out there. They will need to have on hand blank paper, pencils for drawing, straightedges (rulers) and compasses. Compasses can be challenging to use effectively, so it may be best to have at least two students per group working to construct the diagrams required for the Task to allow their drawings to be compared and improved upon. Still, artistic perfection is not the ultimate goal here, so you might ask them to be somewhat forgiving of their drawings.

Next, spend about 10 minutes publicly identifying students' responses to Task 4. There are likely to be common responses. As a focused investigation of the structure of the Table of Chords is the aim of the project, these responses should help direct the students' attention for the remainder of their work on it. Give another 10 minutes to checking that the students get the mechanics of converting decimal numbers into sexagesimal, and the reverse (Task 5). Instructors may find that many handheld calculators have a tool that converts radian measures to degrees/minutes/seconds, a useful tool for conversion of decimal numbers into sexagesimal form. But one should also be aware that it may be too tempting for students to employ them as a means of avoiding the conceptual work of understanding the conversion procedure.

During the rest of the first period, I recommend that the class read (either aloud in common, or alone silently) the section 3.2 commentary. They can then work on parts of the straightforward Tasks 6 and 7. What isn't completed in class can be left as homework, if necessary. Indeed, the work of all seven of these Tasks should be assigned as homework for students to formally write up and submit later. In addition, have students prepare solutions to Tasks 8-10 for discussion in the next period. (Note that Task 9 may be omitted, if needed.)

The second class period should open with students' questions, especially regarding their work on Tasks 8 and 10. It may be useful to summarize this discussion by asking students this question: "which inscribed regular n-gons have associated arcs that appear in Hipparchus' table?" The goal is to lead students to speculate to the table's organization. A copy of Hipparchus' Table of Chords has been included as the last page of these Notes to Instructors should you wish to distribute copies for students to use during this period as they work to figure out how to read it and what it means.

Next, students can work in groups on Task 11. I suggest planning 30 minutes for all this. The rest of the period is for students to work on at least one part of Task 12 and all of Task 13. While Task 14 may be considered optional, Task 15 should certainly be assigned for homework; it is a good exercise to bring the project to a close, as it asks the student to translate the Table of Chords into a more modern form. (You may want to devote a few minutes at the start of the next meeting to tackle questions from them about this.) It is recommended that students write up formal solutions to whichever of Tasks 8-15 is assigned as a final homework assignment.

LaTeXcode of this unit is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The project itself can also be modified by instructors as desired to better suit their goals for the course.

Recommendations for Further Reading

Instructors who want to learn more about the history of trigonometry are recommended to consult Glen van Brummelen's masterful *The Mathematics of the Heavens and the Earth: The early history of trigonometry* (Van Brummelen, 2009), from which much of this work took inspiration.

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Hipparchus' Table of Chords (Reconstruction)

Arcs	**Chords**
[7, 30]	[7, 30]
[15, 0]	[14, 57]
[22, 30]	[22, 21]
[30, 0]	[29, 40]
[37, 30]	[36, 50]
[45, 0]	[43, 51]
[52, 30]	[50, 41]
[60, 0]	[57, 18]
[67, 30]	[63, 40]
[75, 0]	[69, 46]
[82, 30]	[75, 33]
[90, 0]	[81, 2]

Arcs	**Chords**
[97, 30]	[86, 9]
[105, 0]	[90, 55]
[112, 30]	[95, 17]
[120, 0]	[99, 14]
[127, 30]	[102, 46]
[135, 0]	[105, 52]
[142, 30]	[108, 31]
[150, 0]	[110, 41]
[157, 30]	[112, 23]
[165, 0]	[113, 37]
[172, 30]	[114, 21]
[180, 0]	[114, 35]