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Summer 2017

### Construction of the Figurate Numbers

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# Construction of the Figurate Numbers

Jerry Lodder\*

May 22, 2020

## 1 Nicomachus of Gerasa

Our first reading is from the ancient Greek text an *Introduction to Arithmetic* [4, 5] written by Nicomachus of Gerasa, probably during the late first century CE. Although little is known about Nicomachus, his *Introduction to Arithmetic* was translated into Arabic and Latin and served as a textbook into the sixteenth century [2, p. 176] [7]. Book One of Nicomachus' text deals with properties of odd and even numbers and offers a classification scheme for ratios of whole numbers. Our excerpt, however, is from Book Two, which deals with counting the number of dots (or alphas) in certain regularly-shaped figures, such as triangles, squares, pyramids, etc. Note that alpha ( $\alpha$ ) is the ancient Greek symbol for the number one, iota ( $\iota$ ) the symbol for ten, etc. The numbers that arise from these figures are called figurate numbers and form various classes, such as the triangular numbers, the square numbers, the pyramidal numbers, etc. Today, figurate numbers have found applications not only in arithmetic, but also in algebra, geometry, probability, calculus and computer science. Many counting arguments can be reduced to certain combinatorial properties of these numbers. We begin with the natural series of numbers, which Nicomachus also calls "linear numbers."

## 2 The Natural Series of Numbers

Let's read in English translation [4]:



Nicomachus, from  
*Introduction to Arithmetic*  
BOOK TWO  
CHAPTER VI

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[. . .] The subjects which we must first survey and observe are concerned with linear, plane, and solid numbers, [. . .] . [T]he germs of these ideas are taken over into arithmetic, as the science which is the mother of geometry and more elementary than it.

First, however, we must recognize that each letter by which we indicate a number, such as iota, the sign for 10, kappa for 20, and omega for 800, designates that number by man's convention and agreement, not by nature. On the other hand, the natural, unartificial, and therefore simplest indication of numbers would be the setting forth one beside the other of the units contained in each. For example, the writing of one unit by means of one alpha will be the sign for 1; two units side by side, that is, a series of two alphas, will be the sign for 2; when three are put in a line it will be the character for 3, four in a line for 4, five for 5, and so on. For by means of such a notation and indication alone could the schematic arrangement of the plane and solid numbers mentioned be made clear and evident, thus:

The number 1,  $\alpha$   
The number 2,  $\alpha \alpha$   
The number 3,  $\alpha \alpha \alpha$   
The number 4,  $\alpha \alpha \alpha \alpha$   
The number 5,  $\alpha \alpha \alpha \alpha \alpha$

and further in similar fashion.

Unity, then, occupying the place and character of a point will be the beginning of intervals and of numbers, but not itself an interval or a number, just as the point is the beginning of a line, or an interval, but is not itself line or interval. [. . .]

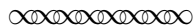
The first dimension is called 'line,' for 'line' is that which is extended in one direction. Two dimensions are called 'surface,' for a 'surface' is that which is extended in two directions. Three dimensions are called 'solid,' for a 'solid' is that which is extended in three directions, and it is by no means possible to conceive of a solid which has more than three dimensions, depth, breath, and length. [. . .]

## CHAPTER VII

The point, then, is the beginning of dimension, but not itself a dimension, and likewise the beginning of a line, but not itself a line; the line is the beginning of a surface, but not surface; and the beginning of the two-dimensional, but not itself extended in two directions. Naturally, too, surface is the beginning of body, but not itself body, and likewise the beginning of the three-dimensional, but not itself extended in three directions.

Exactly the same in numbers, unity is the beginning of all number that advances unit by unit in one direction; linear number is the beginning of plane number, which spreads out like a plane in one more dimension; and plane number is the beginning of solid number, which possesses a depth in the third dimension, besides the original ones. To illustrate and classify, linear numbers are all those which begin with 2 and advance by the addition of 1 in one and the same dimension; and plane numbers are those that begin with 3 as their most elementary root and proceed through the next succeeding numbers. They receive their names also in the same order; for there are first

the triangles, then the squares, the pentagons after these, then the hexagons, the heptagons, and so indefinitely, and, as we said, they are named after the successive numbers beginning with 3. [...]



**Exercise 2.1.** Why, in your opinion, does Nicomachus believe that arithmetic is more elementary than geometry?

**Exercise 2.2.** Explain verbally what is meant by “the natural, unartificial, and . . . simplest indication of numbers.” How are the counting numbers represented in this way?

**Exercise 2.3.** What is the natural series of numbers?

**Exercise 2.4.** After reading Chapters VI and VII, answer the following:

- (a) What is meant by “unity”?
- (b) What is a one-dimensional object called?
- (c) What is a two-dimensional object called?
- (d) What is a three-dimensional object called?
- (e) According to Nicomachus, are there objects in dimensions higher than the third dimension? Cite a relevant passage to justify your answer.

**Exercise 2.5.** According to Nicomachus, does a point have a dimension? Today a point is often considered to be in dimension zero. Is this consistent with Nicomachus’ statement that a point “is the beginning of dimension”?

**Exercise 2.6.** (a) What is a heptagon?  
 (b) Sketch a regular heptagon, which has all sides being of equal length and all (interior) angles of equal measure.

**Exercise 2.7. The Linear Numbers.** Implicit in Nicomachus’ description of the “natural indication of numbers” is a correspondence between line segments and counting numbers, that can be called “linear numbers,” to use his own words. Recall that he writes: “when three [alphas] are put in a line it will be the character for 3, four in a line for 4, . . . .” In this exercise we develop the linear numbers for this correspondence, denoted  $L_1, L_2, L_3$ , etc. Although Nicomachus would probably begin the linear numbers with a segment representing the number two, we will consider the first linear number,  $L_1$ , as a beginning configuration representing just one dot:

$$L_1, \bullet$$

As an equation,  $L_1 = 1$ . Following Nicomachus’ pattern, let  $L_2$  denote the number of dots in a segment with two dots:

$$L_2, \bullet \bullet$$

In the form of an equation,  $L_2 = 2$ . Let  $L_3$  denote the number of dots in a segment with three dots:

$$L_3, \quad \bullet \bullet \bullet$$

Thus,  $L_3 = 3$ . Suppose that this pattern continues.

- (a) Compute a numerical value for  $L_4$ , the fourth linear number, i.e., fill in the blank  $L_4 = \square$ .
- (b) Draw a segment containing the number of dots that corresponds to  $L_4$ .
- (c) What geometric relation is reflected by the equation

$$L_4 = L_3 + 1 ?$$

More specifically, what does the equation  $L_4 = L_3 + 1$  tell us about how the fourth linear number is constructed from the third linear number using dots?

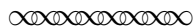
- (d) For an arbitrary counting number  $n$ , compute a numerical value for  $L_n$ , the  $n$ th linear number. You may assume that  $L_n$  corresponds to a line segment containing  $n$ -many dots.
- (e) If  $L_{n-1}$  denotes the linear number preceding (just before)  $L_n$ , find a numerical value for  $L_{n-1}$ .
- (f) **The Recursive Formula for the Linear Numbers.** Find an equation relating  $L_n$  and  $L_{n-1}$  so that

$$L_n = L_{n-1} + \square.$$

This is sometimes called the recursive formula for the linear numbers.

- (g) What does the recursive formula for the linear numbers, part (f), tell us about how a linear number is constructed from the preceding linear number using dots? How does this compare to Nicomachus' statement that "[the] linear numbers ... advance by the addition of 1 in one and the same dimension."

### 3 The Triangular Numbers

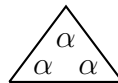


#### CHAPTER VIII

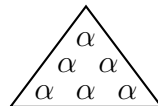
Now a triangular number is one which, when it is analyzed into units, shapes into triangular form the equilateral placement of its parts in a plane. 3, 6, 10, 15, 21, 28, and so on, are examples of it; for their regular formations, expressed graphically, will be at once triangular and equilateral. As you advance you will find that such a number series as far as you like takes the triangular form, if you put as the most elementary form the one that arises from unity, so that unity may appear to be potentially a triangle, and 3 the first actually.

Their sides will increase by the successive numbers, for the side of the one potentially first is unity; that of the one actually first, that is, 3, is 2; that of 6, which is actually second, 3; that of the third 4; the fourth, 5; the fifth, 6, and so on.

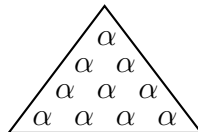
The triangular number is produced from the natural series of number set forth in a line, and by the continued addition of successive terms, one by one, from the beginning; for by the successive combinations and additions of another term to the sum, the triangular numbers in regular order are completed. For example, from this natural series, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, I take the first term and have the triangular number which is potentially first, 1,  $\triangle_{\alpha}$ ; then adding the next term I get the triangle actually first, for 2 plus 1 equals 3. In its graphic representation it is thus made up: Two units, side by side, are set beneath one unit, and the number three is made a triangle:



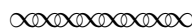
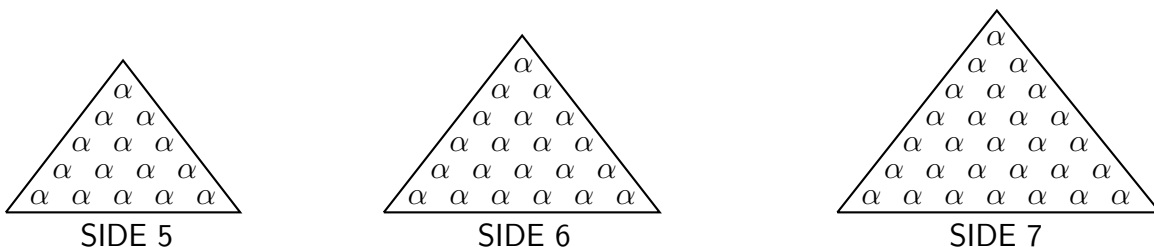
Then when next after these the following number, 3, is added, simplified into units, and joined to the former, it gives 6, the second triangle in actuality, and furthermore, it graphically represents this number:




Again, the number that naturally follows, 4, added in and set down below the former, reduced to units, gives the one in order next after the aforesaid, 10, and takes a triangular form:

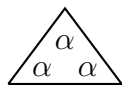


5, after this, then 6, then 7, and all the numbers in order, are added, so that regularly the sides of each triangle will consist of as many numbers as have been added from the natural series to produce it:

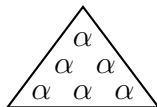


**Exercise 3.1.** In this exercise we introduce the triangular numbers, which count the number of dots (or the number of alphas,  $\alpha$ ) in regularly-shaped, equilateral triangles. Although Nicomachus is somewhat reluctant to state that the value of the first triangular number is 1, since the triangle drawn around one alpha, , is only potentially the first triangle, we will begin the triangular numbers with the value of one. Let  $T_1$  denote the first triangular number. Then  $T_1 = 1$ .

- (a) Let  $T_2$  denote the second triangular number. Compute the numerical value of  $T_2$  by counting the number of alphas ( $\alpha$ ) in the triangle:



- (b) Let  $T_3$  denote the third triangular number. Compute the numerical value of  $T_3$  by counting the number of alphas in the triangle:



- (c) Let  $T_4$  denote the fourth triangular number,  $T_5$  the fifth triangular number,  $\dots$ , and let  $T_n$  denote the  $n$ th triangular number for a natural number  $n$ . Fill in the following table for the numerical value of the first eight triangular numbers:

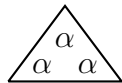
$n$	1	2	3	4	5	6	7	8
$T_n$	1							

What strategy did you devise to compute the triangular numbers?

- (d) Carefully explain how  $T_8$  is computed. Does Nicomachus include a picture of this eighth triangle in his writing?

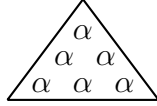
**Exercise 3.2. The Iterative Formula for the Triangular Numbers.** Nicomachus writes that “The triangular number is produced from the natural series of number  $\dots$  by the continued addition of successive terms, one by one, from the beginning  $\dots$  .”

- (a) Write  $T_2$ , the second triangular number, as the sum of two successive whole numbers by using the number of alphas in each row of the second triangle:



Thus,  $T_2 = \square + \square$ .

- (b) Write  $T_3$ , the third triangular number, as the sum of three successive whole numbers by using the number of alphas in each row of the triangle:



Thus,  $T_3 = \square + \square + \square$ .

- (c) Following the above format, write  $T_4$  as the sum of four successive whole numbers so that:

$$T_4 = \square + \square + \square + \square.$$

- (d) What ten successive whole numbers would be needed to be added together to produce  $T_{10}$ , the 10th triangular number?
- (e) Let  $n$  denote a generic natural number. Find a formula for  $T_n$  that expresses the  $n$ th triangular number as the sum of  $n$  successive whole numbers. What geometric idea does this formula represent in terms of the number of alphas in each row of the  $n$ th triangle? Today this formula is called the iterative formula for  $T_n$ .

**Exercise 3.3. The Recursive Formula for the Triangular Numbers.**

- (a) After finishing the construction of (what we denote)  $T_2$ , Nicomachus describes the construction of the next triangle as “the following number 3, is added, . . . , and joined to the former [triangle].” Using  $T_2$  to denote the former triangular number and  $T_3$  to denote the next triangular number, find an equation relating  $T_3$  to  $T_2$  to that

$$T_3 = T_2 + \square.$$

- (b) To find the next triangular number, Nicomachus states “the number that naturally follows, 4, added in and set down below the former [triangle], . . . , gives . . . the next [triangle].” Letting  $T_3$  denote the former triangular number and  $T_4$  the next triangular number, find an equation relating  $T_4$  to  $T_3$  so that

$$T_4 = T_3 + \square.$$

- (c) Following the above format, find an equation relating  $T_5$  and  $T_4$ .
- (d) For an arbitrary natural number  $n$ , find a formula relating  $T_n$  to the preceding triangular number  $T_{n-1}$ . Express this formula verbally describing how the preceding triangle is used to construct the next triangle. This formula is sometimes called the recursive formula for the triangular numbers.
- (e) Using part (d), find an equation relating  $T_n$  to the sum of  $T_{n-1}$  and a certain linear number, i.e., fill in the blank so that:

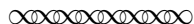
$$T_n = T_{n-1} + L_{\square}.$$

**Exercise 3.4. Extra.** Arrange two copies of  $T_n$  to produce a rectangle with  $n$  rows and  $(n + 1)$  columns. Use this to answer the following.

- (a) Find a simple formula for  $T_n$  in terms of  $n$ .
- (b) Find a simple formula for  $1 + 2 + 3 + 4 + \dots + n$  using part (a).

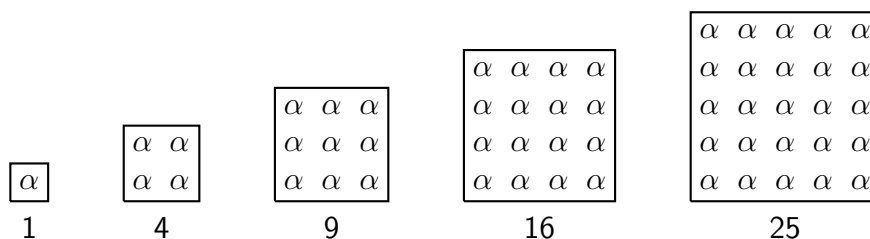


## 4 The Square Numbers



### CHAPTER IX

The square is the next number after this, which shows us no longer 3, like the former, but 4, angles in its graphic representation, but is nonetheless equilateral. Take, for example, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100; for the representations of these numbers are equilateral, square figures, as here shown; and it will be similar as far as you wish to go:



It is true of these numbers, as it was also of the preceding, that the advance in their sides progresses with the natural series. The side of the square potentially first, 1, is 1; that of 4, the first in actuality, 2; that of 9, actually the second, 3; that of 16, the next, actually the third, 4; that of the fourth, 5; of the fifth, 6, and so on in general with all that follow.

This number also is produced if the natural series is extended in a line, increasing by 1, and no longer the successive numbers are added to the numbers in order, as was shown before, but rather all those in alternate places, that is, the odd numbers. For the first, 1, is potentially the first square; the second, 1 plus 3, is the first in actuality; the third, 1 plus 3 plus 5, is the second in actuality; the fourth, 1 plus 3 plus 5 plus 7, is the third in actuality; the next is produced by adding 9 to the former numbers, the next by the addition of 11, and so on.

In these cases, also, it is a fact that the side of each consists of as many units as there are numbers taken into the sum to produce it.



**Exercise 4.1.** The square numbers count the number of dots (or the number of alphas,  $\alpha$ ) in regularly-shaped squares, where the side length of the squares increases by one unit. Again, Nicomachus is reluctant to represent the first square by one alpha, since a square drawn around a single alpha is only “potentially” a square. Nonetheless, we will let the first square number be 1. In symbols, let  $S_1$  denote the first square number,  $S_2$  the second square number,  $S_3$  the third square number, etc. Thus,  $S_1 = 1$ .

- (a) Compute the value of  $S_2$  by counting the number of alphas ( $\alpha$ ) in the square:



Thus,  $S_2 = \square$ .

- (b) For an arbitrary natural number  $n$ , let  $S_n$  denote the number of alphas in the  $n$ th square. Fill in the following table giving the first ten square numbers.

$n$	1	2	3	4	5	6	7	8	9	10
$S_n$	1									

**Exercise 4.2.** Nicomachus claims that the square numbers are produced by adding the numbers that are “in alternate places, that is, the odd numbers.”

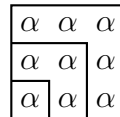
- (a) Use the following suggestive diagram



to write  $S_2$ , the second square number, as the sum of two consecutive odd numbers:

$$S_2 = \square + \square.$$

- (b) Use the following suggestive diagram



to write  $S_3$  as the sum of three consecutive odd numbers:

$$S_3 = \square + \square + \square.$$

- (c) Following the above format, write  $S_4$  as the sum of four consecutive odd numbers.

**Exercise 4.3. The First Difference.** The odd numbers identified geometrically in Exercise (4.2) can be realized arithmetically by computing first differences, i.e., the differences of consecutive squares.

- (a) Compute  $D_1 = S_2 - S_1$ .
- (b) Identify  $D_1$  geometrically in the figure drawn in Exercise (4.2), part (a).
- (c) Compute  $D_2 = S_3 - S_2$ .
- (d) Identify  $D_2$  geometrically in the figure drawn in Exercise (4.2), part (b).
- (e) Compute  $D_3 = S_4 - S_3$ .
- (f) Compute  $D_4 = S_5 - S_4$ .

- (g) What series of numbers is formed by  $D_1, D_2, D_3, D_4$ ? Would you include  $S_1$  as the first term of this series? Why or why not?

**Exercise 4.4. The Second Difference.** The pattern formed by the first differences  $D_1, D_2, D_3, D_4$  is fairly obvious in the above exercise. Should this not be the case, another difference, the second difference, can be computed.

- (a) Compute  $D_2 - D_1$ .  
 (b) Compute  $D_3 - D_2$ .  
 (c) Compute  $D_4 - D_3$ .  
 (d) Verbally state a pattern formed by the second differences above.

**Exercise 4.5. Extra.** Let  $n$  denote an arbitrary natural number.

- (a) **The Recursive Formula for the Square Numbers.** If  $S_{n+1}$  denotes the  $(n + 1)$ st square number, then  $S_n$  denotes the previous square number. How many new dots (or alphas) need to be added to  $S_n$  to produce  $S_{n+1}$ ? Hint: Thinking geometrically, add  $n$ -many dots along right-hand column of  $S_n$ , add another  $n$ -many dots along the top of  $S_n$ , and add one more dot to the upper right-hand corner of  $S_n$  to finally produce  $S_{n+1}$ . Fill in the blank with the total number of new dots so that

$$S_{n+1} = S_n + \boxed{\phantom{000}},$$

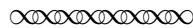
and express this result verbally.

- (b) **The Iterative Formula for the Square Numbers.** Find a formula for  $S_{n+1}$  in terms of the sum of consecutive odd whole numbers so that:

$$S_{n+1} = 1 + 3 + 5 + 7 + \dots + \boxed{\phantom{000}}.$$

Hint: Compare with part (a) above.

## 5 The Pentagonal Numbers



### CHAPTER X

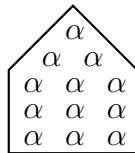
The pentagonal number is one which likewise upon its resolution into units and depiction as a plane figure assumes the form of an equilateral pentagon. 1, 5, 12, 22, 35, 51, 70, and analogous numbers are examples. Each side of the first actual pentagon, 5, is 2, for 1 is the side of the pentagon potentially first, 1; 3 is the side of 12, the second of those listed; 4, that of the next, 22; 5, that of the next in order, 35, and 6 of the succeeding one, 51, and so on. In general the side contains as many units as are the numbers that have been added together to produce the pentagon, chosen out of the natural arithmetical series set forth in a row. For in a like and similar manner, there are added together to produce the pentagonal numbers the terms beginning with 1 to any extent whatever that are two places apart, that is, those that have a difference of 3. Unity is the first pentagon, potentially, and is thus depicted:



5, made up of 1 plus 4, is the second, similarly represented:



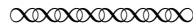
12, the third, is made up out of the two former numbers with 7 added to them, so that it may have 3 as a side, as three numbers have been added to make it. Similarly the preceding pentagon, 5, was the combination of two numbers and had 2 as its side. The graphic representation of 12 is this:



The other pentagonal numbers will be produced by adding together one after another in due order the terms after 7 that have the difference 3, as, for example, 10, 13, 16, 19, 22, 25, and so on. The pentagons will be 22, 35, 51, 70, 92, 117, and so forth.

## CHAPTER XII

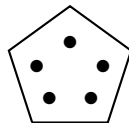
Concerning the nature of plane polygons this is sufficient for a first *Introduction*. That, however, the doctrine of these numbers is to the highest degree in accord with their geometrical representation, and not out of harmony with it, would be evident, [ . . . ] .



**Exercise 5.1.** Nicomachus represents the pentagonal numbers by arranging alphas ( $\alpha$ ) into equilateral pentagons. To help in the visual representation of these numbers, we use equilateral, equiangular pentagons, i.e., regular pentagons. We also use the convention that the first pentagonal number is represented by one dot (or one  $\alpha$ ):



Letting  $E_1$  denote the first pentagonal number<sup>1</sup>, we have  $E_1 = 1$ . The second pentagonal number,  $E_2$ , is formed by placing two dots parallel to each side of a regular pentagon:

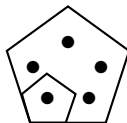



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<sup>1</sup>The symbol  $P$  is reserved for the pyramidal numbers in this project.

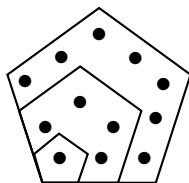
By counting the number of dots in the above figure, find the value of  $E_2$ , i.e.,  $E_2 = \square$ .

**Exercise 5.2.** To understand how the second pentagon is constructed from the first, recall that Nicomachus writes: “5 made up of 1 plus 4, is the second,” where “the second” refers to the second pentagon. Geometrically this can be represented as:



Write an equation relating  $E_2$  and  $E_1$  that reflects the above geometry.

**Exercise 5.3.** Nicomachus writes “12, the third [pentagon] is made up out of the former numbers with 7 added to them, so that it [the third pentagon] may have 3 as a side . . . .” Geometrically this can be represented as:



- (a) Compute the value of the third pentagonal number,  $E_3$ , by counting the total number of dots in the above figure, i.e.,  $E_3 = \square$ .
- (b) Express  $E_3$  as the sum three natural numbers, where two of these are “the former numbers” appearing in Exercise (5.2) and the third number is 7, i.e.,

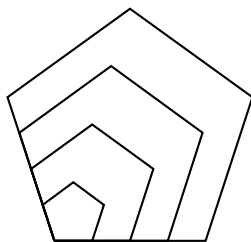
$$E_3 = \square + \square + \square.$$

Hint: Count the number of dots in the various ringed sections of the above figure.

- (c) Find an equation relating  $E_3$  and  $E_2$  based on the above figure.

**Exercise 5.4.** What numbering scheme for the pentagons is Nicomachus tacitly using when he refers to the second and third pentagons as quoted in Exercises (5.2) and (5.3)? Does this match the convention used in this project for numbering the pentagons?

**Exercise 5.5.** Compute the value of the fourth pentagonal number,  $E_4$ , by filling in the following figure with dots around the ringed pentagonal sections.



Be sure to show your work.

**Exercise 5.6.** Compute the value of the fifth pentagonal number,  $E_5$ . Be sure to explain your work and include any supporting diagrams.

**Exercise 5.7.** Fill in the following table giving the first seven pentagonal numbers.

$n$	1	2	3	4	5	6	7
$E_n$	1						

**Exercise 5.8. The First Difference.** Nicomachus produces the pentagonal numbers by adding together certain terms that can be interpreted as first differences.

- (a) Compute the first difference  $D_1 = E_2 - E_1$ .
- (b) Identify  $D_1$  geometrically in the figure drawn in Exercise (5.2).
- (c) Compute the difference  $D_2 = E_3 - E_2$ .
- (d) Identify  $D_2$  geometrically in the figure drawn in Exercise (5.3).
- (e) Compute the difference  $D_3 = E_4 - E_3$ .
- (f) Compute the difference  $D_4 = E_5 - E_4$ .
- (g) What terms are formed by  $D_1$ ,  $D_2$ ,  $D_3$  and  $D_4$ ? These terms are called the first differences of the pentagonal numbers.
- (h) Would you include  $E_1$  as the first term of the series  $D_1, D_2, D_3, D_4$ ? Why or why not?

**Exercise 5.9. The Second Difference.** After completing Exercise (5.8), consider the series

$$E_1, D_1, D_2, D_3, D_4, \dots$$

Do you see a pattern in these numbers? If not, let's compute the difference of these new terms, forming "a second difference."

- (a) Compute the second difference  $D_1 - E_1$ .
- (b) Compute the second difference  $D_2 - D_1$ .
- (c) Compute the second difference  $D_3 - D_2$ .
- (d) Compute the second difference  $D_4 - D_3$ .
- (e) State verbally a pattern in the second differences.
- (f) To what extent does this pattern in part (e) determine the pentagonal numbers? Compare this with Nicomachus' description: "... there are added together to produce the pentagonal numbers the terms beginning with 1 to any extent whatever that are two places apart, that is, those that have a difference of 3."

**Exercise 5.10. Extra.** With the help of the second difference found in Exercise (5.9), answer the following.

- (a) **The Recursive Formula for the Pentagonal Numbers.** Let  $E_n$  denote the  $n$ th pentagonal number, and  $E_{n+1}$  the next pentagonal number. Fill in the blank so that

$$E_{n+1} = E_n + \boxed{\phantom{000}}.$$

- (b) Find a geometric explanation for the recursive formula found in part (a).  
 (c) **The Iterative Formula for the Pentagonal Numbers.** Use part (a) to fill in the blank so that

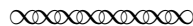
$$E_{n+1} = 1 + (1 + 3) + (1 + 3 \cdot 2) + (1 + 3 \cdot 3) + (1 + 3 \cdot 4) + \dots + \boxed{( \phantom{000} )}.$$

- (d) Show that for  $E_n$ , we have

$$E_n = n + 3(1 + 2 + 3 + \dots + n - 1).$$

- (e) Find a simple formula for  $E_n$  without using “...” anywhere in your equation. Hint: Find a formula for  $(1 + 2 + 3 + \dots + n - 1)$  by using the result at the end of Exercise (3.4). Check that your formula gives the correct values for  $E_1, E_2, E_3$  and  $E_4$ .

## 6 The Pyramidal Numbers



### CHAPTER XIII

From this it is easy to see what the solid number is and how its series advances with equal sides; for the number which, in addition to the two dimensions contemplated in graphic representation in a plane, length and breadth, has a third dimension, which some call depth, others thickness, and some height, that number would be a solid number, extended in three directions and having length, depth, and breath.

The first makes its appearance in the so-called pyramids. These are produced from rather wide bases narrowing to a sharp apex, first after the triangular form from a triangular base, [...]

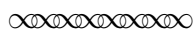
Exactly so among the geometrical solid figures; if one imagines three lines from the three angles of an equilateral triangle, equal in length to the sides of the triangle, converging in the dimension height to one and the same point, a pyramid would be produced, bounded by four triangles, equilateral and equal one to the other, one the original triangle, and the other three bounded by the aforesaid three lines. [...]

So likewise among numbers, each linear number increases from unity, as from a point, as for example, 1, 2, 3, 4, 5, and successive numbers to infinity; and from these numbers, which are linear and extended in one direction, combined in no random manner, the polygonal and plane

numbers are fashioned—the triangles by combination of root-numbers immediately adjacent, the square by adding every other term, the pentagons every third term, and so on. In exactly the same way, if the plane polygonal numbers are piled one upon the other and as it were built up, the pyramids that are akin to each of them are produced, the triangular pyramid from the triangles, [ . . . ] .

The pyramids with a triangular base, then, in their proper order, are these: 1, 4, 10, 20, 35, 56, 84, and so on; and their origin is the piling up of the triangular numbers one upon the other, first, 1, then 1, 3, then 1, 3, 6, then 10 in addition to these, and next 15 together with the foregoing, then 21 besides these, next 28, and so on to infinity.

It is clear that the greatest number is conceived of as being lowest, for it is discovered to be the base; the next succeeding one is on top of it, and the next on top of that; until unity appears at the apex and, so to speak, tapers off the completed pyramid into a point.

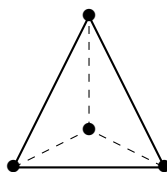


**Exercise 6.1.** Nicomachus describes the construction of a three-dimensional pyramid with a triangular base as: “[I]f one imagines three lines from the three angles of an equilateral triangle, equal in length to the sides of the triangle, converging in the dimension height to one and the same point, a *pyramid* would be produced, bounded by four triangles, equilateral and equal one to the other, one the original triangle, and the other three bounded by the aforesaid three lines.”

- (a) Sketch the “pyramid” that is produced.
- (b) Indicate in your sketch the “three lines converging in the dimension height to one and the same point.”
- (c) Indicate in your sketch the “four triangles, equilateral and equal one to the other.”
- (d) Indicate in your sketch the apex of this pyramid.

**Exercise 6.2.** Nicomachus begins his list of pyramidal numbers with the number 1. We follow the same convention and letting  $P_1$  denote the first pyramidal number, set  $P_1 = 1$ . He constructs the pyramids (with a triangular base) by “the piling up of the triangular numbers one upon the other, first 1, . . . .” Review the list of triangular numbers developed at the end of Exercise (3.1).

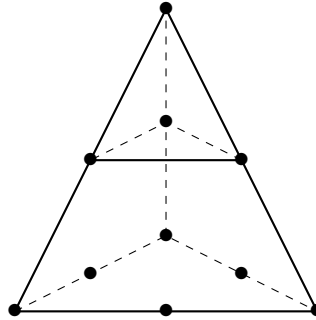
- (a) The second pyramid is constructed by piling “then 1, 3.” Nicomachus refers to the triangular numbers by their numerical values, without the use of separate symbols. Pictorially the second pyramid can be represented as:





Identify the first and second triangles (triangular numbers) in the above figure. Then compute the numerical value of  $P_2$ , the second pyramidal number, by counting the total number of dots in the figure. Thus,  $P_2 = \square$ .

- (b) The third pyramid is constructed by piling “then 1, 3, 6,” which are the first three triangular numbers. Pictorially the third pyramid can be represented as:



Identify the first, second and third triangles (triangular numbers) in the above figure. Compute the numerical value of  $P_3$ , the third pyramidal number, by counting the total number of dots in the figure. Thus,  $P_3 = \square$ .

- (c) Carefully sketch the fourth pyramid by stacking the first four triangles. Compute the numerical value of  $P_4$ , the fourth pyramidal number, by determining the number of dots in your figure. Explain the strategy that you devised to compute  $P_4$ .
- (d) Let  $P_5$  denote the fifth pyramidal number,  $P_6$  the sixth pyramidal number, etc. Fill in the following table for the numerical values of the first eight pyramidal numbers.

$n$	1	2	3	4	5	6	7	8
$P_n$	1							

**Exercise 6.3. The Iterative Formula for the Pyramidal Numbers.** Nicomachus writes that the pyramids (with a triangular base) are formed by “the piling up of the triangular numbers one upon another.” In this exercise we work with the symbol  $T_n$ , representing the  $n$ th triangular number.

- (a) Write  $P_2$ , the second pyramidal number, as the sum of two successive triangular numbers so that

$$P_2 = T_{\square} + T_{\square}.$$

- (b) Write  $P_3$ , the third pyramidal number, as the sum of three successive triangular numbers so that

$$P_3 = T_{\square} + T_{\square} + T_{\square}.$$

- (c) Write  $P_4$  as the sum of four successive triangular numbers so that

$$P_4 = T_{\square} + T_{\square} + T_{\square} + T_{\square}.$$

- (d) What ten successive triangular numbers would be needed to be added together to produce  $P_{10}$ , the 10th pyramidal number?
- (e) Let  $n$  denote a generic natural number. Find a formula for  $P_n$  that expresses the  $n$ th pyramidal number as the sum of  $n$  successive triangular numbers. What geometric idea does this formula represent? Today this formula is called the iterative formula for  $P_n$ .

**Exercise 6.4. The Recursive Formula for the Pyramidal Numbers.** We examine another formula for computing a pyramidal number that uses the value of the preceding pyramidal number.

- (a) Note that the second pyramid can be formed by placing the first pyramid (one dot) on the second triangle. Find a triangular number  $T_{\square}$  so that

$$P_2 = P_1 + T_{\square}.$$

- (b) Note that the third pyramid can be formed by placing the second pyramid on the third triangle. Find a triangular number  $T_{\square}$  so that

$$P_3 = P_2 + T_{\square}.$$

- (c) Note that the fourth pyramid can be formed by placing the third pyramid on the fourth triangle. Find a triangular number  $T_{\square}$  so that

$$P_4 = P_3 + T_{\square}.$$

- (d) Following the above format, find an equation relating  $P_5$  and  $P_4$ .
- (e) For an arbitrary natural number  $n$ , find a formula relating  $P_n$  to the preceding pyramidal number  $P_{n-1}$ . Express this formula verbally describing how the preceding pyramid can be used to construct the next pyramid. This formula is sometimes called the recursive formula for the pyramidal numbers.

**Exercise 6.5. The First Difference.** In this exercise we compute the first difference of the pyramidal numbers. Does knowing the values of these differences help in finding a concise formula for  $P_n$  depending only on  $n$ ?

- (a) Compute the first difference  $D_1 = P_2 - P_1$ .
- (b) Identify  $D_1$  geometrically in the figure in Exercise (6.2), (a).
- (c) Compute the difference  $D_2 = P_3 - P_2$ .

- (d) Identify  $D_2$  geometrically in the figure in Exercise (6.2), (b).
- (e) Compute the difference  $D_3 = P_4 - P_3$ .
- (f) Identify  $D_3$  geometrically in your sketch in Exercise (6.2), (c).
- (g) Compute the difference  $D_4 = P_5 - P_4$ .
- (h) What numbers are formed by  $D_1, D_2, D_3, D_4$ ?
- (i) Would you include  $P_1$  as the first term of the series  $D_1, D_2, D_3, D_4$ ? Why or why not?
- (j) Does knowing the first difference of the pyramidal numbers help in finding a formula for  $P_n$ ? Why or why not?
- (k) Since the pyramidal numbers are formed by adding successive triangular numbers, why would you anticipate that the first difference of the pyramidal numbers is a triangular number? Use this observation to find a formula for  $P_n - P_{(n-1)}$ .

## 7 Fermat's Formulas for the Triangular and Pyramidal Numbers

In letters to both Marin Mersenne (1588–1648) and Gilles Persone de Roberval (1602–1675), Pierre de Fermat (1601–1665) verbally states a pattern in the figurate numbers that cannot be anticipated from their first differences. Studying their quotients, however, will reveal the secret. To see why, consider what would happen if the square numbers were divided by their side lengths. Compute  $16/4$  or  $25/5$  for example. In a letter to Roberval dated November 4, 1636, Fermat [1, p. 83–87] phrases the pattern for the triangular numbers, in Latin, as:

○○○○○○○○○○

Ultimum latus in latus proxime majus facit duplum trianguli.

○○○○○○○○○○

We will decipher this in the following exercise. Fermat offers no verification of this pattern, but leaves its justification for the reader.

**Exercise 7.1.** (a) For review, fill in the following table for the triangular numbers,  $T_n$ .

$n$	1	2	3	4	5	6	7	8
$T_n$	1							

- (b) Compute the following quotients

$$\frac{T_1}{2} = \quad , \quad \frac{T_2}{3} = \quad , \quad \frac{T_3}{4} = \quad , \quad \frac{T_4}{5} =$$

$$\frac{T_5}{6} = \quad , \quad \frac{T_6}{7} = \quad , \quad \frac{T_7}{8} = \quad , \quad \frac{T_8}{9} = \quad .$$

(c) To simplify the fractions, compute

$$\frac{2T_1}{2} = \quad , \quad \frac{2T_2}{3} = \quad , \quad \frac{2T_3}{4} = \quad , \quad \frac{2T_4}{5} = \\ \frac{2T_5}{6} = \quad , \quad \frac{2T_6}{7} = \quad , \quad \frac{2T_7}{8} = \quad , \quad \frac{2T_8}{9} = \quad .$$

(d) Based on the above computations, guess a pattern for

$$\frac{2T_n}{n+1}.$$

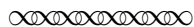
Why do you think that your guess is correct? Do you think that your guess is valid for all natural numbers  $n$ ? Why or why not?

(e) From part (d), find a formula for  $2T_n$  and state this formula in words.

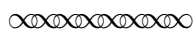
(f) Fermat's rule "Ultimum latus in latus proxime majus facit duplum trianguli" has been translated as "The last side multiplied by the next larger makes twice the triangle" [3, p. 230]. Does this translation agree with your formula for  $2T_n$ ? Why or why not?

(g) Finally, find a formula for just  $T_n$ .

In the above letter to Roberval, Fermat further states that he has found a rule for computing the pyramidal numbers, which, in his words, provides a very nice formula for the sum of the triangular numbers. Fermat writes [1, p. 85]:



Ultimum latus in triangulum lateris proxime majoris facit triplum pyramidis.



Let's explore the meaning of this in the following exercise, again using quotients to reach into the next (third) dimension.

**Exercise 7.2.** For a natural number  $n$ , recall that  $P_n$  denotes the value of the  $n$ th pyramidal number.

(a) For review, fill in the following table giving the pyramidal numbers. You may use the values of the triangular numbers already in the table.

$n$	1	2	3	4	5	6	7	8	9
$T_n$	1	3	6	10	15	21	28	36	45
$P_n$	1								

(b) Compute the following quotients

$$\frac{P_1}{T_2} = \quad , \quad \frac{P_2}{T_3} = \quad , \quad \frac{P_3}{T_4} = \quad , \quad \frac{P_4}{T_5} = \\ \frac{P_5}{T_6} = \quad , \quad \frac{P_6}{T_7} = \quad , \quad \frac{P_7}{T_8} = \quad , \quad \frac{P_8}{T_9} = \quad .$$

(c) To simplify the fractions, compute

$$\frac{3P_1}{T_2} = \quad , \quad \frac{3P_2}{T_3} = \quad , \quad \frac{3P_3}{T_4} = \quad , \quad \frac{3P_4}{T_5} = \\ \frac{3P_5}{T_6} = \quad , \quad \frac{3P_6}{T_7} = \quad , \quad \frac{3P_7}{T_8} = \quad , \quad \frac{3P_8}{T_9} = \quad .$$

(d) Based on the above computations, guess a pattern for

$$\frac{3P_n}{T_{n+1}}.$$

Why do you think that your guess is correct? Do you think that your guess is valid for all natural numbers  $n$ ? Why or why not?

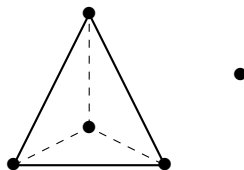
(e) Find a formula for  $3P_n$  and state verbally what this formula is.

(f) Fermat's rule "Ultimum latus in triangulum lateris proxime majoris facit triplum pyramidis" has been translated as "The last side multiplied by the triangle of the next larger makes three times the pyramid" [3, p. 230]. Does this represent the formula that you found for  $3P_n$ ? Why or why not?

(g) **Extra.** Find a formula for just  $P_n$  that depends only on  $n$ , without using the symbol  $T_{n+1}$ . Hint: Substitute a formula for  $T_{n+1}$  based on your knowledge of the triangular numbers.

## 8 The Triangulo-Triangular Numbers

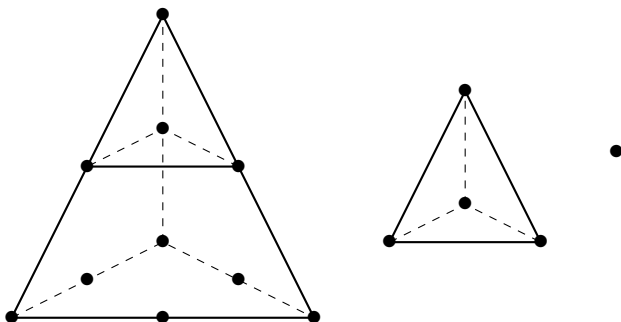
Recall that Nicomachus states [4] "a 'solid' is that which is extended in three dimensions, and it is by no means possible to conceive of a solid which has more than three dimensions, depth, breadth, and length," indicating that, for the ancient Greeks, there are no figurate numbers beyond the third dimension. Fermat, however, finds a formula for the summation of the pyramidal numbers, which he calls "triangulo-triangular" numbers, forming figures that could be described to exist in four dimensions. Each mention of the word "triangle" carries two dimensions, so that a "triangulo-triangular" would be an object having  $2 + 2 = 4$  dimensions. For this exposition, let  $Q_1$  denote the first triangulo-triangular number,  $Q_2$  the second,  $Q_3$  the third, etc. We set  $Q_1 = 1$  with  $Q_1$  representing one dot. Then  $Q_2$  is formed by juxtaposing the first and second pyramids as follows:



One could imagine that  $P_1$  and  $P_2$  are joined in the fourth dimension. Now,

$$Q_2 = P_1 + P_2 = 1 + 4 = 5.$$

The third triangulo-triangle is formed by juxtaposing the first three pyramids. One can imagine that  $P_1$ ,  $P_2$  and  $P_3$  are joined in the fourth dimension as well.



Thus,

$$Q_3 = P_1 + P_2 + P_3 = 1 + 4 + 10 = 15.$$

Also,

$$Q_3 = (P_1 + P_2) + P_3 = Q_2 + P_3 = 5 + 10 = 15.$$

As with the triangular and pyramidal numbers, there are two main methods for computing the triangulo-triangular numbers, one iterative and one recursive. Let  $n$  denote a natural number. Then the iterative formula for the  $n$ th triangulo-triangular number,  $Q_n$ , is an expression of  $Q_n$  as the sum of the first  $n$  pyramidal numbers. The recursive formula for  $Q_n$  is an expression of  $Q_n$  in terms of the previous triangulo-triangular number and the  $n$ th pyramidal number. We now develop these formulas.

**Exercise 8.1. The Iterative Formula for the Triangulo-Triangular Numbers.** For a natural number  $n$ , let  $Q_n$  denote the  $n$ th triangulo-triangular number.

- Draw the first four pyramids,  $P_1$ ,  $P_2$ ,  $P_3$ , and  $P_4$  next to each other. What is the total number of dots contained in all four of these pyramids?
- Write  $Q_4$  as the sum of four successive pyramidal numbers so that

$$Q_4 = P_{\square} + P_{\square} + P_{\square} + P_{\square}.$$

- Compute the numerical value of  $Q_4$ .
- Compute the numerical value of  $Q_5$  and explain the strategy that you devised to compute this number.
- What ten successive pyramidal numbers would be needed to be added together to produce  $Q_{10}$ , the 10th triangulo-triangular number?

- (f) Let  $n$  denote a generic natural number. Find a formula for  $Q_n$  that expresses the  $n$ th triangulo-triangular number as the sum of  $n$ -many successive pyramidal numbers. This is called the iterative formula for  $Q_n$ .

**Exercise 8.2. The Recursive Formula for the Triangulo-Triangular Numbers.** For a natural number  $n$ , let  $Q_n$  denote the  $n$ th triangulo-triangular number.

- (a) Note that the second triangulo-triangular number can be formed by placing the first triangulo-triangular number (one dot) next to the second pyramid. Find a pyramidal number  $P_{\square}$  so that

$$Q_2 = Q_1 + P_{\square}.$$

- (b) Note that the third triangulo-triangle can be formed by placing the second triangulo-triangle next to the third pyramid. Find a pyramidal number  $P_{\square}$  so that

$$Q_3 = Q_2 + P_{\square}.$$

- (c) Note that the fourth triangulo-triangle can be formed by placing the third triangulo-triangle next to the fourth pyramid. Find a pyramidal number  $P_{\square}$  so that

$$Q_4 = Q_3 + P_{\square}.$$

- (d) Following the above format, find an equation relating  $Q_5$  and  $Q_4$ .
- (e) For an arbitrary natural number  $n$ , find a formula relating  $Q_n$  to the preceding triangulo-triangular number  $Q_{n-1}$ . Express this formula verbally describing how the preceding triangulo-triangle can be used to construct the next triangulo-triangle. This formula is called the recursive formula for the triangulo-triangular numbers.

- (f) Use the recursive formula for  $Q_n$  to fill in the following table for the first seven triangulo-triangular numbers. You may use the values of the pyramidal numbers already in the table.

$n$	1	2	3	4	5	6	7
$P_n$	1	4	10	20	35	56	84
$Q_n$	1						

In the above letter to Roberval, Fermat identifies a pattern for the value of the triangulo-triangular numbers, which he states as [1, p. 85]:

∞∞∞∞∞∞∞∞∞∞

Ultimum latus in pyramidem lateris proxime majoris facit quadruplum triangulotrianguli.

∞∞∞∞∞∞∞∞∞∞

From our previous work with the figurate numbers, the reader may be able to decipher the meaning of this statement. If not, let's study the quotient of a triangulo-triangular number by the next pyramidal number.

**Exercise 8.3.** Use the notation of Exercise (8.2) and the values of  $Q_n$  given in the last part of that exercise.

(a) Compute the following quotients

$$\begin{array}{ccc} \frac{Q_1}{P_2} = & \frac{Q_2}{P_3} = & \frac{Q_3}{P_4} = \\ \frac{Q_4}{P_5} = & \frac{Q_5}{P_6} = & \frac{Q_6}{P_7} = \end{array}$$

(b) To simplify fractions, compute

$$\begin{array}{ccc} \frac{4Q_1}{P_2} = & \frac{4Q_2}{P_3} = & \frac{4Q_3}{P_4} = \\ \frac{4Q_4}{P_5} = & \frac{4Q_5}{P_6} = & \frac{4Q_6}{P_7} = \end{array}$$

(c) Based on the above computations, guess a pattern for

$$\frac{4Q_n}{P_{n+1}}.$$

Why do you think that your guess is correct? Do you think that your guess is valid for all natural numbers  $n$ ? Why or why not?

(d) Find a formula for  $4Q_n$  and state verbally what this formula is.

(e) Fermat's statement "Ultimum latus in pyramidem lateris proxime majoris facit quadruplum triangulotrianguli" has been translated as "The last side multiplied by the pyramid of the next greater makes four times the triangulo-triangle" [3, p. 230]. Does this represent the formula that you found for  $4Q_n$ ? Why or why not?

(f) **Extra.** Find a formula for just  $Q_n$  that depends only on  $n$ , without using the symbol  $P_{n+1}$ . Hint: Substitute a formula for  $P_{n+1}$  based on your knowledge of the pyramidal numbers.

## 9 The Five-Dimensional Figurate Numbers

After stating the pattern for the triangulo-triangular numbers, Fermat [1, p. 85] writes "Et eo in infinitum progressu," meaning "And so on by the same progression in infinitum" [3, p. 230]. What is the progression that Fermat has identified? If the figurate numbers are to continue into the next dimension and beyond, what are these higher dimensional numbers? In this final section we examine the five-dimensional figurate numbers by generalizing previous



constructions. Let  $R_1$  be the first of the five-dimensional numbers,  $R_2$  the second,  $R_3$  the third, etc. Set  $R_1 = 1$ , which represents one dot. Then  $R_2$  is formed by juxtaposing the first two triangulo-triangles, i.e.,

$$R_2 = Q_1 + Q_2 = 1 + 5 = 6.$$

Similarly,  $R_3$  is formed by juxtaposing the first three triangulo-triangles, i.e.,

$$R_3 = Q_1 + Q_2 + Q_3 = 1 + 5 + 15 = 21.$$

Also,

$$R_3 = (Q_1 + Q_2) + Q_3 = R_2 + Q_3 = 6 + 15 = 21.$$

As with the previous figurate numbers, there are two main methods for computing the five-dimensional numbers, one iterative and one recursive. Let  $n$  denote a natural number. Then the iterative formula for the  $n$ th five-dimensional number,  $R_n$ , is an expression of  $R_n$  as the sum of the first  $n$  triangulo-triangular numbers. The recursive formula for  $R_n$  is an expression of  $R_n$  in terms of the previous five-dimensional number and the  $n$ th triangulo-triangular number. We now develop these formulas.

**Exercise 9.1. The Iterative Formula for the Five-Dimensional Numbers.** For a natural number  $n$ , let  $R_n$  denote the  $n$ th five-dimensional number.

- (a) Write  $R_4$  as the sum of four successive triangulo-triangular numbers so that

$$R_4 = Q_{\square} + Q_{\square} + Q_{\square} + Q_{\square}.$$

- (b) Compute the numerical value of  $R_4$ .
- (c) Compute the numerical value of  $R_5$  and explain the strategy that you devised to compute this number.
- (d) What ten successive triangulo-triangular numbers would be needed to be added together to produce  $R_{10}$ ?
- (e) Let  $n$  denote a generic natural number. Find a formula for  $R_n$  that expresses the  $n$ th five-dimensional number as the sum of  $n$ -many successive triangulo-triangular numbers. This is called the iterative formula for  $R_n$ .

**Exercise 9.2. The Recursive Formula for the Five-Dimensional Numbers.** For a natural number  $n$ , let  $R_n$  denote the  $n$ th five-dimensional number.

- (a) Note that the second five-dimensional number can be formed by placing the first five-dimensional number (one dot) next to the second triangulo-triangle. Find a triangulo-triangular number  $Q_{\square}$  so that

$$R_2 = R_1 + Q_{\square}.$$

- (b) Note that the third five-dimensional number can be formed by placing the second five-dimensional number next to the third triangulo-triangle. Find a triangulo-triangular number  $Q_{\square}$  so that

$$R_3 = R_2 + Q_{\square}.$$

- (c) Note that the fourth five-dimensional number can be formed by placing the third five-dimensional number next to the fourth triangulo-triangle. Find a triangulo-triangular number  $Q_{\square}$  so that

$$R_4 = R_3 + Q_{\square}.$$

- (d) Following the above format, find an equation relating  $R_5$  and  $R_4$ .
- (e) For an arbitrary natural number  $n$ , find a formula relating  $R_n$  to the preceding five-dimensional number  $R_{n-1}$ . Express this formula verbally describing how the preceding five-dimensional number can be used to construct the next five-dimensional number. This formula is called the recursive formula for the five-dimensional figurate numbers.
- (f) Use the recursive formula for  $R_n$  to fill in the following table for the first seven five-dimensional numbers. You may use the values of the triangulo-triangular numbers already in the table.

$n$	1	2	3	4	5	6	7
$Q_n$	1	5	15	35	70	126	210
$R_n$	1						

**Exercise 9.3.** Use the notation of Exercise (9.2) and the values of  $R_n$  given in the last part of that exercise. Can you guess a formula for  $R_n$ ? If not, try the following.

- (a) Compute the following quotients

$$\begin{array}{ccc} \frac{R_1}{Q_2} = & \frac{R_2}{Q_3} = & \frac{R_3}{Q_4} = \\ \frac{R_4}{Q_5} = & \frac{R_5}{Q_6} = & \frac{R_6}{Q_7} = \end{array}$$

- (b) To simplify fractions, compute

$$\begin{array}{ccc} \frac{5R_1}{Q_2} = & \frac{5R_2}{Q_3} = & \frac{5R_3}{Q_4} = \\ \frac{5R_4}{Q_5} = & \frac{5R_5}{Q_6} = & \frac{5R_6}{Q_7} = \end{array}$$

- (c) Based on the above computations, guess a pattern for

$$\frac{5R_n}{Q_{n+1}}.$$

Why do you think that your guess is correct? Do you think that your guess is valid for all natural numbers  $n$ ? Why or why not?

- (d) Find a formula for  $5R_n$  and state verbally what this formula is.
- (e) **Extra.** Find a formula for just  $R_n$  that depends only on  $n$ , without using the symbol  $Q_{n+1}$ . Hint: Substitute a formula for  $Q_{n+1}$  based on your knowledge of the triangular numbers.
- (f) **Extra, Extra, Extra.** Let  $Z_{n,k}$  denote the  $n$ th figurate number in dimension  $k$ . Find a formula for  $Z_{n,k}$  that depends only on  $n$  and  $k$ . This formula would be an ultimate expression for Fermat's pattern that continues infinitely far.

## Notes to the Instructor

This project is written for an entry-level general education course in undergraduate mathematics for students with very little background in algebra. It could also be used in high school at the junior or senior level, particularly to interest students in mathematics and pattern recognition. The project explores certain combinatorial numbers that historically arose by counting the number of dots in regularly-shaped figures, such as triangles, squares, pyramids, etc. These numbers are known as figurate numbers and include the binomial coefficients, although this latter point of view is not explored in the project. There are sections on linear, triangular, square, pentagonal, pyramidal, triangulo-triangular and five-dimensional numbers. Covered in its entirety, this material would form about a four or five week module within a general education course. The primary sources for the project are Nicomachus' *Introduction to Arithmetic* [4, 5] and Fermat's letters to Mersenne and Roberval [1]. Since no algebra appears in these writings, they form suitable material for students with little background in algebra.

Familiarity with counting and the basic arithmetical operations of addition, subtraction, multiplication, and division are assumed knowledge for students. Also, students should have a working knowledge of subscript notation, such as  $T_1$ ,  $T_2$ ,  $T_3$  respectively for the first, second and third terms of a sequence or series. A generic counting (whole) number is denoted  $n$  and appears as a generic subscript in expressions like  $T_n$ . Summation notation is not used, although could be introduced if desired. The project stays close to the quoted original source material and explores the development and computation of the figurate numbers. More subtle algebraic formulas for these numbers appear in the exercises marked "Extra," which may count as extra credit at the discretion of the instructor. Instructors with additional time could develop applications of the figurate numbers to the binomial theorem or the combinatorics of choosing  $n$ -many objects from  $k$ ,  $k \geq n$ , topics that are not covered in this project. For a more algebraic treatment of this material as well as results with related summations, see David Pengelley's project "Figurate numbers and sums of numerical powers: Fermat, Pascal, Bernoulli" [6], written for an upper-divisional undergraduate course in discrete mathematics or combinatorics.

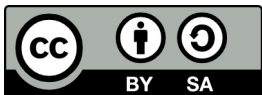
To pace coverage of the material, the instructor should have a good sense of the students' ability. The instructor should also study the original source excerpts before class and work exercises before assignment. Nicomachus writes loquaciously and the exercises for this source are designed to reiterate and explore his statements. The solutions to certain of these exercises can be found in his own words. In his correspondence, Pierre de Fermat, however, makes very concise and sweeping statements about the figurate numbers without justification. The exercises on this material are designed to lead the reader to discover the patterns stated by Fermat. In the project a few of Fermat's statements are written in the original Latin, with translations appearing in the exercises, after students have worked some of the discovery problems.

Not all sections of the project need be covered. For a rudimentary understanding of the figurate numbers, cover the sections on the linear, triangular and pyramidal numbers. To observe how first and second differences can be used to find patterns in the figurate numbers, cover the sections on the square and pentagonal numbers. For those wishing to

concentrate on the combination numbers (binomial coefficients) or those wishing to use the Pascal project for a general education course after this project, cover the linear, triangular, pyramidal, triangulo-triangular and five-dimensional numbers.

LaTeX code of this entire PSP is available from the author by request. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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For more information about TRIUMPHS, visit <http://webpages.ursinus.edu/nscoville/TRIUMPHS.html>.

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