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# Fourier's Heat Equation

Kenneth M. Monks

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It is often said that Joseph Fourier gave birth to modern climate science. His 1827 paper *Mémoire sur les Températures du Globe Terrestre et des Espaces Planétaires*, (*On the Temperatures of the Terrestrial Sphere and Interplanetary Space*, translated in [Pierrehumbert, 2004]) contained the following very influential passage:

> The Earth is heated by solar radiation... Our solar system is located in a region of the universe of which all points have a common and constant temperature, determined by the light rays and the heat sent by all the surrounding stars. This cold temperature of the interplanetary sky is slightly below that of the Earth's polar regions. The Earth would have none other than this same temperature of the Sky, were it not for ... causes which act ... to further heat it.

The above passage is widely considered to be the first demonstration of the existence of the greenhouse effect. His claim above was far from speculative; rather, it was based on his groundbreaking study of heat published five years earlier, *Théorie analytique de la chaleur (The Analytical Theory of Heat)* [Fourier, 1822b]. The goal of this project is to give the reader an insight into the techniques Fourier employed therein, as they have become the basis of modern thermodynamics, as well as enormously consequential in mathematics itself. The sections tell the following story:

- **Section 1.** We see what Fourier's starting assumptions were for his heat investigation.
- **Section 2.** We retrace one of Fourier's primary examples: determining the temperature of a square prism of infinite length. Part of the way through, we find Fourier snapping his fingers and solving a differential equation in just one step.
- **Section 3.** The magical incantation he used was some old magic due to Leonhard Euler. In this section, we read this technique, in Euler's own words.
- **Section 4.** We return to the infinite square prism problem and apply Euler's work to solve it.
- **Section 5.** We present Fourier's more general heat equation. Note we do not present the full derivation of this equation (which is in the original, in Chapter II, Section V, for interested readers) but just an intuitive explanation.

*Department of Mathematics, Front Range Community College – Boulder County Campus, Longmont, CO 80537; `kenneth.monks@frontrange.edu`.

- **Section 6.** In this section, Fourier's work took a very surprising turn; in studying the heat dispersion in a particular infinite solid, he invented a whole new theory of infinite series, now called *Fourier series*. In a way, it is similar to power series, in which one writes a function as a sum of powers of $x$; what is different is that he instead used sums of cosines of varying frequency (and in the modern theory typically a mix of sines and cosines are used).

- **Section 7.** In the last section, we talk through some of the ripple effects of Fourier's work in *The Analytical Theory of Heat*, leading into foundational questions of rigor in analysis.

All primary source excerpts that follow are from the 1878 translation of *The Analytical Theory of Heat* from Alexander Freeman[1], with the exception of the Euler excerpts in Section 3.

## 1 Introducing Fourier and Fourier's Introduction

Jean-Baptiste Joseph Fourier (1768–1830) was born into a working-class family in Auxerre, France. He soon found himself in unfortunate circumstances: at the age of eight he became an orphan. Luckily, he obtained admission to a local military school, where he received an education from the Benedictine monks of Saint-Maur. In 1790, they gave him a mathematics teaching appointment at their school in Auxerre, where he also taught rhetoric, history, and philosophy. He later became a founding faculty member at the Ècole Polytechnique[2] in Paris, where Napoleon sometimes attended lectures. This led to Napoleon's request for Fourier's help in the administration of Egypt after its occupation by France in 1798. Upon his return to France, Fourier served as the prefect of the Department of Isère, where he led extensive infrastructure projects to quell chronic infections that were emanating from marshes in the area. In 1817, he was elected to the Académie des Sciences, and five years later he became their perpetual secretary. (For more on Fourier's life, see [Hutchins, 1952].)

Thus, one can hardly imagine someone with a broader background than Fourier, more uniquely situated to simultaneously tackle problems of pure thought as well as in the physical world around him, perhaps in the same stroke of the pen. In the introduction of *The Analytical Theory of Heat*, he made no secret about the fact that he intended to do just that, with mathematics as his language and tool. Of mathematics, he said the following: [Fourier, 1822a, 7-8]

> Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena most diverse, and discovers the hidden analogies which unite them.

[1] Alexander Freeman (1838–1897) was born in Blackheath, Surrey, England. He was an astronomer and mathematics teacher. In addition to providing the English translations of Fourier's work we use here, he was a frequent correspondent of James Clark Maxwell (1831–1879), the creator of the modern theories of electricity and magnetism [Hutchins, 2004].

[2] Founded during the French Revolution in 1794 (the same year as Fourier's arrest for having defended a member of a particular faction) in part by mathematician Gaspard Monge (1746 – 1818), Ècole Polytechnique remains one of the most well-respected institutions of mathematics in the world today.

He also made clear the necessity of looking at heat through a mathematical lens [Fourier, 1822a, 1].

> Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys. The theory of heat will hereafter form one of the most important branches of general physics.

He then made it clear what problem he was actually trying to solve [Fourier, 1822a, 14]!

> The problem of the propagation of heat consists in determining what is the temperature at each point of a body at a given instant, supposing that the initial temperatures are known.

Lastly, he laid out some starting assumptions regarding how heat transfer can be modeled (and commented that his assumptions were verified by experiment and observation). His primary assumption is often referred to today as *Newton's Law of Cooling* or the first law of thermodynamics, which loosely says that the rate of transfer of heat between two objects will be proportional to the difference in temperature between the two objects. Fourier stated this in Section III of his first chapter, calling it the "Principle of the communication of heat". In his own words, he said the following:

> ... the action of two molecules, or the quantity of heat which the hottest communicates to the other, is the difference of the two quantities which they give up to each other.

## 2 Fourier's Section V, a Square Prism of Infinite Length

Fourier began by studying heat transfer in a figure which has a square base (which he called $A$) that then extends from that base in **just one direction** off to infinity, creating a prism of infinite length. Although the shape itself is three-dimensional, he throws away all unknowns here except one, looking for a temperature function $v(x)$ where $x$ represents the distance from the square base.

§73 ... A metal bar, whose form is that of a rectangular parallelepiped infinite in length, is exposed to the action of a source of heat which produces a constant temperature at all points of its extremity $A$. It is required to determine the fixed temperatures at the different sections of the bar.

The section perpendicular to the axis is supposed to be a square whose side $2l$ is so small that we may without sensible error consider the temperatures to be equal at different points of the same section. The air in which the bar is placed is maintained at a constant termperature $0$, and carried away by a current with uniform velocity.

Within the interior of the solid, heat will pass successively all the parts situate to the right of the source, and not exposed directly to its action; they will be heated more and more, but the temperature of each point will not increase beyond a certain limit. This maximum temperature is not the same for every section; it in general decreases as the distance of the section from the origin increases: we shall denote by $v$ the fixed temperature of a section perpendicular to the axis, and situate at a distance $x$ from the origin $A$.

**Task 1** There are really four unknowns here: time $t$ and three spatial dimensions $x$,$y$, and $z$. How did Fourier manage to simplify the situation, resulting in representing the temperature function $v$ we seek to being a function only of $x$?

Before every point of the solid has attained its highest degree of heat, the system of temperatures varies continually, and approaches more and more to a fixed state, which is that which we consider. This final state is kept up of itself when it has once been formed. In order that the system of temperatures may be permanent, it is necessary that the quantity of heat which, during unit of time, crosses a section made at a distance $x$ from the origin, should balance exactly all the heat which, during the same time, escapes through that part of the external surface of the prism which is situate to the right of the same section. The lamina whose thickness is $dx$, and whose external surface is $8ldx$, allows the escape into the air, during unit of time, of a quantity of heat expressed by $8hlv.dx$, $h$ being the measure of the external conducibility of the prism[3].

[3] Note that in this time and place, it was fairly common to use a lower dot (period) as a multiplication symbol in mathematics writing instead of a centered dot as we more often do today.

**Task 2** Draw a diagram that illustrates the geometric setup Fourier presented above, and verify his claim that the "external surface is $8ldx$".

Hence taking the integral $\int 8hlv.dx$ from $x = 0$ to $x = \infty$, we shall find the quantity of heat which escapes from the whole surface of the bar during unit of time; and if we take the same integral[4] from $x = 0$ to $x = x$, we shall have the quantity of heat lost through the part of the surface included between the source of heat and the section made at the distance $x$. Denoting the first integral by $C$, whose value is constant, and the variable value of the second by $\int 8hlv.dx$; the difference $C - \int 8hlv.dx$ will express the whole quantity of heat which escapes into the air across the part of the surface situate to the right of the section.

**Task 3** Why could Fourier consider the first integral to be a constant value $C$?

On the other hand, the lamina of the solid, enclosed between two sections infinitely near at distances $x$ and $x + dx$, must resemble an infinite solid, bounded by two parallel planes, subject to fixed temperatures $v$ and $v + dv$, since, by hypothesis, the temperature does not vary throughout the whole extent of the name section. The thickness of the solid is $dx$, and the area of the section is $4l^2$: hence the quantity of the heat which flows uniformly, during one unit of time, across a section of this solid, is, according to the preceding principles, $-4l^2K\frac{dv}{dx}$, $K$ being the specific internal conducibility:

**Task 4** Draw a diagram of the prism and the section of the solid with boundaries at $x$ and $x + dx$. Label the side lengths. Verify Fourier's claim that "the area of the section is $4l^2$", as well as the claim that "the quantity of the heat which flows uniformly, during one unit of time, across a section of this solid, is $\ldots -4l^2K\frac{dv}{dx}$". (Note the original translation is inconsistent with regards to use of capital versus lowercase $K$; here we have changed them to all be capital for readability.)

[4] What Fourier did here is considered invalid (or at least a very poor choice of notation) today, using the same letter $x$ both as the independent variable of the function being integrated as well as in the bounds. Today one would replace the independent variable of the function with some other letter, like $\tau$, and then take the integral from $\tau = 0$ to $\tau = x$. It may be helpful to think of it this way in the calculations that follow.

. . . we must therefore have the equation

$$-4l^2K\frac{dv}{dx} = C - \int 8hlv.dx,$$

whence

$$Kl\frac{d^2v}{dx^2} = 2hv.$$

**Task 5** (a) Where did Fourier's first equation above come from? He came up with two formulas that both represent what quantity?

(b) What did Fourier do to get the second equation from the first? (Hint: You will need to apply a very famous theorem from your first-semester Calculus course in order to justify that little "whence"!)

§76. The integral of the preceding equation is

$$v = Ae^{-x\sqrt{\frac{2h}{lK}}} + Be^{+x\sqrt{\frac{2h}{lK}}},$$

$A$ and $B$ being two arbitrary constants[5]; . . .

After he obtained the equation

$$kl\frac{d^2v}{dx^2} = 2hv, \tag{1}$$

Fourier solved it without even pausing to take a breath. He showed no work as to how he obtained the general solution. This was certainly not due to incomplete exposition, however. Rather, it was because the technique for solving such a differential equation was already quite well-known in Fourier's time, having been worked out almost eighty years earlier by none other than the great Leonhard Euler[6] (1707–1783).

[5]Be aware that this $A$ is just a generic real number and has nothing to do with the $A$ that represents the plate serving as origin of the heat. Sometimes, $A \neq A$.

[6]Leonhard Euler was born in Basel, Switzerland to Marguerite (née Brucker) and Paul Euler, a Protestant minister who had attended Johann Bernoulli's lectures at University of Basel. Paul wished for his son to follow him into the ministry, but Johann persuaded Paul to allow Leonhard to study mathematics instead after witnessing his incredible potential for mathematics.

## 3 Euler's Solution to Linear Homogeneous Constant-Coefficient Differential Equations

Though much of the groundwork was done in his correspondence with Johann Bernoulli (1667–1748), Euler's method for solving such differential equations was fully written up in the paper *De integratione aequationum differentialium altiorum graduum (On the integration of differential equations of higher orders)* [Euler, 1743]. Here we present very (very!) selected excerpts which provide just the tiny slice of his method that Fourier actually used above.[7] The first excerpt we show is Euler's statement of the problem.

§28

If a differential equation of order $n$ of this kind was propounded

$$0 = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \cdots + N\frac{d^ny}{dx^n}$$

in which the element $dx$ is put constant, and the letters $A, B, C, D, \ldots, N$ denote arbitrary constant coefficients, to find the integral of this equation in finite real terms.

**Task 6** Take Fourier's Equation (1) and write it in the form Euler gives above. What is $n$? What are "the letters $A, B, C, D, \ldots, N$"?

§12 ...the following algebraic equation will result

$$0 = A + Bp + Cp^2 + Dp^3 + Ep^4 + \cdots + Np^n$$

if from which any value of $p$ is found, one will at the same time have a particular integral $y = e^{px}$ satisfying the propounded differential equation.

In modern differential equations courses, the equation that Euler showed above is typically called either the *characteristic equation* or the *auxilary equation*. Because of connections to linear algebra and eigenvalues, typically the symbol $\lambda$ is used in place of $p$.

[7] For a more comprehensive treatment, see Adam Parker's *Leonhard Euler and Johann Bernoulli Solving Homogenous Higher Order Linear Differential Equations With Constant Coefficients* [Parker, 2020].

**Task 7** (a) Find the characteristic equation corresponding to Fourier's Equation (1).

(b) What are the values of $p$ that solve this characteristic equation?

(c) Euler claimed that "one will at the same time have a particular integral[8] $y = e^{px}$ satisfying the propounded differential equation." For each of your values of $p$, substitute it into the formula $y = e^{px}$, and check that it is a valid solution to Fourier's Equation (1) as claimed.

Euler then described how to stitch together separate solutions to build a general solution, which he called "the complete value for $y$".

§15 Therefore, if all roots of this algebraic equation of order $n$ were real, then the complete value for $y$ will ... be the aggregate of $n$ exponential formulas of this kind $\alpha e^{qx:p}$, ...

Be aware that the passage lies in a slightly earlier section of the same paper, and Euler shifted notation a bit throughout. To make it consistent with what we read in Euler's §28, you may replace $e^{qx:p}$ with just $e^{px}$.

**Task 8** Notice that Euler above allowed us to attach a constant $\alpha$ to the front of our exponential function (and for exponential functions corresponding to different roots, you could have a different constant in front). He also says we should take the "aggregate of $n$ exponential formulas" to obtain "the complete value for $y$". Verify that performing these steps results in Fourier's general solution to Equation (1).

## 4 Back to the Future[9]

Having seen where that solution came from, we return again to Fourier's §76.

§76. The integral of the preceding equation is

$$v = Ae^{-x\sqrt{\frac{2h}{lK}}} + Be^{+x\sqrt{\frac{2h}{lK}}},$$

$A$ and $B$ being two arbitrary constants; now, if we suppose the distance $x$ infinite, the value of the temperature $v$ must be infinitely small; hence the term $Be^{+x\sqrt{\frac{2h}{lK}}}$ does not exist in the integral: thus the equation $v = Ae^{-x\sqrt{\frac{2h}{lK}}}$ represents the permanent state of the solid; the

[8]Note that Euler uses the word "integral" in this context as today we would use the word "solution". Note that if a differential equation is of the form $y' = f(x)$, then the solution of the differential equation and the indefinite integral of $f(x)$ are one and the same, hence the interchangeable words.

[9]Though this future is 200 years in the past.

temperature at the origin is denoted by the constant $A$, since that is the value of $v$ when $x$ is zero.[10]

**Task 9** Let us analyze the above passage in modern terminology and notation. Thinking of $v$ as a function of $x$, what did Fourier claim about the value of $\lim_{x\to\infty} v(x)$? What then does that claim imply about the value of $B$, and why?

§80. It is easy to ascertain how much heat flows during unit of time through a section of the bar arrived at its fixed state: this quantity is expressed by $-4Kl^2\frac{dv}{dx}$, or $4A\sqrt{2Khl^3}.e^{-x\sqrt{\frac{2h}{KL}}}$, and if we take its value at the origin, we shall have $4A\sqrt{2Khl^3}$ as the measure of the quantity of heat which passes from the source into the solid during unit of time; thus the expenditure of the source of heat is, all other things being equal, proportional to the square root of the cube of the thickness.

We should obtain the same result on taking the integral $\int 8hlv.dx$ from $x$ nothing to $x$ infinite.

**Task 10** We proceed to verify Fourier's claims from the section above.

(a) Take the formula for $v$ established in §76 and substitute it into $-4Kl^2\frac{dv}{dx}$ in order to verify Fourier's claim about "how much heat flows during unit of time through a section of the bar" as well as "the measure of the quantity of heat which passes from the source into the solid during unit of time".

(b) Take that same formula for $v$ established in §76, and substitute it into the integral

$$\int_{x=0}^{x=\infty} 8hlv.dx$$

and verify that we "obtain the same result" as Fourier claimed.

[10] In the same section, Fourier mentioned that this solution was in fact verified empirically! He wrote the following. "This law according to which the temperatures decrease is the same as that given by experiment; several physicists have observed the fixed temperatures at different points of a metal bar exposed at its extremity to the constant action of a source of heat, and they have ascertained that the distances from the origin represent logarithms, and the temperatures the corresponding numbers."

## 5 Fourier's Heat Equation

In Section 2, the differential equation that modeled the given heat problem ended up being an *ordinary differential equation* (ODE). It is called "ordinary" because the derivatives are "ordinary". More specifically, it is a differential equation in which the solution is a function of just one independent variable, and thus all derivatives (be it a first derivative or a higher-order derivative) are calculated with respect to that one variable. This was possible because the temperature of our object was really only dependent on how far you were from the heat source, represented by $x$. Fourier realized, however that this would be insufficient for more complicated scenarios, and proceeded to introduce *partial differential equations*, equations in which the solution will be a function of several independent variables, and the derivatives may be taken with respect to any number of those independent variables.

In his Chapter II, "Equations of the Movement of Heat", he states his much-celebrated heat equation:

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right).$$

The derivation is based on a simple idea: any change in the temperature of a location with respect to time (the left-hand side of his equation) must somehow correspond to heat moving in or out of that location through any one of three directions, $x, y$, or $z$ (the right-hand side of his equation). The quantities $K, C$, and $D$ represent properties of the material being studied.

## 6 Fourier Series

In Chapter III, "Propogation of Heat in an Infinite Rectangular Solid", Fourier's work took a surprising turn. Solving the heat problem in this "infinite rectangular solid", which in terms of a geometric object seems maybe only slightly more complicated, did not amount to simply applying a century-old algorithm handed to him by Euler. Rather, he invented a whole new theory, Fourier series, in order to study this shape. Loosely speaking, it is the idea of constructing infinite series built out of trigonometric functions [11].

We shall see in the following passage how and why Fourier constructed these!

Note that in the passage below, one may imagine an $x, y$, and $z$ axis in three-dimensional space. The line of intersection of planes $A$ and $B$ is parallel to the $z$-axis (and similarly the line of intersection of planes $A$ and $C$). Thus, the figure is completely uniform in the $z$ direction, hence Fourier's remark

[11] It should be noted that Fourier was not the first to construct an infinite series out of trigonometric functions. The first such example actually appeared in 1744 in a letter from Euler to the German mathematician and lawyer Christian Goldbach (1690–1764). In [Euler and Goldbach, 2015, 834], he says "§I. Given an arbitrary arc $a$ of a circle, let its sine be $= \alpha$, the sine of the double arc be $= \beta$, the sine of the triple arc $= \gamma$, the sine of the quadruple $= \delta$, of the quintuple $= \epsilon$, and so on: I say that the sum of the infinite series

$$\frac{1}{2}a + \alpha + \frac{1}{2}\beta + \frac{1}{3}\gamma + \frac{1}{4}\delta + \frac{1}{5}\epsilon + \cdots$$

always expresses the length of an arc of $90°$ in the same circle." Why then, one may ask, are these called Fourier series and not Euler series? Well, on one hand Euler did an unbelievable amount of work with infinite series, so it would be hard to name any one particular type after him. But perhaps the better reason is that Euler's perspective was quite different: he presented this trigonometric series as subdivisions of an arc of a circle into chords. Fourier on the other hand, was thinking of it as it is more often thought of today: one has a function in mind (in Fourier's case, the solution to a partial differential equation) which is then expressed as a summation of trigonometric functions, much as one finds a power series for a function by expressing it as a summation of powers of the independent variable.

that "abstraction is made of the co-ordinate $z$", essentially meaning that you can draw just one two-dimensional slice of the region (that slice being perpendicular to the $z$-axis) and have all the information you need.

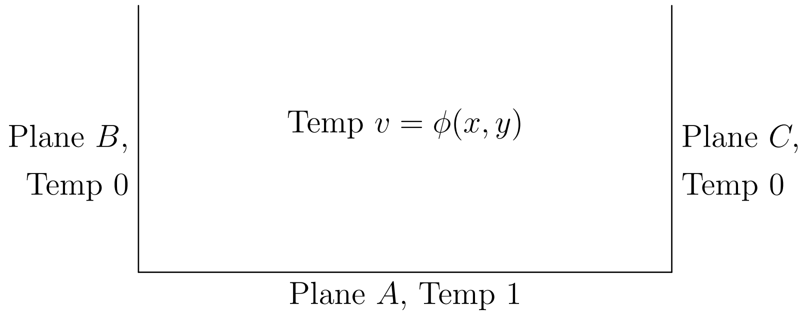
§164 Suppose a homogeneous solid mass to be contained between two planes $B$ and $C$ vertical, parallel, and infinite, and to be divided into two parts by a plane $A$ perpendicular to the other two . . . ; we proceed to consider the temperatures of the mass $BAC$ bounded by the three infinite planes $A$,$B$, $C$. The other part $B'AC'$ of the infinite solid is supposed to be a constant source of heat, that is to say, all its points are maintained at the temperature $1$, which cannot alter. The two lateral solids bounded, one by the plane $C$ and the plane $A$ produced, the other by the plane $B$ and the plane $A$ produced, have at all points constant temperature $0$, some external cause maintaining them always at that temperature; lastly, the molecules of the solid bounded by $A, B$ and $C$ have the initial temperature $0$. Heat will pass continually from the source $A$ into the solid $BAC$, and will be propagated there in the longitudinal direction, which is infinite, and at the same time will turn towards the cool masses $B$ and $C$, which will absorb great part of it. The temperatures of the solid $BAC$ will be raised gradually: but will not be able to surpass nor even to attain a maximum of temperature, which is different for different points of the mass. It is required to determine the final and constant state to which the variable state continually approaches.

If this final state were known, and were then formed, it would subsist of itself, and this is the property which distinguishes it from all other states. Thus the actual problem consists in determining the permanent temperatures of an infinite rectangular solid, bounded by two masses of ice $B$ and $C$, and a mass of boiling water $A$; the consideration of such simple and primary problems is one of the surest modes of discovering the laws of natural phenomena, and we see, by the history of the sciences, that every theory has been formed in this manner.

. . .

It is supposed that there is no loss of heat at the surface of the plate, or, which is the same thing, we consider a solid formed by superposing an infinite number of plates similar to the preceding: the straight line $Ax$ which divides the plate into two equal parts is taken as the axis of $x$, and the co-ordinates of any point $m$ are $x$ and $y$; lastly, the width $A$ of the plate is represented by . . . $\pi$,. . .

We now pause the reading of the primary source to draw and label a diagram to illustrate the region that Fourier described above. Recall that Fourier's region extends uniformly along the $z$-axis, so what we draw below is a representation of a cross section perpendicular to the $z$-axis. (One may think of this as the $xy$-plane, for example.)




Plane $B$ can be represented by the equation $y = -\frac{\pi}{2}$ and plane $C$ by $y = \frac{\pi}{2}$. Plane $A$ is given by $x = 0$. All three extend indefinitely in the $z$ direction (which is perpendicular to the page).

Imagine a point $m$ of the solid plate $BAC$, whose co-ordinates are $x$ and $y$, to have the actual temperature $v$, at that the quantities $v$, which correspond to different points, are such that no change can happen in the temperatures, provided that the temperature of every point of the base $A$ is always 1, and that the sides $B$ and $C$ retain at all their points the temperature 0.

§166 To apply the general equation

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right),$$

we must consider that, in the case in question, abstraction is made of the co-ordinate $z$, so that the term $\frac{d^2v}{dz^2}$ must be omitted; with respect to the first member $\frac{dv}{dt}$, it vanishes, since we wish to determine the stationary temperatures; thus the equation... is the following[12]:

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0 \ldots\ldots\ldots\ldots (a).$$

The function[13] of $x$ and $y$, $\phi(x, y)$, which represents the permanent state of the solid $BAC$, must, 1st, satisfy the equation $(a)$; 2nd become nothing when we substitute $-\frac{1}{2}\pi$ or $+\frac{1}{2}\pi$ for $y$, whatever the value of $x$ may be; 3rd, must be equal to unity when we suppose $x = 0$ and $y$ to have any value included between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Further, this function $\phi(x, y)$ ought to become extremely small when we give to $x$ a very large value, since all the heat proceeds from the source $A$.

For ease of reference, we now list out the four conditions that Fourier has specified above on a solution $v = \phi(x, y)$ to this heat problem. Specifically, $v = \phi(x, y)$ must satisfy the following:

[12] This special case of Fourier's heat equation is sometimes called the two-dimensional Laplace equation, named after the great French mathematician and scientist Pierre-Simon, marquis de Laplace (1749–1827).

[13] Note that Fourier used $v$ and $\phi(x, y)$ somewhat interchangeably, with the subtle difference being that $v$ represented the unknown function and $\phi(x, y)$ represented the solution for $v$.

1. Satisfy the partial differential equation $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$

2. "become nothing when we substitute $-\frac{1}{2}\pi$ or $+\frac{1}{2}\pi$ for $y$"

3. "must be equal to unity when we suppose $x = 0$ and $y$ to have any value included between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$"

4. "ought to become extremely small when we give to $x$ a very large value"

In the modern theory of partial differential equations (PDEs), the first of the above conditions is called a *partial differential equation*, while conditions 2 through 4 are called *boundary conditions*. This generalizes the theory of ODEs, in which one is often given an Initial Value Problem consisting of an ordinary differential equation and an initial condition. Note that today we often use a slightly different symbol for those derivatives, replacing the $d$ with $\partial$, to indicate a *partial derivative*. So in today's notation, Fourier's equation above would be written

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

where one interprets $\frac{\partial^2 v}{\partial x^2}$ as the second derivative of $v$ with respect to $x$, treating $y$ as a constant, and similarly for $\frac{\partial^2 v}{\partial y^2}$, mutatis mutandis.

§167. . . Functions of two variables often reduce to less complex expressions, . . . which in this particular case, take the form of the product of a function of $x$ by a function of $y$. . . . We shall then write $v = F(x)f(y)$; substituting in equation $(a)$ and denoting $\frac{d^2F(x)}{dx^2}$ by $F''(x)$ and $\frac{d^2f(y)}{dy^2}$ by $f''(y)$, we shall have

$$\frac{F''(x)}{F(x)} + \frac{f''(y)}{f(y)} = 0;$$

we then suppose[14] $\frac{F''(x)}{F(x)} = m^2$ and $\frac{f''(y)}{f(y)} = -m^2$, $m$ being any constant quantity, and as it is proposed only to find a particular value of $v$, we deduce from the preceding equations $F(x) = e^{-mx}$, $f(y) = \cos my$.

**Task 11** Notice that above, Fourier was not claiming to have found general solutions to the equations[15]

$$\frac{F''(x)}{F(x)} = m^2$$

[14] Note that in the original, Fourier simply wrote $m$ and not $m^2$ in the equations that follow. We made this minor change to the translation to improve readability for the student; since otherwise it requires following Fourier's slightly awkward step of essentially redefining $m$ as the its own square root in his particular solutions.

[15] To do so would require Euler's technique from Section 3, as well as more techniques from that same paper that we did not present here.

and

$$\frac{f''(y)}{f(y)} = -m^2,$$

as he wanted "only to find a particular value of $v$". Verify that each of the solutions he showed for $F(x)$ and $f(y)$ satisfies the corresponding equation.

At this point, Fourier built the form of a particular solution to his heat equation:

$$v = e^{-mx} \cos my.$$

Next, he wished to understand more about what $m$ could be. To accomplish this, he used his four conditions!

We could not suppose $m$ to be a negative number, ...

**Task 12** Which of Fourier's four conditions justified the above claim, and why?

The exponent $m$ which enters into the function $e^{-mx} \cos my$ is unknown, and we may choose for this exponent any positive number: but, ..., $m$ must be taken to be one of the terms of the series, $1, 3, 5, 7, \&c.$;

**Task 13** Which of Fourier's four conditions justified the above claim, and why?

A more general value of $v$ is easily formed by adding together several terms similar to the preceding, and we have

$$v = ae^{-x}\cos(y) + be^{-3x}\cos(3y) + ce^{-5x}\cos(5y) + de^{-7x}\cos(7y) + \&c. \ldots\ldots\ldots (b).$$

It is evident that the function $v$ denoted by $\phi(x, y)$ solves the equation $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$, and the condition $\phi(x, \pm\pi) = 0$.

**Task 14** Verify the two claims Fourier made about the function $v$. On our list of four conditions, which three are now satisfied? Which one remains to be used?

A third condition remains to be fulfilled, which is expressed thus, $\phi(0, y) = 1$, and it is essential to remark that this result must exist when we give to $y$ any value whatever included between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Amazingly, just the one condition stated above ended up being enough information to solve for the infinitely many unknowns $a, b, c, d, \ldots$!

Equation $(b)$ must therefore be subject to the following condition:

$$1 = a\cos y + b\cos 3y + c\cos 5y + d\cos 7y + \&c.$$

The coefficients $a, b, c, d, \&c$, whose number is infinite, are determined by means of this equation.

Fourier proceeded to solve for all infinitely many unknowns via some very clever, and very messy, infinite series manipulations. The end results are actually surprisingly clean, producing

$$a = \frac{4}{\pi}, b = -\frac{4}{3\pi}, c = \frac{4}{5\pi}, d = -\frac{4}{7\pi}, e = \frac{4}{9\pi}, \ldots$$

and so on. While this work was very impressive, (and is quite worth reading sometime!) we instead follow the suggestion of our translator, Alexander Freeman, who suggested a different approach in a footnote to his translation!

**Task 15** Freeman says the coefficients can instead be determined "by multiplying both sides of the first equation by $\cos y$, $\cos 3y$, $\cos 5y$, &c., respectively, and integrating from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$." Apply this method[16] and verify that the same coefficients are produced. You don't need a rigorous argument that the pattern continues forever; just verify that the first few match. **Hint!** The product-to-sum identity for cosine,

$$\cos(A)\cos(B) = \frac{\cos(A+B) + \cos(A-B)}{2},$$

[16]Freeman attributes this method to the Scottish mathematician Duncan Gregory (1813–1844). As it is more algorithmic and requires far less cleverness than what Fourier did, this is the standard method for finding coefficients in a Fourier series taught in mathematics courses today. Here you may assume that an integral of an infinite series can be calculated by integrating each term separately.

will be very helpful! Be warned that for each coefficient, you will need to integrate this series term-by-term, producing infinitely many integrals. However, do not fret; all but one will be zero!

**Task 16** Substitute these values into Fourier's "following condition" written above, and deduce from it his celebrated identity

$$\frac{\pi}{4} = \cos y - \frac{1}{3}\cos 3y + \frac{1}{5}\cos 5y - \frac{1}{7}\cos 7y + \frac{1}{9}\cos 9y - \&c.$$

Note at this point, Fourier had essentially solved his heat problem, having found the last remaining unknowns in the solution $v$. However, he did not even stop to substitute them into his form for the solution $v$. Rather, in his Section III, "Remarks on these series." Fourier paused to admire what just happened, completely ignoring the mission of solving heat problems for a little while[17]. Let us bask in this warm glow with him a bit.

Note that in this section he traded out $y$ for $x$ as his variable, and worked instead with the equation

$$\frac{\pi}{4} = \cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \frac{1}{9}\cos 9x - \&c.$$

**Task 17** Use a graphing utility to graph the following functions:

(a) $\cos x$

(b) $\cos x - \frac{1}{3}\cos 3x$

(c) $\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x$

(d) $\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x$

(e) $\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \frac{1}{9}\cos 9x$

On what interval (containing $x = 0$) does this sequence of functions appear to be converging to the constant function $\pi/4$? How does this correspond to how Fourier built the heat model that this function came from?

Fourier then had plenty of fun (which we now get to have with him) substituting different values of $x$ and seeing what popped out!

The case where $x$ is nothing is verified by Leibnitz' series,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$$

[17] Though he did then return to study heat further in Chapter IV.

**Task 18** Set $x = 0$ and verify Fourier's remark above. Also, verify that the same series can be produced by choosing $x = \pi/4$ into the power series for arctangent[18].

§181. If in this equation we assume $x = \frac{1}{2}\frac{\pi}{2}, \ldots$

**Task 19** How dare we spoil the surprise for you! Go ahead and take Fourier's suggestion and use that particular value for $x$. From the resulting equation, construct an infinite series expansion for the quantity

$$\frac{\pi}{2\sqrt{2}}.$$

The resulting series and that total were in fact discussed by none other than Isaac Newton (1642–1726) and Gottfried Leibniz in their correspondence.[19]

...by giving to the arc $x$ other particular values, we should find other series, ..., several of which have been already published in the works of Euler. If we multiply [the] equation ...by $dx$, and integrate it, we have

$$\frac{\pi x}{4} = \sin(x) - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \&c.$$

Making in the last equation $x = \frac{1}{2}\pi$, we find

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c,$$

a series already known.

[18] This series was actually first discovered by fourteenth-century Kerala school mathematician Mādhava (1350–1425). However, there is no clear evidence that such works were transmitted to Europe by Fourier's time, hence his referring to it as "Leibnitz' series", as Gottfried Leibniz (1646–1716) had independently come up with the same result, albeit roughly two centuries later. For a student project on the infinite series of Mādhava, see [Monks, 2020], or for much more detail, see [Plofker, 2009].

[19] Read more about this in the student project [Klyve, 2018].

Though Fourier did not explicitly say it, the series

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots \tag{2}$$

is in fact equivalent to one of the most famous infinite series problems of all time, the sum of the reciprocals of the squares of all positive naturals [20]. Specifically, the goal was to find an exact value for

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Jacob and Johann Bernoulli studied this series, but were only able to prove that it converged to some number less than 2, proven by Jacob Bernoulli in [Bernoulli, 1713]. Euler's 1735 solution to that problem which had stumped the esteemed Bernoulli's thus was very famous. So, Fourier's reference to series "which have been already published in the works of Euler" almost surely includes the one shown above.

**Task 20** Let us examine how Fourier's less famous series for $\frac{\pi^2}{8}$ is in fact equivalent to Euler's solution to the Basel Problem[21].

(a) Take Fourier's Equation (2) and create infinitely many new equations from it: the first obtained by multiplying both sides by 1/4, the second obtained by multiplying both sides by 1/16, the third obtained by multiplying both sides by 1/64, and so on for all powers of one-fourth.

(b) Add up all the equations! On the left-hand side, you will get a sum of multiples of $\frac{\pi^2}{8}$. On the right-hand side, you will get an infinite sum of infinite sums (yikes!).

(c) Evaluate the left-hand side using the infinite geometric series formula.

(d) Explain why the right-hand side is simply equal to $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (Hint: Think about which denominators are multiples of 4, which are multiples of 16, which are multiples of 64, and so on.)

(e) Thus, what is the value for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

Well, that was wild! Wait, what did this all have to do with heat again?

**Task 21** Remember the original problem Fourier was trying to solve: finding the temperature function for a region bounded on the sides by masses of ice (planes $B$ and $C$) with a boiling plate beneath (plane $A$). Recall that he found that the temperature was given by a function of the form given in Equation ($b$), but then we diverged from his methods for a bit to find the values of the coefficients $a, b, c, d, \ldots$. Now that we have those values, substitute those into Equation ($b$) to find the actual formula for $v$ (as an infinite series). Plot a few partial sums for $v$ in a

[20] The fact that the mathematicians (Jacob Bernoulli, Johann Bernoulli, and Euler) who studied the evaluation of the sum of the reciprocals of the squares came from Basel has led to the problem being referred to as *The Basel Problem.* [O'Connor and Robertson, 1998]

[21] Note that Fourier's method for obtaining a solution to the Basel Problem was not even slightly similar to what Euler did. To see Euler's argument, see [Monks].

3D graphing utility (for example Geogebra 3D or similar). What do these graphs represent in terms of the original temperature problem? On what domain do the graphs make sense?

## 7 Epilogue

Though it has become the basis of modern thermodynamics, it should be noted that Fourier's *The Analytical Theory of Heat* was not immediately celebrated by everyone. Some in the mathematical community questioned its claims. For example, consider the following passage from the brilliant Norwegian mathematician Niels Abel (1802–1829). This is taken from a letter he wrote back to his teacher in Norway during his time in Paris, as quoted in [Bottazzini, 1986, 87-89].

Once can [22] rigorously demonstrate that

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \text{etc.}$$

for all values of $x$ smaller than $\pi$.

...

By taking derivatives, one has

$$\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \text{etc.}$$

A completely false result, because this series is divergent.

**Task 22** Take the derivative of both sides on the first equation Abel states above and verify that it produces the second as he claims. Then, investigate Abel's frustration a bit more by using $x = 0$ in the second series. Are things any better if you set $x = \pi/4$? How about $x = \pi/2$?

Bothered by examples like the one above, Abel ranted about a lack of rigour in the field of analysis in the same passage.

[22] ...and in fact Fourier did, in §182 of *The Analytical Theory of Heat*...

In general the theory of infinite series, up to the present, is very poorly established. One performs every kind of operation on infinite series, as if they were finite, but is it permissible? . . . Where has it been demonstrated that one can obtain the derivative of an infinite series by taking the derivative of each term?

**Task 23** In light of Abel's example and corresponding rant, what step of Fourier's work in establishing Equation (2) do you think would have made him very uncomfortable?

Abel, along with a few other contemporaries including Augustin-Louis Cauchy (1789–1857), Karl Weierstrass (1815–1897), Bernard Bolzano (1781–1848), and Richard Dedekind (1831–1916), proceeded to lay out some of the more rigorous foundations of mathematics that have become standard today[23]. Note that their concerns did not invalidate any of Fourier's work; rather, Fourier's work served as fertile ground for the very fruitful thought exercises that followed.

In conclusion, it is hard to overstate the influence of Fourier's work *The Analytical Theory of Heat*. On one hand, it laid the groundwork for one of today's standard branches of physics and engineering, thermodynamics, along with climate science and the study of global warming. On the other, it led to fabulous advances in pure mathematics, both through the work contained therein and future mathematicians responses to it!

**Task 24** To see the massive flow of ideas that wove together into and then tendriled out of Fourier's work here, draw a timeline, and place each mathematician mentioned in this project on the timeline, represented as an interval marked by their birth and death years. In the cases where the date of a specific work is mentioned, plot that date as well (and if you are feeling ambitious and curious, see if you can find dates for some of the other relevant works not mentioned in this project too). Where does the actual publication of *The Analytical Theory of Heat* lie?

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## References


Janet Heine Barnett. Why be so Critical? Nineteenth Century Mathematics and the Origins of Analysis. 2016. URL `https://digitalcommons.ursinus.edu/triumphs_analysis/1`.

Jakob Bernoulli. *Ars conjectandi, opus posthumum: accedit Tractatus de seriebus infinitis, et epistola Gallice scripta de ludo pil reticularis.* Basile: Impensis Thurnisiorum, 1713. Available in English, translated by Edith Dudley Sylla, from Baltimore: Johns Hopkins University Press, 2006.

Umberto Bottazzini. *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass.* Springer-Verlag, New York, 1986. ISBN 978-0387963020.


[23] For more on their work, see the Primary Source projects [Barnett, 2016] and [Ruch, 2016].

Leonard Euler and Christian Goldbach. *Correspondence of Leonhard Euler with Christian Goldbach, Part I*. Springer Basel, 2015. ISBN 978-3-0348-0892-7. Edited by Franz Lemmermeyer and Martin Mattmüller.

Leonhard Euler. De integratione aequationum differentialium altiorum graduum (On the integration of differential equations of higher orders). 1743. Translated by Alexander Aycock as part of the Euler Circle - Mainz project. Eneström number E62.

Joseph Fourier. *Théorie Analytique de la Chaleur (The Analytical Theory of Heat)*. Cambridge University Press, Cambridge UK, 1822a. Translated by Alexander Freeman in 1878.

Joseph Fourier. *Théorie analytique de la chaleur*. F. Didot, 1822b. URL `https://books.google.com/books?id=1TUVAAAAQAAJ`.

Robert Maynard Hutchins, editor. *Britannica Great Books of the Western World*. William Benton, 1952.

Roger Maynard Hutchins. *Oxford Dictionary of National Biography*. Oxford University Press, 2004. URL `https://www.oxforddnb.com/`.

Dominic Klyve. How to Calculate pi: Machin's Inverse Tangents. 2018. URL `https://digitalcommons.ursinus.edu/triumphs_calculus/6`.

Kenneth M Monks. Euler's calculation of the sum of the reciprocals of the squares. *Convergence*. DOI:10.4169/convergence20190801.

Kenneth M Monks. Bhāskara's approximation to and mādhava's series for sine. 2020. URL `https://digitalcommons.ursinus.edu/triumphs_calculus/15`.

J J O'Connor and E F Robertson. Leonhard Euler. *MacTutor History of Mathematics*, University of St Andrews, Scotland. `http://www-history.mcs.st-andrews.ac.uk/Biographies/Euler.html`, 1998.

Adam Parker. Leonhard Euler and Johann Bernoulli Solving Homogenous Higher Order Linear Differential Equations with Constant Coefficients. 2020. URL `https://digitalcommons.ursinus.edu/triumphs_differ/`.

Raymond Pierrehumbert. Warming the world. *Nature*, page 677, 2004. Translation of *Mémoire sur les Températures du Globe Terrestre et des Espaces Planétaires*, originally published in *Annales de Chimie et de Physique, Tome* XXVII (pp.136-167). Available at `users.physics.ox.ac.uk/~pierrehumbert/papers/Fourier1827Trans.pdf`.

Kim Plofker. *Mathematics in India*. Princeton University Press, 2009. ISBN 978-0691120676.

Dave Ruch. Abel and Cauchy on a Rigorous Approach to Infinite Series. 2016. URL `https://digitalcommons.ursinus.edu/triumphs_analysis/4`.

# Notes to Instructors

## PSP Content: Topics and Goals

This Primary Source Project (PSP) is intends to give students a guided tour through Fourier's incredibly influential book, *The Analytical Theory of Heat*. The mathematics that they will work through is quite broad, and includes the following:

- Higher-order linear homogeneous constant-coefficient differential equations
- Fourier series
- Infinite series
- Power series
- Partial differential equations
- Improper integrals
- Limits to infinity
- Foundational questions of rigor in analysis

## Student Prerequisites

Basic techniques of differentiation, integration, and limit calculation are the only assumed prerequisites for this project, along with some basic idea of what a differential equation is and what a solution to a differential equation is. Hence, this project is quite appropriate near the end of a second-semester calculus course (assuming there was a brief intro to ODEs) as well as in a course in differential equations.

## PSP Design, and Task Commentary

This PSP gives some context and background in the introduction and Section 1. Section 2 is Fourier's solution to the heat problem in the case of a square prism of infinite length, which results in a second-order linear differential homogeneous equation. This technique is presented in Section 3 straight from the mouth of Euler. In Section 4, we then apply this technique to Fourier's problem from Section 2. Section 5 introduces Fourier's general PDE with respect to time and three-dimensional space, and Section 6 then applies it to an infinite rectangular solid. This application motivated Fourier's development of Fourier series, and in Section 7 we look at the mathematical community's reaction to these results as a sneak peek into the foundations of rigor in analysis.

None of the manipulations are terribly messy in this project, except for perhaps the integrals in Task 15. One might wish to hint to their students that the product-to-sum identities for cosine can help make these integrals easier.

## Suggestions for Classroom Implementation

This project could be implemented any of a number of ways, and would probably take a very different form in a second-semester calculus course versus in differential equations. In calculus 2, it could actually make a very good project-based assessment, replacing perhaps a cumulative test, since it covers such a broad plethora of topics from the calculus sequence, including limits, definite integrals, improper integrals, geometric series, and power series. In differential equations, it might be a little too easy for a test, but could be used to introduce the content of homogeneous linear constant-coefficient differential equations in a very contextualized applied setting.

The author is happy to provide LaTeX code for this project. It was created using Overleaf which makes it very convenient to copy and share projects and can allow instructors to adapt this project in whole or in part as they like for their course.

### Sample Implementation Schedule (based on a 50-minute class period)

The author recommends two full 50-minute class periods for implementation of this PSP in a differential equations class.

- The readings and tasks of the PSP up to and including Section 1 can be assigned as preparation for class.
- Start class with 10 minutes of followup discussion on the first two sections. Perhaps get them ready for Section 2 by reading together and drawing some diagrams together.
- The next 30 minutes could consist of students working in small groups, complete as much as they can of the remaining PSP, as the instructor and any learning assistants assist.
- During the last 10 minutes of class, perhaps have some discussion around common difficulties and sticking points.
- The second session could proceed similarly, picking up from where the groups left off in the first.
- The completion of the remainder of the PSP can be assigned for homework.

In a second-semester calculus class, a third day may be necessary, since there may be additional time spent discussing what exactly a differential equation is, what a solution to a differential equation is, and so on. In a differential equations class, the two periods would likely suffice.

### Connections to other Primary Source Projects

There are six PSPs that relate very directly to the content of this PSP (available at the URLs listed in the bibliography). The ways in which they relate are discussed in footnotes throughout this document.

Furthermore, this differential equations PSP is just one of a series of such projects that include student projects on first-order linear DEs, Bernoulli DEs, exact DEs, higher-order linear DEs, Wronskians, and more! Find them at the URL below:

`https://digitalcommons.ursinus.edu/triumphs_differ/`

### Recommendations for Further Reading

The six PSPs mentioned above are perfect further reading for the curious student! For an advanced student looking for a deeper treatment of the subjects discussed here, the author recommends *Fourier Analysis: An Introduction* by Stein and Shakarchi. It takes Fourier's foundational ideas and extends them in incredibly surprising ways (for example using them to prove Dirichlet's Theorem), and is quite readable for an advanced undergraduate.

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For more information about the NSF-funded project TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), visit
`http://blogs.ursinus.edu/triumphs/`.