Summer 2018

The Radius of Curvature According to Christiaan Huygens

Jerry Lodder
New Mexico State University, jlodder@nmsu.edu

The Radius of Curvature According to Christiaan Huygens

Jerry Lodder*

November 9, 2020

1 The Longitude Problem

An outstanding problem of navigation during the sixteenth, seventeenth and eighteenth centuries was determining longitude at sea. Countless lives were lost and ships were wrecked simply because the ship's captain did not know how far east or west the ship was with respect to an imaginary line (semi-circle) called the Prime Meridian, or with respect to any given meridian for that matter. This problem is chillingly described in *The Illustrated Longitude* [9], from which we relate one incident. In September, 1740, Captain George Anson (1697–1762) of the British Royal Navy set sail for the South Pacific aboard the *Centurion*, under orders to disrupt the Spanish trading monopoly in the Pacific. On March 7, 1741, the *Centurion* entered the Pacific Ocean from the Atlantic by passing through the Straits Le Maire at the southern end of South America. Then a violent storm blew in from the west, lasting some 58 days, unduly delaying the voyage and disorienting the *Centurion*. Many of the captain's men began to die of scurvy. Having rounded Cape Horn and now off the western coast of South America, the captain set sail for Juan Fernández Island for fresh food and water. Sailing north, Anson reached the proper latitude of the island (about 35°S) on May 24, 1741, but he could only guess whether the island lay east or west of his current position. The captain guessed west and sailed in that direction for four days only to find nothing. Changing course by 180°, he then sailed east for about two days only to sight the steep cliffs of Spanish controlled Chile. Anson turned around again, sailed west and reached Juan Fernández Island on June 9, 1741, with only about half of his men remaining [9, pp. 21–25]. What was the root cause of Captain Anson not being able to find the island? Anson knew his latitude, but he was unable to determine his longitude, namely how far east or west he was.

The determination of latitude can be reduced to a simple trigonometrical calculation given the length of a shadow of an object (of known height) at high noon, when the sun is at its highest elevation for the day.

*Mathematical Sciences; Dept. 3MB, Box 30001; New Mexico State University; Las Cruces, NM 88003; jlodder@nmsu.edu.

Exercise 1.1. Suppose that on the day of Spring Equinox (or Autumn Equinox), a 1' pole casts a shadow of 0.4' at high noon. Determine the latitude of the observer at this location (in the Northern Hemisphere). Be sure to explain your answer. (For days other than an equinox, the tilt of the Earth could be looked up in a table and would be used to adjust the above calculation.)

Hint: Since the sun is at such a great distance from the Earth, the rays of sun light are essentially parallel to each other when they reach the Earth. At high noon during an equinox, these rays are directly over head at the Equator. See Figure 1.

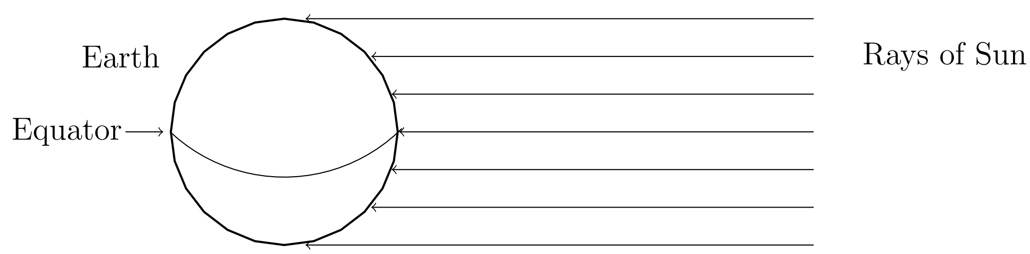


Figure 1: High Noon During an Equinox.

Consider now the 1' pole perpendicular to the Earth, and imagine a line connecting the pole to the center of the Earth, O. The length of the pole is exaggerated in Figure 2, where angle α denotes the latitude of the observer in the Northern Hemisphere. Now draw a separate right triangle with legs being the pole and its shadow, and the hypotenuse formed by a ray of sun light. Finish Exercise (1.1).

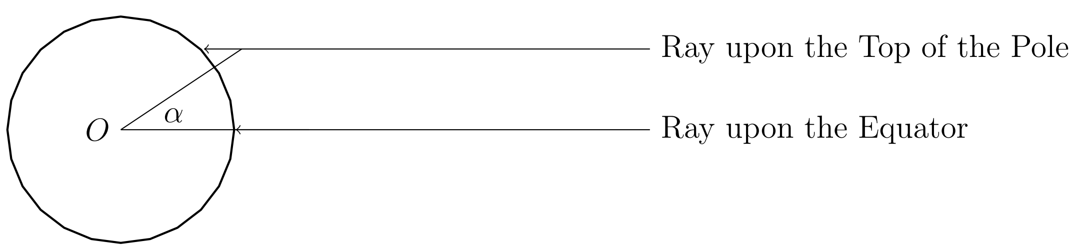


Figure 2: Determining Latitdue.

Could the determination of longitude be reduced to a simple calculation as in Exercise (1.1)? Certainly the sun (in its apparent motion) travels from east to west across the sky. Could the position of the sun be used to determine longitude? How? When the sun reaches its highest elevation on a given day, shadows are at their shortest, marking high noon. If a sailor knew when high noon occurred (or will occur) at the port of embarkment, then the difference between when noon occurs aboard ship and at the port of embarkment could be converted into a longitude reading, indicating how far west (or east) the ship is from its port.

Exercise 1.2. Suppose that a ship sets sail from England with a clock set to give time in Greenwich, England, through which the Prime Meridian passes. After sailing for several days on the open sea, suppose further that noon occurs on ship when the clock giving time in Greenwich reads 1:00 P.M.

(a) Is the ship sailing west or east of Greenwich? Justify your answer.

(b) How many degrees is the ship west of Greenwich? Explain your answer. Hint: There are 360° in a circle and the sun circles the Earth once every 24 hours (in its apparent motion).

The reader should now verify that one minute discrepancy between the time aboard ship and the time at the port of embarkment (or any given, fixed location) corresponds to 0.25° of longitude. Using $R = 3960$ miles as the radius of the Earth, then 0.25° of longitude at the Equator (a great circle on the Earth) corresponds to $(0.25)(\pi/180)R \approx 17$ miles, a non-trivial distance. However, a clock that is one minute off, even during a lengthy voyage, could result in a navigational error as large as 17 miles.

One solution to the longitude problem would be to construct a very accurate clock, set the clock to give time at Greenwich, England, and then send the clock to sea aboard ship. This project discusses Christiaan Huygens's (1629–1695) work on constructing a pendulum clock that theoretically keeps perfect time. Huygens described the path of the pendulum bob as one "whose curvature is marvelously and quite rationally suited to give the required equality to the pendulum" [3, p.11]. With these words, Huygens has identified a key concept, *curvature*, used to this day in physics and calculus to describe motion along a curved path. Alas, a pendulum clock behaved erratically on the high seas, and did not solve the longitude problem for naval navigation, although it remained the standard for terrestrial time keeping until the development of spring balance timepieces. So acute was the need to determine longitude that the British government had already issued the Longitude Act in 1714 that offered a first prize of £20,000 for a method to determine longitude to an accuracy of half a degree of a great circle [9, p.66]. There were also second and third prizes. Using the idea of a spring balance, John Harrison (1693–1776) perfected a sea clock that did solve the longitude problem. Due to prejudice against a clock method and in favor of a lunar method to solve the longitude problem, Harrison received marginal credit for his work during his lifetime, much of it spent perfecting his various timekeepers [9].

In the next section we turn to a detailed study of Huygens's work on the isochronous pendulum, a pendulum clock that theoretically keeps perfect time. To describe the path of a pendulum bob in terms of its curvature, Huygens first offers a geometric construction for what is known as the *radius of curvature* (see below). Since his work predates the development of calculus by Issac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716), Huygens relies on known techniques from Euclidean geometry, such as similar triangles, to capture the ratios of certain side lengths that play the role of certain derivatives. The geometric idea of a tangent line had been known well before Huygens, which he uses adroitly. The goal of this project is to follow Huygens's clever geometric arguments and then, through guided exercises, rewrite the expression for the radius of curvature in terms of derivatives, and finally to reconcile this with the modern equation for curvature found in many calculus textbooks. For further details on the construction of an isochronous pendulum, see Huygens's *Horologium oscillatorium* (The Pendulum Clock) [4, pp. 86–365] originally published in 1673, translated into English [3], and presented as part of a book chapter on curvature [6, pp. 174–178].

2 Huygens and The Radius of Curvature

Holland during the seventeenth century was a center of culture, art, trade and religious tolerance, nurturing the likes of Harmenszoon van Rijn Rembrandt (1606–1669), Johannes Vermeer (1632–1675), Benedict de Spinoza (1632–1677) and René Descartes (1596–1650). Moreover, the country was the premier center of book publishing in Europe during this time with printing presses in Amsterdam, Rotterdam, Leiden, The Hague and Utrecht, all publishing in various languages, classical and contemporary [2, p. 88]. Into this environment was born Christiaan Huygens, son of a prominent statesman and diplomat.

The young Huygens showed an interest in astronomy, developed improved methods of grinding and polishing lenses for telescopes, and made notable discoveries about the rings of Saturn and the length of the Martian day [8, p. 801]. During a visit to Paris in 1655, the Dutchman began to study probability, and authored the book *De Ratiociniis in Aleae Ludo* (On the Calculations in Games of Chance), published in 1657 [5, p. 456]. At the invitation of Jean-Baptiste Colbert (1619–1683), minister of King Louis XIV (1638–1715), Huygens moved to Paris in 1666 as a member of the newly established *Académie des Sciences*, where he resided for the next 15 years. Aside from his work on pendulum clocks, he formulated a principle for the conservation of energy for an elastic collision of two bodies, and correctly identified the centripetal force of an object moving in circular motion, closely related to the radius of curvature. Newton held the work of Huygens in high regard, and used the Dutch scholar's results in some of his own investigations [8, p. 802]. Unfortunately, growing religious intolerance for Protestants in Paris prompted Huygens to return to The Hague in 1681. Later in life, he launched a study of microscopy in loose connection with Anton van Leeuwenhoek (1632–1723), and developed highly original ideas in protozoology. In 1690 Huygens published his *Traité de la Lumière* (Treatise on Light), in which he proposed a wave theory of light. His final publication, *Cosmotheoros*, appeared posthumously, and contains a summary of what was known about the universe at the time. We turn now to the master's work on horology (the study of clocks and clock making).

In a burst of creativity during 1659 Christiaan Huygens developed a pendulum clock that theoretically keeps perfect time [3, 10]. In the years prior to his landmark discovery, Huygens had studied the simple pendulum, which consisted of a bob attached by a thread to a fixed point. The bob then oscillated in a circular arc. As a timekeeper, the simple pendulum is not entirely accurate, since the time required to complete one oscillation depends on the amplitude of the swing. The greater the swing, the more time is needed for an oscillation. Huygens's genius was to discover a curve for which the time of an oscillation is independent of the swing amplitude, an idea which at first glance seems a virtual impossibility.

Such a curve is described either as isochronous or as tautochronous, both terms referring to the "same-time" property at which the bob reaches its lowest point, regardless of the

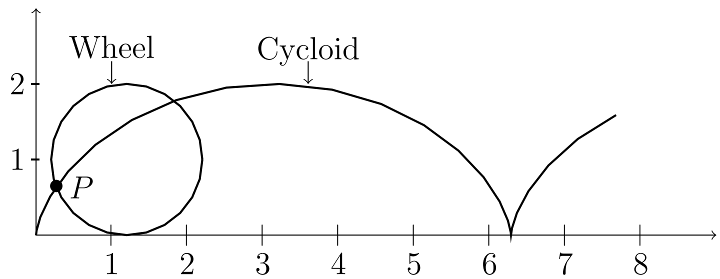


Figure 3: A Cycloid.

amplitude. Astonishingly Huygens showed that the shape of the isochrone is given by a curve that had been studied intensely and independently during the seventeenth century, namely a cycloid. Consider a point P on the circumference of a wheel and suppose that the wheel begins to roll along a flat surface. The curve traced by the point P is called a cycloid (Figure 3). For use in the pendulum, this curve could simply be turned upside down (inverted), which would then serve as the path of the bob. The cycloid had already occupied the minds of influential scholars such as Galileo, Torricelli, Mersenne, Roberval, Fermat, Descartes, Pascal, and others [1], yet none of them discovered its isochronous property.

Of course, once the shape of the isochrone had been determined, the problem of forcing a pendulum bob to oscillate along such a curve remained. This the Dutch scholar solved by placing two curved metal or wooden plates at the fulcrum of the pendulum (Figure 4, II). As the bob swings upward, the thread winds along the plates, forcing the bob away from the path of a perfect circle, and as the bob swings downward, the thread unwinds. This leads then to another problem in what today would be called mathematical physics—what should be the shape of the metal plates? Huygens called the curve for the plates an evolute of the cycloid or *evolutus* (unrolled) in the original Latin, and went on to discuss the mathematical theory of evolutes for general curves, not just cycloids. The key idea for the construction of the evolute is this. Suppose (Figure 5) that the thread leaves the plate at point A, the bob is at B, and segment AB is taut. Although B is no longer traversing a circle, the bob is instantaneously being forced around a circle whose center is A and radius is AB. To find A and AB, simply determine the circle which best matches the (lower) cycloid at point B. The length of AB became known as the radius of curvature of the cycloid at point B, while A became known as the center of curvature. Surprisingly, the evolute of a cycloid is another congruent cycloid, shifted in position.

Huygens made ready avail of the geometric idea of a tangent line, which was part of the mathematical culture at the time. Specifically Huygens wishes to study the curvature of arc ABF (see Figure 6) at the point B. To do so, he considers the circle, say $\mathcal{C}_1$, that best matches the curve at the point B. Suppose this circle has center D and radius BD. Huygens then considers another point F very close to B. Let $\mathcal{C}_2$ be the circle that best matches the curve at point F with its own radius of curvature given by FE. Extend lines BD and FE to meet at point G, assuming that arc ABF is concave down. Then BG plays the role of the radius of curvature at either points B or F, and as F approaches B, BG

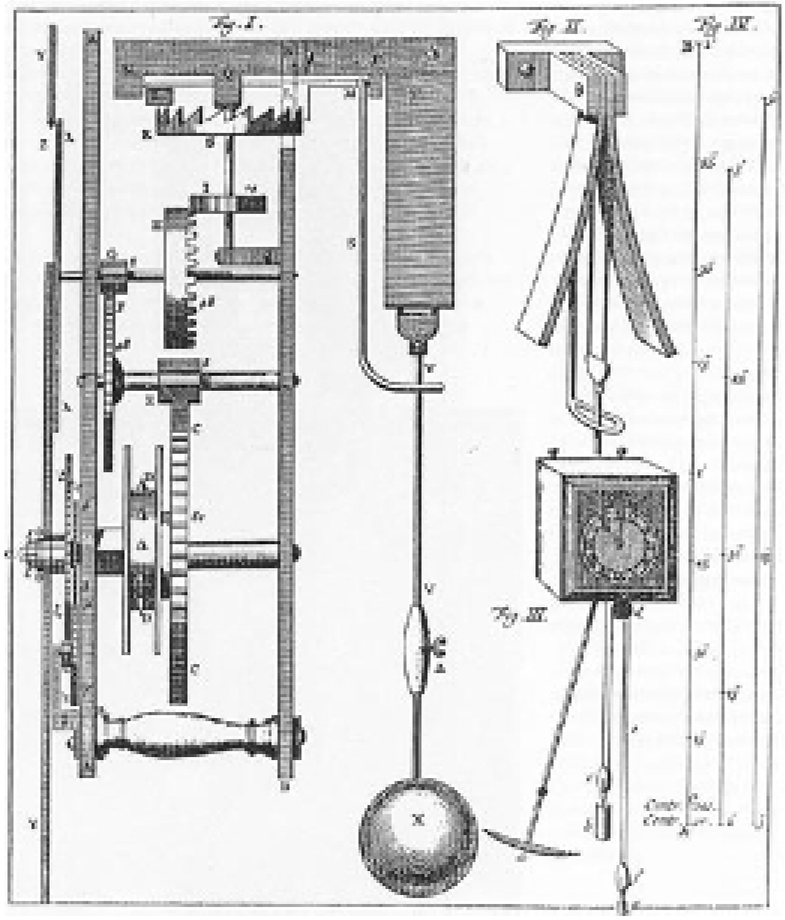


Figure 4: Huygens's Pendulum.

approaches the exact value of the radius of curvature at B (or F). Since the tangent to a circle is perpendicular to its radius, and the circle $\mathcal{C}_1$ approximates the curve very well at the point B, the tangent to the curve, BH, is perpendicular to BG, ($BH \perp BG$), a fact that Huygens uses liberally.

Exercise 2.1. Given a circle of radius 1 and a circle of radius 10, which would you describe as more sharply curved? Today, a circle of radius r is said to have a value of curvature given by $k = 1/r$ at all of its points. What is the curvature of the circle $\mathcal{C}_1$?

Huygens wishes to find an expression for BG, except he does so entirely in terms of geometric quantities. Let's read and verify a few of Huygens's original statements as translated from his 1673 treatise *Horologium oscillatorium* (The Pendulum Clock) [3, 4]. In proposition XI from this treatise Huygens demonstrates how an evolute of a given curve (curve ABF in Figure 6) can be constructed from its radii of curvature.

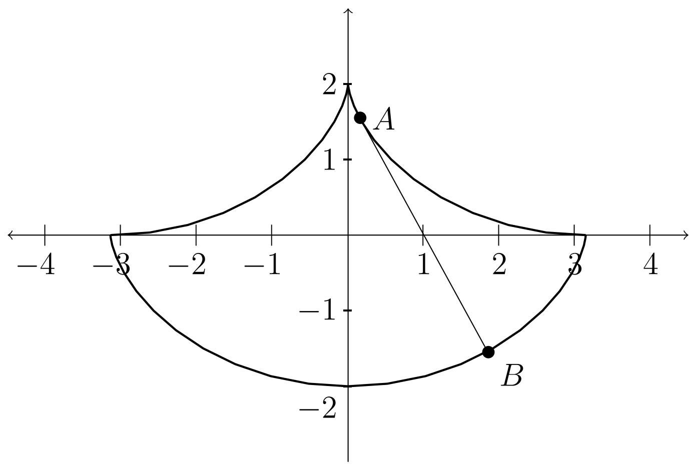


Figure 5: The Isochronous Pendulum.

Huygens, from
The Pendulum Clock

PROPOSITION XI

Given a curved line, find another curve whose evolution describes it. ...

Let ABF (see Figure 6) be any curved line, or part thereof, which is curved in one direction. And let KL be a straight line to which all points are referred. We are required to find another curve, for example DE, whose evolution will describe ABF. ...

Next select the points B and F, which are close to each other. ... BD and FE must intersect since they are perpendicular to the curve BF on its concave side. ...

Exercise 2.2. Why are BD and FE perpendicular to the curve BF?

And if the interval BF is taken to be infinitely small, these three points [D, G, E] can be treated as one. As a result the line BH, after having been drawn, is tangent to the curve at B and also can be thought of as tangent at F. Let BO be parallel to KL, and let BK and FL be perpendiculars to KL. FL cuts the line BO at P, and let M and N be the points where the lines BD and FE meet KL. Since the ratio of BG to GM is the same as that of BO to MN, then when the latter is given, so is the former.

Exercise 2.3. Verify that $BG/GM = BO/MN$. Hint: Use similar triangles.

And when the line BM is given in length and in position, so is the point G on the extension of BM, and also D on the curve CDE, since we have taken G and D to be one. ...

Now since the ratio of BO to MN is composed of the ratio of BO to BP ... and the ratio of BP to MN ....

Exercise 2.4. Write an equation for the above statement about ratios. How is Huygens using the word "and" (translated from the Latin)? What would be the modern phrasing of the above statement? Be sure that you have the correct answer to this before proceeding.

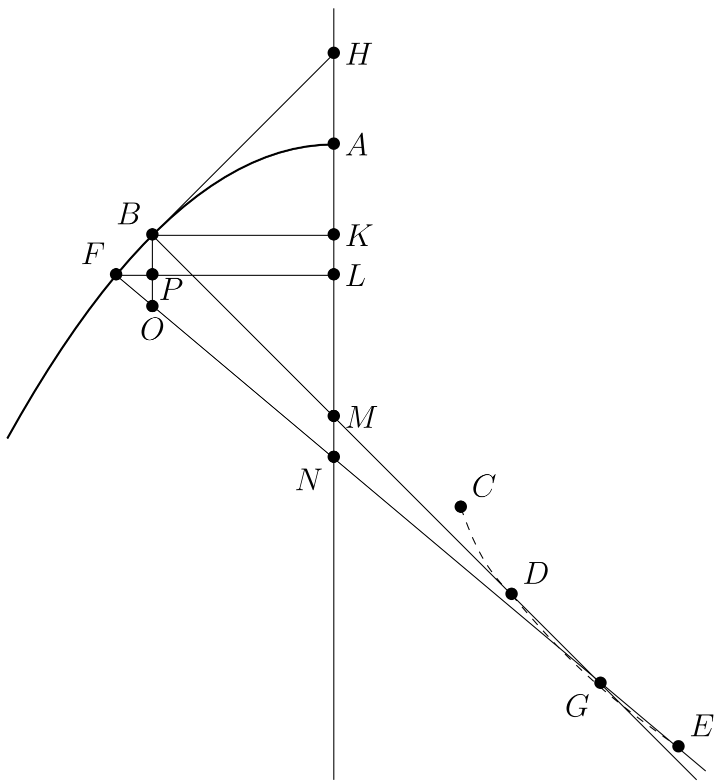


Figure 6: The Radius of Curvature.

... the ratio of BO to BP or of NH to LH ...

Exercise 2.5. Verify that $BO/BP = NH/LH$. Carefully justify your answer and identify all similar triangles used. Consider using two pairs of similar triangles for a total of four triangles.

... the ratio of BP or KL to MN ...

Exercise 2.6. Verify that $BP/MN = KL/MN$. Briefly justify your answer.

Exercise 2.7. Finally, verify that $BO/MN = (NH/LH)(KL/MN)$, and state why

$$\frac{BG}{GM} = \frac{HN}{HL}\frac{KL}{MN}\,.$$

Huygens ends his derivation of BG with what amounts to a verbal description of the above equality about ratios. In a specific example knowledge of the point M and the ratios HN/HL, KL/MN would determine the position of G and the distance BG, which is the radius of curvature.

3 The Modern Equation for Curvature

We wish to compare the modern equation for curvature with Huygens's expression of ratios. Since his work occurs just before the dawn of calculus, infinitesimals will be used in the sequel. Notice how the vertical line KL in Figure 6 serves as a reference line for forming the ratios needed for BG. Today, such a vertical line would be called the y-axis. Treating this as the axis for the independent variable, we could write curve ABF as $x = f(y)$, i.e., x is a function of y. Suppose that $B(x_1, y_1)$ and $F(x_2, y_2)$ are two points on the graph of $f(y)$ which are infinitesimally close. With this interpretation, increasing values of y point downward and increasing values of x point to the left, so that point B is reached before point F, when traversing arc ABF, starting at A and moving toward F. Let

$$dy = y_2 - y_1, \quad dx = x_2 - x_1.$$

Then the derivative of $f(y)$ at (x_1, y_1) or at (x_2, y_2) (with respect to y) is given by $\frac{dx}{dy}$. Moreover, the length of the line segment joining (x_1, y_1) and (x_2, y_2) is

$$ds = \sqrt{(dx)^2 + (dy)^2},$$

and this segment may be considered tangent to $x = f(y)$ at either of the two points. Line FL can be considered as the x-axis and point L taken as the origin, although any horizontal line could be considered as the x-axis.

Exercise 3.1. From the equation

$$\frac{HN}{HL} = \frac{HN}{FH}\,\frac{FH}{HL},$$

conclude that

$$\frac{HN}{HL} = \left(\frac{ds}{dy}\right)^2$$

by arguing that

$$\frac{FH}{HL} = \frac{ds}{dy} \quad \text{and} \quad \frac{HN}{FH} = \frac{ds}{dy}.$$

Hint: Try similar triangles.

Exercise 3.2. Next show that

$$\frac{MN}{KL} = 1 + \left(\frac{LN - KM}{KL}\right).$$

Exercise 3.3. Use geometry to conclude that $LN = x_2 \frac{dx}{dy}$. Explain conceptually why

$$\frac{LN - KM}{KL} = \frac{d}{dy}\left(x\,\frac{dx}{dy}\right).$$

Hint: Interpret $LN - KM$ as the first difference of a certain function and interpret KL as a small change along the y-axis, i.e., dy. Here x denotes the first coordinate of a generic point (x, y) on the curve ABF, i.e., on the graph $x = f(y)$.

Exercise 3.4. Use the product rule to compute

$$\frac{d}{dy}\left(x\,\frac{dx}{dy}\right).$$

Exercise 3.5. Using the geometry of $\triangle BKM$ and $\triangle BPF$, show that $BM = x\frac{ds}{dy}$ and substitute this into the equation

$$MG = BG - BM\,.$$

Exercise 3.6. From Exercise (2.7) compute BG in terms of infinitesimals. Note that $\dfrac{d^2x}{dy^2}$ may be treated as $\dfrac{d(dx)}{(dy)(dy)}$, where $d(dx)$ is the second difference of the quantity x.

Exercise 3.7. Find an expression for $k = 1/BG$, the curvature of curve ABF at the point B in terms of infinitesimals, and compare this with the derivative equation for the curvature of a plane curve given in a calculus text. Why might you now wish to switch dx and dy? Does a minus sign occur in front of your equation for $1/BG$? Do you know why?

Exercise 3.8. Graph $y = 4 - x^2$ in the xy-plane (x is the independent variable, as usual). Using the derivative equation for curvature, compute the curvature of this function at the values:

(a) $x = -1$,

(b) $x = 0$,

(c) $x = 1$.

Extra Credit A. Build an actual isochronous pendulum with a bob attached to a thread that is constrained by a cycloid. The fulcrum of the pendulum should be at the cusp of the cycloid. One method to do this would be to print a graph of a cycloid on paper, transfer the paper to a cork board and use push pins to outline the cycloid. Another method would be to use a 3-D printer. Build a second (identical) isochronous pendulum. Release the two pendulum bobs at different points and see whether they reach the vertical position at the same time.

Extra Credit B. Report on the brachistochrone problem, to which the cycloid is also a solution. Given two points A and B in the xy-plane, not on the same vertical line (but A with a larger y-component than B) construct a physical brachistochrone from A to B along which a marble could be rolled. Physically demonstrate that a marble released from point A reaches B along the brachistochrone sooner than it would if rolled along a flat line segment from A to B.

Notes to the Instructor

This Primary Source Project is written for a calculus course covering the curvature of plane curves, usually taught in conjunction with understanding motion (velocity and acceleration) of vector-valued functions. Many calculus textbooks define curvature as the magnitude of the rate of change of the unit tangent vector with respect to arc length, although such an opaque definition offers little insight into what curvature was designed to capture [7], not to mention its rich historical origins. The project offers Christiaan Huygens's (1629–1695) geometric construction of the radius of curvature and discusses its use in Huygens's design of an isochronous pendulum clock. A perfect timekeeper, if one could be constructed to operate at sea, would solve the longitude problem for naval navigation during The Age of Exploration [9]. John Harrison (1693–1776) did construct a sea clock that solved the longitude problem, although he used springs instead of a pendulum. He received only marginal credit for his work during his lifetime [9].

The project is divided into three sections. The first is an introduction describing the longitude problem and a potential solution with an accurate clock. The first two exercises, (1.1) and (1.2) are elementary and could be assigned as warm-up homework for in-class discussion. Depending on the students' background, the instructor should probably review the meaning of latitude, longitude, the Prime Meridian, a great circle, and arc length along a great circle of the Earth. The second section contains excerpts from Huygens's original work *Horologium oscillatorium* (The Pendulum Clock) [3, 4]. Understanding Huygens's geometric construction requires a careful study of Figure 6, particularly curve ABF and points H, K, L, M, and N, all lying on the same vertical line. The exercises involve verifying Huygens's statements about the various segments in this figure, which comprise the radius of curvature. Huygens relies on similar triangles, substitution and the geometric idea of a tangent line. The solution to some exercises simply involves writing Huygens's verbal description of ratios as algebraic equations, such as Exercise (2.4), where $BO/MN = (BO/BP)(BP/MN)$. Other exercises require the identification of several pairs of similar triangles, particularly Exercise (2.5). For ease of reference, Figure 6 is reproduced on a separate page following these notes, which could be photocopied and distributed to the class.

Allow about one week to cover sections one and two. From Huygens's final description of the radius of curvature, segment BG in Figure 6, a numerical calculation for its value, in a specific example, could be covered in class. For the curve $y = 4 - x^2$, given point B as $(-1, 3)$ and F as $(-1.1, 2.79)$, segment lengths HN, HL, KL, MN and BM could be computed numerically by placing the y-axis along line HL. This appears as an exercise in *Mathematical Masterpieces* [6, p. 176], reproduced below for the reader's convenience.

Exercise 3.9. [**Optional**] In this exercise the radius of curvature of $y = 4 - x^2$ is estimated at the point $B(-1, 3)$ using the geometric ideas of Huygens. In modern terminology, let the y-axis be placed along the line HL (Figure 6), and suppose that the x-axis is parallel to the line FL so that B has coordinates $(-1, 3)$. What is the y-coordinate of the point K? Using a modern equation for the slope of the tangent line to $y = 4 - x^2$ at B, find the y-coordinate of H. Find the y-coordinate of M from the equation of the line perpendicular to $y = 4 - x^2$ at B. Let $F(-1.1, 2.79)$ be another point on the parabola $y = 4 - x^2$, close to B. From F, determine the y-coordinates of L and N. From the equations

$$BG = BM + MG\,, \qquad \frac{BG}{MG} = \frac{HN}{HL}\,\frac{KL}{MN}$$

estimate BG, the radius of curvature of the parabola at B. Repeat the construction using the same point B, now considering F as $(-1.01,\, 2.9799)$. $\square$

Section three contains a construction for the modern equation of curvature by assigning infinitesimals (dx, dy, ds) to Huygens's work. This culminates in Exercise (3.7), which requires the use of nearly all the previous exercises for this section. Allow one week also to cover this section in its entirety. Once the derivative formula for curvature has been completed, this could be used to compute the curvature of $y = 4 - x^2$ at $x = -1$ and compared to the geometric (numerical) example above. For further details about curvature and in particular Newton's derivation for the radius of curvature in terms of his fluxion notation (not given in this project), see the text *Mathematical Masterpieces* [6]. In fact, the chapter "Curvature and the Notion of Space" from this text [6] could serve as a semester-long undergraduate course in differential geometry or the history of mathematics.

The following is a sample timeline for covering the project over a two-week implementation, three class sessions per week.

Day 1. As advance reading, ask students to study pp. 1–3 of the project before class, and prepare questions. During class the instructor may wish to review the meaning of latitude and longitude, and explain the longitude problem for navigation at sea (during the sixteenth, seventeenth, and eighteenth centuries). Field questions about determining latitude and longitude. Assign Exercises (1.1), (1.2).

Day 2. As advance reading, ask students to study pp. 4–7 of the project before class, and prepare questions. During class, the instructor may wish to review the construction of a cycloid, and develop a parametric equation for a cycloid, particularly if such an equation is present in the calculus text being used for the course. As an enrichment activity, the instructor could use graphic software to plot the parametric equation of a cycloid and project this on screen. Field questions about the reading. Review similar triangles, and the equality of ratios of corresponding sides in similar triangles. Discuss Exercises (2.1), (2.2) in class, and perhaps assign Exerciese (2.3), (2.4).

Day 3. As advance reading, ask students to study p. 8 of the project before class, and prepare questions. This is the heart of Huygens's geometric construction for capturing the radius of curvature as a limiting process via certain ratios. As in-class activities, group work, group presentations, or assigned homework, cover Exercises (2.5) through (2.7). After Exercise (2.7) is complete, the instructor may wish to demonstrate how Huygens's work determines the radius of curvature by covering optional Exercise (3.9) above. Time permitting, an extra day could be devoted to a discussion of Exercise (3.9).

Day 4. As advance reading, ask students to study p. 9, particularly Exercises (3.1), (3.2) before class, and prepare questions. Begin a discussion of differentials, and how a ratio of sides in a triangle can be used to capture the quotient of two differentials. Field questions about the Exercises (3.1), (3.2), and assign these as homework. If time permits, begin a discuss of Exercise (3.3), which requires a conceptual understanding of what a derivative is as well as an identification of what function is being differentiated.

Day 5. As advance reading, ask students to study Exercises (3.3) through (3.5) before class, and prepare questions about these exercises. Also, ask students what type of information segment MN may contain in terms of derivatives. Discuss the conceptual underpinnings of Exercise (3.3) during class. Notice that segment MN measures how quickly the radial lines BG and FG are approaching each other. Since a radial line is perpendicular to the tangent line, a radial line contains first derivative information about the original function (or graph). How two radial lines differ would then contain second derivative information about the function. Exactly what expression in terms of second derivatives MN/KL represents follows from Exercises (3.2) through (3.4). Cover Exercises (3.3) through (3.5) as in-class activities, group activities, group presentations, or assigned homework.

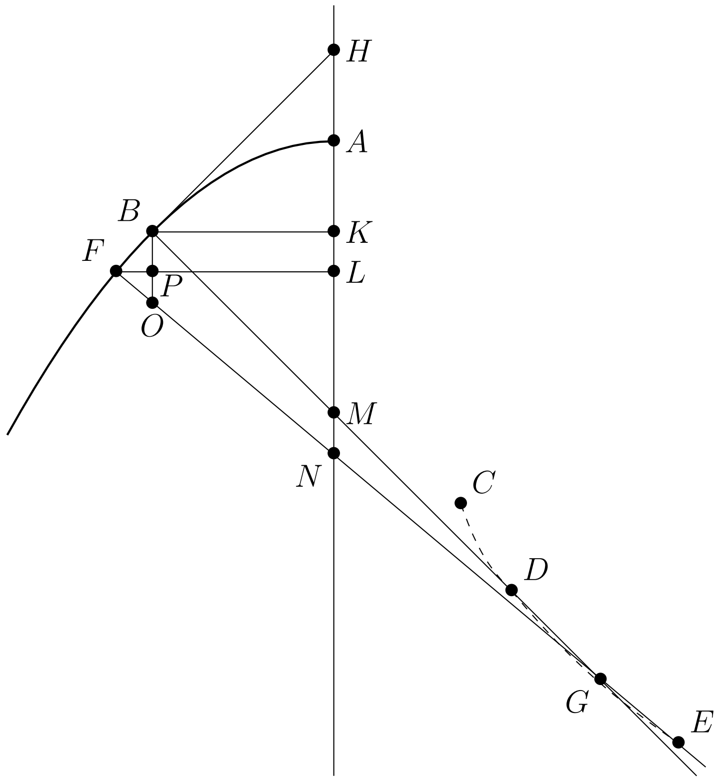
Day 6. As advance reading, ask students to study Exercises (3.6), (3.7), and prepare questions about these exercises before class. Begin a discussion about the needed steps to complete Exercise (3.6), which is rather lengthy. Note that the final derivative formula for BG, the radius of curvature, is preceded by a negative sign, since BG represents a length, and curve ABF is concave down with $\overrightarrow{HL}$ being the axis of the independent variable and $\overrightarrow{LF}$ the axis of the dependent variable. An engaging in-class activity is to ask students to determine the concavity of curve ABF with respect to the axes of the independent and dependent variables. As a final step for Exercise (3.7), the roles of x and y may be switched to reconcile Huygens's construction with the modern convention of letting x denote the independent variable when differentiating functions $y = f(x)$ today. Assign Exercises (3.6), (3.7). If time permits, begin Exercise (3.8), and compare this to Exercise (3.9) above, or discuss a comparison of these exercises during the next class session.

The above syllabus for the project is ambitious, and exercises can be assigned at the instructor's discretion.

The LaTeX code of this entire Primary Source Project (PSP) is available from the author by request (`jlodder@nmsu.edu`). The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

The development of this project has been partially supported by the National Science Foundation's Improving Undergraduate STEM Education Program under Grant Number DUE-1523747. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily reflect the views of the National Science Foundation.

For more information about TRIUMPHS, visit
https://blogs.ursinus.edu/triumphs/



The Radius of Curvature.

References

[1] Boyer, C.B., Merzbach, U.C., *A History of Mathematics*, second edition, John Wiley & Sons, New York, 1989.

[2] Hazard, P., *The European Mind, 1680–1715*, May, J.L., translator, World Publishing Co., Cleveland, Ohio, 1963.

[3] Huygens, C., *The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks*, Blackwell, R.J., translator, The Iowa State University Press, Ames Iowa, 1986.

[4] Huygens, C., *Oeuvres complètes de Christiaan Huygens*, tome 18, Société Hollandaise des Sciences, Amsterdam, 1967.

[5] Katz, V.J., *A History of Mathematics: An Introduction*, second edition, Addison-Wesley, New York, 1998.

[6] Knoebel, A., Laubenbacher, R., Lodder, J., Pengelley, D. *Mathematical Masterpieces: Further Chronicles by the Explorers*, Springer Verlag, New York, 2007.

[7] Lodder, J., "Curvature in the Calculus Curriculum," *The American Mathematical Monthly*, **110** (2003), 593–605.

[8] Simmons, G.F., *Calculus with Analytic Geometry*, McGraw Hill, Inc., New York, 1985.

[9] Sobel, D., Andrewes, J.H., *The Illustrated Longitude: The True Story of a Lone Genius Who Solved the Greatest Scientific Problem of His Time*, Walker and Company, New York, 1995.

[10] Yoder, J.G., *Unrolling Time: Christiaan Huygens and the Mathematization of Nature*, Cambridge University Press, Cambridge, 1988.