Winter 2020

Gauss and Cauchy on Complex Integration

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1 Introduction

Carl Gauss (1777–1855) and Augustin-Louis Cauchy (1789–1857) were two pioneering giants in the field of complex integration. Gauss set out some very big ideas in 1811, but did not immediately publicize his results. Cauchy began developing some related mathematics around the same time, and published his research in 1825 [Cauchy, 1825]. This work is still fundamental in the field of complex variables.

2 Gauss's Letter on Complex Integration

Gauss outlined some brave new ideas on complex integration in a 1811 letter to the mathematician Friedrich Bessel (1784–1846). When reading the following excerpt from this letter, bear in mind that mathematicians of his time frequently used the differential concept when thinking about integration. A differential was considered an infinitely small quantity that could be summed to create an integral. This approach is occasionally still used informally today when learning introductory calculus ideas such as area under a curve and determining work expended in mechanics.

Let's now read an excerpt [Gauss, 1811][1] from Gauss's informal discussion about how integration should work in the complex plane.

> Now what should one think of $\int f(z)\, dz$ for $z = x + yi$? Obviously, if we want to begin from clear concepts, we must assume that z passes through infinitely small increments (each of the form $\alpha + \beta i$) from the value for which the integral is 0 to $z = x + iy$, and then sum all the $f(z)\, dz$. In this way the meaning is completely established. But the passage can occur in infinitely many ways: just as one can think of the entire domain of all real magnitudes as an infinite straight line, so one can make the entire domain of all magnitudes, real and imaginary, meaningful as an infinite plane, wherein each point determined by abscissa $= x$ and ordinate $= y$ represents the quantity $x + iy$, as it were. The continuous passage from one value of z to another $x + yi$ accordingly occurs along a curve and is consequently possible in infinitely

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[1] Translation from the original German done by David Pengelley, Oregon State University, 2019.

many ways. I now assert that the integral always maintains a single value after two different passages, if $f(z)$ nowhere $= \infty$ within the region enclosed between the curves representing the two passages. This is a very beautiful theorem,[2] for which I will give a not difficult proof at a suitable opportunity.

Reread Gauss's footnote about the properties of the function $f(z)$ and the region in the complex plane. The precise meaning of the terms in his note is unfortunately unclear. Mathematicians have since refined this setting to be for an *analytic* function f on a *domain* in the complex plane.

Our first task will focus on illustrating Gauss's ideas.

Task 1 Consider the integral $\int f(z)\, dz$ from point 1 to point i in the complex plane, where $f(z) = \frac{1}{z}$. Consider the following four "passages" (i.e. paths) for this integral:

$\mathcal{P}_1$ along the quarter circle of radius 1 centered at the origin, oriented counterclockwise.

$\mathcal{P}_2$ along the three-quarter circle of radius 1 centered at the origin, oriented clockwise.

$\mathcal{P}_3$ along the line segment from point 1 to point i.

$\mathcal{P}_4$ along the vertical segment from point 1 to point $1 + i$ and then horizontally from $1 + i$ to point i.

(a) Sketch these paths, and label any points where $f(z)$ is "$= \infty$" in the complex plane.

(b) Along which pair(s) of these paths would Gauss claim you get the same value for the integral $\int f(z)\, dz$ from point 1 to point i? Explain.

(c) For which path(s) would Gauss **not** guarantee the same integral value? Explain.

3 Cauchy's Definition of Complex Integration

To motivate his 1825 definition for integrals in the complex plane, Cauchy began by writing his earlier definition of integration on the real line, which he developed and published earlier [Cauchy, 1823]. Cauchy did this before the invention of the "Riemann sum" definition you may recall from your introductory calculus course. In fact, Riemann had read Cauchy's work and generalized his integration ideas to get the Riemann sum version we see nowadays in calculus courses [Riemann, 1854]. After an introductory section in his *Mémoire* [Cauchy, 1825], Cauchy begins Section 2 with the following integral definition. Cauchy often used the term "imaginary" where we would nowadays say "complex".

[2] *Gauss's footnote*: In reality it is here also assumed that $f(z)$ is itself a uniform function of z, or at least that, for those values within the entire region, only one system of values is taken without a break in continuity.

In order to fix generally the meaning of the notation

$$\int_{x_0}^{X} f(x)\, dx, \tag{1}$$

where x_0 and X designate real limits and $f(x)$ a real or imaginary function of the [real] variable x, it is sufficient to consider the definite integral represented by this notation as equivalent to the limit, or one of the limits, towards which the sum

$$(x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (X - x_{n-1}) f(x_{n-1}) \tag{2}$$

converges when the elements of the difference $X_0 - x_0$, namely,

$$x_1 - x_0, x_2 - x_1, \cdots, X - x_{n-1},$$

are quantities having the same sign as the difference, and receive ever smaller numerical values.

This is a very long sentence! Notice that Cauchy leaves open the possibility that an integral of this kind could have multiple values. Indeed, this could happen when $f(x)$ is multivalued, and things could get quite complicated. To avoid this possibility, modern integration theory assumes a single-valued integrand function f, in which case it can be shown that the integral value is unique. We will make this assumption for the duration of the project.

Task 2 The sum (2) is a type of Riemann sum. Consider the example with $x_0 = 0, X = \pi/2$ and $f(x) = \cos x + i \sin x$. Write out this sum for $n = 4$ with evenly spaced x_k values. You do not need to evaluate the sum.

Task 3 Rewrite Cauchy's integral definition using your own words and several sentences. Include and reference a labeled diagram for the x_k values on the real line.

Task 4 Rewrite this integral definition in a modern formulation, using a limit and sigma notation.

Task 5 Consider the example from Task 2 with unspecified n. Split the sum (2) into real and imaginary parts, and take the limit of each part as $n \to \infty$. Evaluate the result.

Hint: To find each limit, use Cauchy's definition of integral, and the Fundamental Theorem of Calculus from introductory calculus.

Bonus. Use a calculator or write a program to evaluate the sum (2) for this example with $n = 4$ and $n = 10$. Compare your answers to the exact value you found in the limit.

Cauchy next generalized this definition for a function f of a *complex* variable and paths in the complex plane, not just along the real line. Carefully read his definition below.

In order to include in the same definition integrals between real limits and integrals taken between imaginary limits, it is necessary to represent by the notation

$$\int_{x_0+iy_0}^{X+iY} f(z)\, dz$$

the limit, or one of the limits, towards which the sum of the products of the form

$$\begin{array}{c}[(x_1-x_0)+(y_1-y_0)\,i]\,f(x_0+y_0 i)\,,\\ [(x_2-x_1)+(y_2-y_1)\,i]\,f(x_1+y_1 i)\,,\\ \text{etc.}\\ [(X-x_{n-1})+(Y-y_{n-1})\,i]\,f(x_{n-1}+y_{n-1} i)\end{array} \tag{3}$$

converge, when, in each of the two series,

$$\begin{array}{c}x_0, x_1, x_2, \cdots, x_{n-1}, X,\\ y_0, y_1, y_2 \cdots, y_{n-1}, Y,\end{array}$$

these being composed of terms that always go on increasing or decreasing from the first to the last, these same terms indefinitely approach each other, and their numbers grow more and more.[3]

To illustrate what Cauchy proposed here, let's use examples from our Gauss investigation.

Task 6 From Task 1, consider the integral $\int \frac{1}{z}\, dz$ along the linear path $\mathcal{P}_3$ from point 1 to point i. Write out Cauchy's "sum of the products" that approximates the integral using $n = 4$ with evenly spaced x_k and y_k values. You do not need to evaluate the sum.

Task 7 From Task 1, consider the integral $\int \frac{1}{z}\, dz$ along the piecewise linear path $\mathcal{P}_4$ from point 1 to point i. Write out Cauchy's "sum of the products" that approximates the integral using $n = 6$ with evenly spaced x_k and y_k values. You do not need to evaluate the sum.

Notice that in his definition, Cauchy does not explicitly insist on the points $x_k + iy_k$ being on any particular path, and he allows for more than one possible value for the integral limit. There are several feasible explanations for this explored in the bonus task below. For clarity and simplicity in this project, we will assume the integrand function f is single-valued and the points $x_k + iy_k$ are on a particular path $\mathcal{P}$. It can then be shown that the integral value is unique for the given path $\mathcal{P}$.

[3] Throughout his paper, Cauchy uses $\sqrt{-1}$ where we have written i for notational ease. It was also common in Cauchy's era to use the term "series" where we now use "sequence".

Task 8 Rewrite Cauchy's integral definition using your own words and several sentences, but require the points $x_k + iy_k$ to be on a particular path $\mathcal{P}$. Include, and make reference to, a labeled diagram for the x_k, y_k, and $x_k + iy_k$ in the complex plane. Then write out this modified Cauchy's definition for $\int_{x_0+iy_0}^{X+iY} f(z)\, dz$ along a path $\mathcal{P}$ in a modern formulation, using a limit and sigma notation.

Task 9 **History bonus**. Why do you think Cauchy did not explicitly insist on the points $x_k + iy_k$ being on any particular path? Was it an accidental omission? Was he thinking of different path possibilities like Gauss mentioned in his letter to Bessel? Or perhaps he was thinking of multi-valued functions? What do you think?

Cauchy next introduced a very clever idea that makes it possible to calculate these integrals without directly using limits. As you read this next passage from his *Mémoire*, his idea should remind you of an important topic from your introductory calculus classes.

To obtain two sequences of this kind, it is sufficient to suppose that

$$x = \phi(t), \quad y = \psi(t),$$

where $\phi(t), \psi(t)$ are two continuous functions of a new variable t, always increasing or decreasing from $t = t_0$ to $t = T$, and satisfying the conditions

$$\left[\begin{array}{ll} \phi(t_0) = x_0, & \psi(t_0) = y_0 \\ \phi(T) = X, & \psi(T) = Y \end{array}\right]$$

and then to represent by

$$x_0, x_1, x_2, \cdots, x_{n-1}, X,$$
$$y_0, y_1, y_2 \cdots, y_{n-1}, Y,$$

the values of x and of y corresponding to the values of t that compose an increasing or decreasing series of the form

$$t_0, t_1, \cdots, T.$$

Once again Cauchy wrote a very long sentence! Do the next task to make sure you understand Cauchy's ideas.

Task 10 Consider the quarter-circular path $\mathcal{P}_1$ from point 1 to point i introduced in Task 1. Find a pair of parametric functions $\phi(t), \psi(t)$ with $t_0 = 0$ and $T = \pi/2$ for $\mathcal{P}_1$. Then write out the complex numbers $x_k + iy_k$ corresponding to $t_0, \cdots, t_n$ for $n = 4$ with evenly spaced t_k values. Make labeled diagrams for both the t_k values on the t-axis, and the $x_k + iy_k$ points on the complex plane.

Hint: Use classic trigonometric functions for $\phi(t)$ and $\psi(t)$.

Task 11 Consider the piecewise linear path $\mathcal{P}_4$ from point 1 to point i introduced in Task 1. Find two pairs of parametric functions $\phi(t), \psi(t)$ for $\mathcal{P}_4$, one pair for the vertical subpath from 1 to $1+i$ and a second pair for the horizontal subpath from $1+i$ to i. For each subpath, write out the complex numbers $x_k + iy_k$ corresponding to $t_0, \cdots, t_n$ for $n = 3$ with evenly spaced t_k values. Make labeled diagrams for both the t_k values on the t-axis, and the $x_k + iy_k$ points on the complex plane.

Cauchy next uses an argument where he replaces the terms $x_k - x_{k-1}$ by $(t_k - t_{k-1})\,\phi'(t_{k-1})$ and the terms $y_k - y_{k-1}$ by $(t_k - t_{k-1})\,\psi'(t_{k-1})$ for each k in the approximating sum (3). This gives him a new integral approximation: the "sum of the products"

$$\begin{array}{c} (t_1 - t_0)\left[\phi'(t_0) + \psi'(t_0)\,i\right] f(\phi(t_0) + \psi(t_0)\,i)\,, \\ (t_2 - t_1)\left[\phi'(t_1) + \psi'(t_1)\,i\right] f(\phi(t_1) + \psi(t_1)\,i)\,, \\ \text{etc.} \\ (T - t_{n-1})\left[\phi'(t_{n-1}) + \psi'(t_{n-1})\,i\right] f(\phi(t_{n-1}) + \psi(t_{n-1})\,i)\,. \end{array} \tag{4}$$

Task 12 Use calculus to explain why replacing the terms $x_k - x_{k-1}$ by $(t_k - t_{k-1})\,\phi'(t_{k-1})$ is a reasonable approximation when $t_k - t_{k-1}$ is small. For simplicity, you may assume the t_k values are evenly spaced.

Hint: Make a labeled plot of the graph $x = \phi(t)$ on a real $t-x$ plane over the interval $[t_{k-1}, t_k]$.

Task 13 Use algebra and Cauchy's replacements $(t_k - t_{k-1})\,\phi'(t_{k-1}), (t_k - t_{k-1})\,\psi'(t_{k-1})$ to explain how Cauchy obtains the approximating sum of products (4) from those in (3).

Task 14 Consider the integral $\int \frac{1}{z}\,dz$ and linear path $\mathcal{P}_3$ from Tasks 1 and 6.

(a) Find a pair of parametric functions $\phi(t), \psi(t)$ with $t_0 = 0$ and $T = 1$ for $\mathcal{P}_3$.

(b) Find the derivatives $\phi'(t), \psi'(t)$.

(c) Write out the approximating sum (4) for this integral and path using $n = 3$ with evenly spaced t_k values. You do not need to evaluate the sum.

Next, Cauchy argues that as the number of terms n grows "more and more", the approximating sum of products (4) approaches the integral

$$\int_{t_0}^{T} \left[\phi'(t) + i\psi'(t)\right] f(\phi(t) + i\psi(t))\,dt$$

which he states is the same as the original integral $\int_{x_0+iy_0}^{X+iY} f(z)\,dz$. Modern mathematicians agree with Cauchy's conclusion that

$$\int_{x_0+iy_0}^{X+iY} f(z)\,dz = \int_{t_0}^{T} \left[\phi'(t) + i\psi'(t)\right] f(\phi(t) + i\psi(t))\,dt \tag{5}$$

along the path defined by the parametric equations $\phi(t), \psi(t)$.

Task 15 In the last excerpt, Cauchy initially claimed it is sufficient to suppose that $\phi(t), \psi(t)$ are continuous functions of t. What stronger condition did Cauchy actually use for his functions $\phi(t), \psi(t)$ in the derivation of formula (5)?

Now let's use Cauchy's new integral formulation (5) to check Gauss's claim for the integral $\int_1^i \frac{1}{z}\,dz$ and the paths from Task 1. To evaluate the integrals with respect to t in the next four tasks, you may use standard calculus integration techniques and assume the Fundamental Theorem of Calculus is valid for functions of a **real** variable.

Task 16 Use your parametric equations from Task 10 and Cauchy's integral formula (5) to find the exact value of $\int_1^i \frac{1}{z}\,dz$ along path $\mathcal{P}_1$.

Task 17 Find parametric equations for path $\mathcal{P}_2$ from Task 1. Then use Cauchy's formula (5) to find the exact value of $\int_1^i \frac{1}{z}\,dz$ along path $\mathcal{P}_2$.

Task 18 Use your parametric equations from Task 14 and Cauchy's formula (5) to find the exact value of $\int_1^i \frac{1}{z}\,dz$ along path $\mathcal{P}_3$.

Note: the integration is challenging for this one. If needed, check with your instructor for a hint, or use a CAS or integral tables.

Task 19 Use your parametric equations from Task 11 and Cauchy's formula (5) to find the exact value of $\int_1^i \frac{1}{z}\,dz$ along path $\mathcal{P}_4$.

Task 20 Compare the integral results you have found along the various paths from point 1 to point i. Are they in agreement with Gauss's claim that "the integral always maintains a single value after two different passages, if $f(z)$ nowhere $= \infty$ within the region enclosed between the curves representing the two passages", or do they conflict with it? Explain your answer.

4 Conclusion

We have followed Cauchy's development of a complex integral along a path in the complex plane, and found that some examples illustrate Gauss' claim that "the integral always maintains a single value after two different passages, if $f(z)$ nowhere $= \infty$ within the region enclosed between the curves representing the two passages". In his *Mémoire*, Cauchy gives a difficult proof that Gauss's claim was indeed correct under certain conditions on the integrand function and region containing the paths. There is much more to learn from these powerful ideas, which are pursued in most courses on complex variables.

References

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Notes to Instructors

This PSP is designed to be used in a course on complex variables. It could also be used in a demanding upper division course on the history of mathematics.

PSP Content: Topics and Goals

1. Explore Gauss's notion of path independence for complex integrals.
2. Develop Cauchy's definition of a complex definite integral.
3. Develop and apply Cauchy's parameterized version of a complex definite integral.
4. Apply Cauchy's parametric form for complex integrals to illustrate Gauss's ideas on path independence for complex integrals.

Student Prerequisites

The PSP is written with very few assumptions about student background beyond the basic sophistication developed in an introductory calculus course sequence. Some familiarity with parametric equations, Riemann sums and integration techniques are important. Instructors may find some just-in-time review helpful.

PSP Design, and Task Commentary

This is designed to take roughly one week of classroom time, with some reading and tasks done outside class. For a complex variables course, the PSP is designed to be used largely in place of the textbook section(s) introducing complex integration.

Complex variables courses are often taught without much emphasis on theory and without assuming much student theoretical background. A number of more theoretical tasks can be downplayed or emphasized, depending on course goals. With this balance in mind, the PSP author chose to summarize Cauchy's work, rather than give more excerpts with exercises, towards the end of the project (between Tasks 11 and 12, and between Tasks 14 and 15).

The PSP author does not know the answer to the history bonus Task 9, but finds it intriguing.

The definite integral mechanics of Tasks 16 - 19 are intended to be handled at an introductory calculus level of sophistication, and should not involve the complex logarithm. The instructor may want to (eventually) point out the connection $e^{it} = \cos t + i \sin t$ in Tasks 16 & 17 if students are familiar with e^{it}, both to ease some computations and to emphasize this important parameterization for the rest of a complex variables course. The integration is challenging in Task 18. Hints or using a CAS may be desirable.

Students familiar with the complex logarithm may be tempted to try using it with the Fundamental Theorem of Calculus in Tasks 16 - 19. Naively substituting the limits of integration i and 1 into the complex logarithm will produce the same answer for all paths, contradicting the result from Task 17. In fact, this apparent paradox could be used by the instructor to *motivate* the need for different branches of the complex logarithm and careful hypotheses for the complex version of the Fundamental Theorem of Calculus.

Suggestions for Classroom Implementation

Advanced reading of the project and some task work before each class is ideal but not necessary. See the sample schedule below for ideas.

Some of the theoretical subtleties such as Gauss's footnote, the uniqueness in Cauchy's integral definition, and the Mean Value Theorem in Task 12 could be basically ignored in an applied course, or strongly emphasized in a more theoretical course.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50 minute class period)

Students read through the Gauss excerpt and do Task 1 before the first class. After first discussing the Gauss material, students read the first Cauchy excerpt and work through Tasks 2 - 5 with some class discussion. Before the second class period, students read the second Cauchy excerpt and do Task 6. During the second class period, groups (or the class as a whole) discuss Task 6 and do Task 8. Depending on the course goals, the history bonus Task 9 could be homework or a class discussion. Then groups read the third Cauchy excerpt and work through Tasks 10 and 12. Before the third class, students do Tasks 7, 11, 13, 14. During the third class, students discuss their work on Tasks 13 and 14, and then complete the project, with Task 18 being assigned as homework.

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