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The Closure Operation as the Foundation of Topology

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The closure operation as the foundation of topology

Nicholas A. Scoville*

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1 Introduction

In the early 1900s, axiomatizing different mathematical disciplines was all the rage. While a discipline like geometry was well established at this time, topology was still quite new when it was being axiomatized, and hence the best way to approach the subject was unclear. We will study the work of two mathematicians who contributed to this study. First, a bit of biographical information.

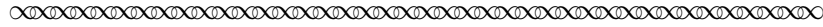
A Polish mathematician and logician, Kazimierz Kuratowski (February 2 1896 – June 18 1980) was born in Warsaw to a well-known lawyer in 1896. In 1913, he was enrolled as an engineering student at the University of Glasgow, evidently because he did not wish to study in Russian. Two years later, he began studying mathematics in Poland after the first World War forced him to remain in Poland. In 1933, he became a professor at Warsaw University and began his studies on applications of topological spaces in other areas of mathematics. One of his most significant contributions to general topology was the axiomatization of the closure operator (1922) in which he used Boolean algebra to characterize the topology of an abstract space without relation to the notion of points. This latter contribution is the main focus of this project.

Born in 1868 to Jewish parents, Felix Hausdorff (November 8, 1868 – January 26, 1942) is known as one of the founders of modern topology, but he also made significant contributions to set theory, descriptive set theory, measure theory, function theory, and functional analysis. Hausdorff studied mathematics and astronomy mainly in the city of Leipzig. He graduated from the University of Leipzig and became a lecturer at the university as well. In 1901, he was appointed as an adjunct assertion in the University of Leipzig. In addition to point-set topology, he also developed Hausdorff spaces and concepts of metric and topological spaces. The development of the idea of closeness independent of the ability to be measured also interested Hausdorff. As a result of the Nazi reign, Hausdorff committed suicide with his wife and her sister after being ordered to move to a concentration camp.

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2 Closed sets and closure

In his foundational 1957 textbook on topology titled “Set Theory” [Hausdorff, 1957, p 257-258], Hausdorff write:



The mathematical discipline concerned with [topological invariance] is called Topology or Analysis Situs. (The latter term, due to Leibniz, was re-introduced by Riemann.) . . . this seems like a suitable occasion to touch, in all brevity, on those point-set theories that emphasize the topological point of view from the very beginning and work only with the topologically invariant concepts. . . What are primary in the topological space X are the sets that are *closed* (in X) and their complements, the *open* sets. . . The closed or open sets can be taken as our starting point and left undefined, or they can be defined from related concepts (limit point, neighborhood), but always derived in such a way as to keep invariant their topological character; the more detailed nature of the space is then determined by *axioms*. . .

[CLOSED] SUM AND INTERSECTION AXIOMS. The closed sets must, regardless of anything else, satisfy the following conditions:

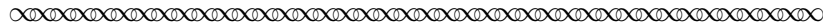
1. The space X and the null set \emptyset are closed.
2. The union of two closed sets is closed.
3. The intersection of any number of closed sets is closed.

As a consequence, the closure \bar{A} can be defined as the intersection of all the closed sets containing A . . . it has the following properties:

CLOSURE AXIOMS.

1. $\bar{\emptyset} = \emptyset$
2. $A \subseteq \bar{A}$
3. $\overline{\bar{A}} = \bar{A}$
4. $\overline{A \cup B} = \bar{A} \cup \bar{B}$

It would also be possible to make these properties our starting point (a la Kuratowski) by taking them as our axioms for the set function \bar{A} .



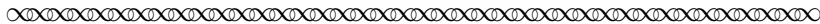
The goal of this project is to prove the last claim made by Hausdorff by following Kuratowski, who lays the foundations of topology via an axiomatic closure operation. We will then show in Task 6 that this is equivalent to the “open set axioms” which you are familiar with. In other words, we will show that the Open Sum and Intersection Axioms (OSIA), which can be found in any modern introductory textbook on topology, are equivalent to the Closure Axioms (CA).

Task 1

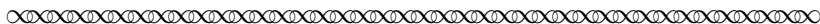
Prove that Hausdorff’s Closed Sum and Intersection Axioms (CSIA) are equivalent to OSIA that we use today. [Hint: Use deMorgan’s law.] When starting with CSIA, “closed” is an undefined term and $A \subseteq X$ is **open** if $X - A$ is closed. Similarly, when starting with OSIA, “open” is undefined and $A \subseteq X$ is **closed** if $X - A$ is open.

Task 2 Let A be any set, and assume CSIA. Prove that \bar{A} is closed.

We now investigate a short section from a paper by Kuratowski entitled “On the Closure Operation” [Kuratowski, 1922] which is described by Kuratowski himself as a:



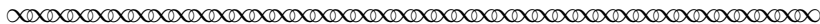
slightly modified [version] of my thesis presented May 12, 1920 at the University of Warsaw for the degree of Doctor of Philosophy.



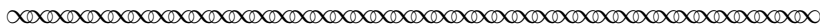
Kuratowski explains the purpose of his paper, as well as the role of the four facts above.



In this paper we analyze these properties [CA] and their consequences. We proceed in an axiomatic fashion by assuming as given both an arbitrary set X , and an operation \bar{A} , such that for every subset $A \subseteq X$, there exists a subset \bar{A} that satisfies CA 1-4.



It is worth pondering exactly what Kuratowski is doing here. As mentioned above, axiomatic systems were all the rage at this time, and Kuratowski was right in the thick of it. Kuratowski is saying that the above properties are so fundamental and so robust, that we can decree that we have an abstract operation, called \bar{A} , that satisfies four axioms. From there, we can prove many things and investigate the theory of this operation. With this in mind, we turn back to Kuratowski:



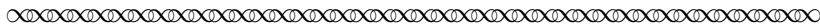
We now establish fundamental properties of the closure operation.

Theorem 1. $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$.

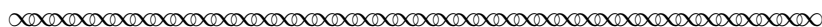
Theorem 2. $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

Theorem 3. $\overline{A - B} \subseteq \bar{A} - \bar{B}$.

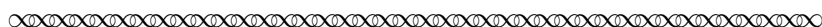
Theorem 4. $\overline{\bar{X}} = X$.



We let Kuratowski prove Theorem 1.

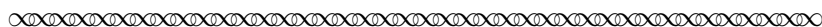


The inclusion $A \subseteq B$ implies $B = A \cup B$, hence $\bar{B} = \overline{A \cup B}$, and by CS4, $\bar{B} = \bar{A} \cup \bar{B}$. Thus $\bar{A} \subseteq \bar{B}$.



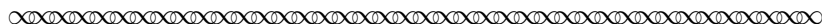
Task 3

Prove Theorems 2-4 assuming CA. [Hint: The following identities from set theory may prove useful in your proof: $A \cap B \subseteq A$, $A \cap B \subseteq B$, $A \subseteq A \cup B$, $A \cup B = (A - B) \cup B$.]



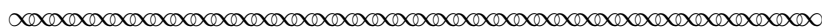
Theorem 2a. Letting $\{A_i\}$ denote an arbitrary family of sets indexed by the variable i , we have

$$\overline{\bigcap_i A_i} \subseteq \bigcap_i \bar{A}_i \text{ and } \bigcup_i \bar{A}_i \subseteq \overline{\bigcup_i A_i}.$$

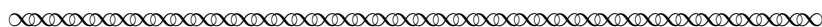


Task 4

Prove Theorem 2a.



A set A is said to be *closed* when $A = \bar{A}$... the union of two closed sets is closed ... and the intersection of any collection of closed sets is closed.



Task 5

Kuratowski has just claimed that his definition of closed set satisfies CSIA2 and CSIA3. Using what you and Kuratowski have shown so far, prove this claim.

Before turning to our main result, it may help to think about the following. We have two sets of axioms (CSIA and CA) and two terms (closed and closure). If we assume the CSIA, which of the two terms is undefined and which one needs a definition? If we assume the CA, which of the two terms is undefined and which one needs a definition? Both definitions are located somewhere in this project. Make sure you locate them and have them on hand for the final question.

Task 6

Finally, prove that a collection of sets satisfies CA if and only if it satisfies OSIA.

References

F. Hausdorff. *Set theory*. Chelsea Publishing Company, New York, 1957. Translated by John R. Aumann, et al.

K. Kuratowski. Sur l'opération \bar{A} de l'analysis situs. *Fund. Math.*, 3:182–199, 1922.

Notes to Instructors

Primary Source Project Content: Topics and Goals

This project is for students who have seen the standard “open set axioms” and provides an equivalent axiomatic system for grounding all of point set topology based on the work of Kuratowski. Other than exposure to and familiarity with the open set axioms, this is a self contained project which can easily be done during a single class period with any unfinished tasks assigned for homework. Because most of the tasks have several claims to prove, this allows the instructor the ability to show one claim, another claim to be worked on in groups during class, and other claims to be done as homework tasks. In addition to the overarching goals of familiarizing students with an axiomatic system and showing them that multiple systems are equivalent, this project has the further content goal of familiarizing the student with the closure operation.

Connections to other Primary Source Projects

There are several other projects in topology written by the author. Project titles along with links are given below. The last two of these are full-lengths PSPs; all others are mini-PSPs that are intended to be completed in 1–2 class days.

- *Topology from Analysis*
https://digitalcommons.ursinus.edu/triumphs_topology/1/
- *The Cantor Set before Cantor*
https://digitalcommons.ursinus.edu/triumphs_topology/2/
- *Connecting Connectedness*
https://digitalcommons.ursinus.edu/triumphs_topology/3/
- *From sets to metric spaces to topological spaces*
https://digitalcommons.ursinus.edu/triumphs_topology/6/
- *A Compact Introduction to a Generalized Extreme Value Theorem*
https://digitalcommons.ursinus.edu/triumphs_topology/5/
- *Nearness Without Distance*
https://digitalcommons.ursinus.edu/triumphs_topology/7/
- *Connectedness: Its Evolution and Applications*
https://digitalcommons.ursinus.edu/triumphs_topology/8/

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