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Completing the Square: From the Beginnings of Algebra

Danny Otero
Xavier University, otero@xavier.edu

Solving Equations and Completing the Square: From the Roots of Algebra

Daniel E. Otero*

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Mathematics students learning how to solve algebraic equations are all too often – and unfortunately so – given the idea that a certain amount of mystery is a necessary and natural component of these methods. One such mystery arises in the general procedure for solving quadratic equations, equations of the form

$$ax^2 + bx + c = 0, \tag{$*$}$$

a procedure known as the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Where does this symbolic "talisman" come from? How does it work?

A fair amount of attention in treatments of beginning algebra is typically given to resolving such equations first through *factoring*, the sometimes difficult process of reversing the multiplication of linear expressions in the unknown. For instance, solving the equation $x^2 + 4x = 21$ would lead to the following steps, in which the factoring takes place in moving from the second equation below into the third one:

$$\begin{gathered} x^2 + 4x = 21 \\ x^2 + 4x - 21 = 0 \\ (x-3)(x+7) = 0 \\ x - 3 = 0 \quad \text{or} \quad x + 7 = 0 \\ x = 3, -7 \end{gathered}$$

If the given quadratic expression is even just slightly more complicated, with a leading coefficient different from one, then the factoring problem often becomes significantly more difficult. For instance,

*Xavier University, Cincinnati, OH, 45207-4441; `otero@xavier.edu`.

consider the use of factoring to solve the equation $2x^2 - 19x + 24 = 0$:

$$
\begin{aligned}
2x^2 - 19x + 24 &= 0 \\
(2x-3)(x-8) &= 0 \\
2x - 3 = 0 \quad \text{or} \quad x - 8 &= 0 \\
x &= \frac{3}{2}, 8
\end{aligned}
$$

The step that carries us from the first equation to the second one here is admittedly not straightforward: one must pair up a factorization of the leading coefficient 2 –which is not difficult, as the prime number 2 has only two factorizations into integers, as the product of either 1 and 2, or of -1 and -2 –with a factorization of the constant term 24 –which is more complicated, as there are many more possible factorizations of 24 as a product of two integers to consider. The two integer factors of 2 will become the leading coefficients of the two algebraic linear factors of $2x^2 - 19x + 24$ (i.e., the expressions $2x - 3$ and $x - 8$), while the two integer factors of 24 must be chosen to be the constant terms of these two algebraic factors. However, these two pairs of integer factors must simultaneously be chosen in such a way that the linear coefficient -19 of the quadratic must appear in the product of the two linear factors. Holding together the necessary arithmetic in one's brain associated with the possible combinations of factors that is required to achieve the desired condition is not easy, and typically requires lots of practice to accomplish successfully.

It is at this stage that problem solvers tend to "cry uncle" and plead for an easier out. Salvation arrives in the form of the aforesaid quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

whose complexity is counterbalanced by the assurance that (1) it is universally applicable for solving all quadratic equations, and (2) it bypasses the entire factoring process, producing immediately the solutions to the original equation $(*)$. But the price one pays to gain these valuable characteristics is manifested in the inscrutability of this formula. How is it that it produces such reliable results? And where does it come from?

What is more, in some cases, the formula will give *complex numbers* as solutions, numbers which don't correspond to familiar values on the (real) number line. This situation can only arise when the problem is either a purely abstract mathematical problem, or when the underlying physical situation that is being modeled by the equation has no measurable solution. Still, can we know in advance when such a circumstance will arise? And what might it mean for the problem to have a complex number as a solution?

There is yet another standard method for solving quadratic equations, called *completing the square*, which enjoys its own level of complexity. Here, the solver is trained to first simplify the equation $(*)$ by dividing through by its leading coefficient a, then to move the resulting constant term to the other side of the equation. Next comes the key step, in which the solver performs a very specific but unmotivated calculation: take half of the coefficient of x, square it, and add this to both sides of the equation. This "incantation" serves to recast the equation into a form that makes it easy to factor and then straightforward to solve completely. Indeed, by carrying out this procedure in its full generality, the quadratic formula itself emerges. That is, by understanding the method of completing the square, the quadratic formula is explained.

This project has been designed to lift the veil on these mysterious procedures. By studying what was written by a man who lived over a thousand years ago and is generally regarded as the "Father of Algebra," the reader will learn how and why the procedures described above work.

1 Algebra from *al-jabr*

Muḥammad ibn Mūsā al-Khwārizmī[1] (ca. 780-850 CE) is one of the great legendary figures in the history of mathematics. What we know of his life is this: he was born into a Persian family near the end of the eighth century in Khwarazm[2], a kingdom in central Asia set within rich agricultural lands along the Amu Darya[3] rivershed as this river runs northwestward from the Pamir[4] Mountains (in what today is the border regions between Afghanistan and Tajikistan) through the Asian countryside and into the Aral Sea.[5]

Just before the time of al-Khwārizmī, this kingdom of Khwarazm, which had enjoyed centuries of stability under Persian Afrighid[6] kings, fell under the rule of the Umayyad[7] caliphate, the second of the major Islamic dynasties to emerge after the rise of Islam from its roots in southern Arabia in the mid-seventh century. It was the Umayyads who expanded their empire to cover a vast region of the world, from the Iberian peninsula (modern Spain and Portugal) and Morocco in the west, across the southern coast of the Mediterranean Sea to Egypt, including all of the modern Middle East from Arabia in the south to the Caucasus mountains–and to Khwarazm–in the north, and then eastward across Persia to the Indus River (encompassing modern-day Iran, Afghanistan and Pakistan, as well as the former Soviet republics of Turkmenistan, Uzbekistan, Tajikistan and Kyrgyzstan).

In around 750, a revolt by the Abbasids brought down the Umayyad dynasty, who had ruled from the ancient city of Damascus. The second of the Abbasid caliphs, Abū Ja'far 'Abd Allāh al-Manṣūr[8] (714-775), so valued the support of his Persian allies to the east because of their invaluable assistance in the successful revolt, and so esteemed their culture as well, that he translated the seat of power of his empire to a new location along the Tigris river in 762, which he then called Madīnat as-Salām[9], the City of Peace. Al-Manṣūr's new court would form the core of what would become the great imperial capital city of Baghdad.

The fifth of the Abbasid caliphs, Hārūn al-Rashīd[10] (763?-809), ascended to power in the year 786 as a young man in his early twenties and ruled for over forty years. He appointed skilled Persian bureaucrats to govern the empire and under him, the empire flourished, both politically and

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[1] Pronounced *mu-HAM-ad ib'n MOO-suh ahl-qua-RIZ-mee*.

[2] Pronounced *HWA-raz'm*.

[3] Pronounced *AH-mu DAR-ya*.

[4] Pronounced *pah-MEER*.

[5] For biographical information on al-Khwārizmī, the reader is directed to [Katz, 1998], [Lindberg, 1992] or [Saliba, 2007]. For a more general history of Islamic science in the middle ages than what appears in the following paragraphs, see, in addition to the above, [Berggren, 2003] and [Katz et al., 2007].

[6] Pronounced *AF-ri-gid*.

[7] Pronounced *OO-mah-yad*.

[8] Pronounced *ah-BOO jah-FAR ahb-DULL-uh AHL man-SOOR*.

[9] Pronounced *ma-dee-NAT ahss-sa-LAM*.

[10] Pronounced *hah-ROON ahl-rah-SHEED*. The epithet 'al-Rashīd' means 'the Upright' or 'the Right-Thinking' in Arabic, indicating the esteem in which the caliph was held by contemporaries – or possibly, simply the immense political power he held over them! Hārūn al-Rashīd is also a recurring main character in *One Thousand and One Nights*, the famous collection of Middle Eastern folk tales compiled and translated into English in the eighteenth century [Lindberg, 1992]. (For more on Hārūn al-Rashīd, consult the Wikipedia article on his life.)

economically as well as culturally. He and his son, Abu al-Abbas al-Ma'mūn[11] (786-833), who also served as caliph after him, were great patrons of scholarship as well as of the arts. Hārūn's personal library of manuscripts, the Khizānat Kutub al-Ḥikma[12] (Storehouse of the Books of Wisdom), was extensive, and it increased by the year through an active program of acquisition, especially in works of philosophy, mathematics, astronomy, geography and cartography, alchemy and medicine collected from around the world.

Al-Ma'mūn, who became caliph in Baghdad in 813 after winning a protracted civil war against his half-brother, was an even greater devotee of scholarship than his father. It was Al-Ma'mūn who turned the library into a public academy called the Bayt al-Ḥikma[13] (House of Wisdom). Here, translations into Arabic of the many works amassed in the caliph's library–works from Persia, India and China to the east, and from Syria and the Greek-speaking areas of the Mediterranean coast to the west–were made, studied, commented upon and extended. There are reports of how Al-Ma'mūn regularly participated in debates among the scholars at the Bayt al-Ḥikma and how generous a patron he was of their works. In particular, the geometrical works of Euclid, Archimedes and Apollonius from the golden ages of Greek mathematics, laid a foundation among those scholars interested in mathematics for continuing this type of scholarship within this new Persian-Arabic culture.

Around the year 820, al-Ma'mūn summoned al-Khwārizmī to Baghdad to join his team of scholars, and there he remained for the rest of his life. Al-Khwārizmī made substantial contributions to the study of mathematical astronomy and their applications to preparing calendars and almanacs to predict the times of celestial events, the dates of Muslim feast days and the *qibla* (direction) for daily prayer,[14] and he authored works in cartography and geography to map the known world. In pure mathematics also he made key advances. He was an important figure in spreading the use of Hindu arithmetical calculation techniques through a now-lost work, whose reputation endures today through a twelfth century Latin version of the text titled *Algorithmi de numero Indorum* (*al-Khwārizmī on the Hindu Art of Reckoning*). This Latin version is almost certainly a substantially rewritten work. But the story illustrates that the Latinized version of his name, *Algorithmus*, became the root of the English word *algorithm*, now used to describe any systematic procedure, typically for computational purposes.

The most important of al-Khwārizmī's works, and the source of the texts we will encounter in this project, is the work now considered to be the first written work in algebra. Written about the year 825, it has the title *al-Kitāb al-mukhtaṣar fī hisāb al-jabr wal-muqābala*[15] (*The Compendious Book on Calculation by Restoration and Reduction*), and it enjoyed immediate and lasting success as a work in quantitative problem solving. While its later chapters deal with solving problems arising from the loan of money at interest, from the calculation of areas of land for real estate transactions, and from the fair distribution of legacies among heirs of the deceased, its initial chapters represented an important advance in the art of arithmetical calculation, in that it offered not just methods of reckoning to work out sums, differences, products, ratios and extraction of roots of numbers for the solutions of many standard practical problems, but it classified many problem types and laid out

[11]Pronounced *ah-BOO'L-ah-BAHSS ahl-mah'-MOON.*

[12]Pronounced *kee-ZAH-NAHT koo-TOOB ahl-HIK-mah.*

[13]Pronounced *BITE ahl-HIK-mah.*

[14]Muslims are required to pray five times each day – at dawn, at noon, at mid-afternoon, at sunset, and at night. These prayers are to be performed by facing in the direction of Mecca, a direction called the *qibla*. Of course, this direction differs depending on where on the earth the supplicant stands or prostrates, and it is determined for each place by using methods of mathematical astronomy.

[15]Pronounced *ahl ki-TAHB ahl-muk-TAH-sahr fih hih-SAHB ahl-JAB'r wal-moo-KAH-bah-lah.*

systematic methods for solving such problems by means of this classification. For the first time, methods for computational problem solving were presented not problem by problem on an ad hoc basis, but for entire families of problems at once.

In the pages that follow, we present selections from this work by al-Khwārizmī, using two English translations, one by the nineteenth century orientalist Frederic Rosen [al Khwārizmī, 1831], and the other a more recent scholarly edition by Roshdi Rashed[16] [Rāshid, 2009].

The two methods referred to in the title of al-Khwārizmī's book, *al-jabr* and *al-muqābala* were important features of this systematic approach to problem solving. We explain their meanings here by invoking the first example of a problem given by al-Khwārizmī in his chapter on legacies.

> A man dies and leaves behind two sons; he bequeaths one-third of his possessions to a stranger, and leaves behind assets of ten *dirhams* and a sum of ten *dirhams* owed to him by one of his two sons.[17]

The situation described here called for ascertaining how the deceased man's "assets" should be disbursed among his three heirs. The situation was complicated by the fact that the man "bequeaths one-third of his possessions to a stranger," and that there was also "a sum of ten *dirhams* owed to him by one of his two sons." Since the amount the indebted son owed his father was clearly greater than whatever share the son could expect from his father's legacy (since he owes his father an amount already equal to his father's current assets), the son in debt would only be required by Islamic law to restore to his father's assets an amount precisely equal to the amount of his share of the legacy. This amount was called by al-Khwārizmī "what is taken out from the debt" in his description below of the solution to the problem.

> We thus infer that: Make what is taken out from the debt one thing, and add it to the assets, which are ten *dirhams*; the result is ten plus one thing. Then set aside one-third of it, for he has bequeathed one-third of his possessions [to the stranger], that is, three *dirhams* plus one-third of a *dirham* plus one-third of a thing. There remain six *dirhams* plus two-thirds of a *dirham* plus two-thirds of a thing. Divide this among the two sons, and each of the sons thus receives three *dirhams* plus one-third of a *dirham* plus one-third of a thing, which is equal to the thing taken away. Therefore, reduce this: you take away one-third of a thing for one third of a thing, and there remain two-thirds of a thing, equal to three *dirhams* plus one-third. You will then need to complete the thing, so add to the two-thirds of a thing their half, and their result is then five *dirhams*, which is the value of the thing taken away from the debt.

[16] Rashed (b. 1936), also identified as Rushdi Rashid, is an eminent Egyptian historian of mathematics.

[17] In Baghdad, likely the most populous city on the planet at the time, the *dirham* was the standard coin of the day and typically struck from silver. The Arabic word *dirham* was derived from the name of the ancient Greek coin, the *drachma*, which would also have been standard currency in the Middle East in pre-Islamic times.

Task 1 (a) Note at least three striking features (to you) of al-Khwārizmī's "solution" to this problem. Are any of these features surprising to you?

(b) How much of al-Khwārizmī's solution do you follow? Can you tell how much, in *dirhams*, the stranger and the two sons receive in disbursement of this legacy? Explain how you know that your answer to this question is correct.

To modern eyes, this legacy problem is precisely the kind of "word problem" that begs for an equation to solve with algebra. So let's do that!

Task 2 Here is the text of the legacy problem again, this time with circled numbers interspersed to help you identify the eight steps of al-Khwārizmī's algebraic solution.

> We thus infer that: ① Make what is taken out from the debt one thing, and ② add it to the assets, which are ten *dirhams*; the result is ten plus one thing. Then ③ set aside one-third of it, for he has bequeathed one-third of his possessions [to the stranger], that is, three *dirhams* plus one-third of a *dirham* plus one-third of a thing. There remain six *dirhams* plus two-thirds of a *dirham* plus two-thirds of a thing. ④ Divide this among the two sons, and each of the sons thus receives three *dirhams* plus one-third of a *dirham* plus one-third of a thing, ⑤ which is equal to the thing taken away. ⑥ Therefore, reduce this: you take away one-third of a thing for one third of a thing, and there remain two-thirds of a thing, equal to three *dirhams* plus one-third. ⑦ You will then need to complete the thing, so add to the two-thirds of a thing their half, ⑧ and their result is then five *dirhams*, which is the value of the thing taken away from the debt.

Turn each of the eight pieces of text into an algebra step and proceed in sequence, carefully describing what is happening at each step to solve the problem.

This legacy problem was a quite common sort of problem that in al-Khwārizmī's day was handled by anyone skilled in calculation. Al-Khwārizmī himself had clearly mastered this art. In his solution, he made reference to two important procedural steps. The first of these occurred at ⑥, where he instructed the solver to "reduce this," namely the "equation" he has formulated at this stage of the solution. Using the quantity x to stand for the unknown amount he was looking for, the "thing" in his problem, the equation he considered just before step ⑥ had the form $x = 3\frac{1}{3} + \frac{1}{3}x$. (Refer to your solution to Task 2 above for why this is so.) For al-Khwārizmī, "reduction," or *al-muqābala* as it appears in the Arabic title of his book, meant balancing the two amounts of the "thing" on either side of the equation–the two terms involving x –so as to reduce it to an equation in which "thing" only appears on one side. Specifically, removing $\frac{1}{3}x$ from both sides of the equation to reduce the equation to the simpler form $\frac{2}{3}x = 3\frac{1}{3}$ was a form of *muqābala*.

Another type of problem simplification arose whenever a difference of quantities appeared on one side of an equation; adding the subtracted quantity to both sides would involve its "restoration" (*al-jabr*) to the opposite member of the equation, as in the case of transforming the equation $100-2x = 40$ into the equivalent equation $100 = 2x + 40$.

These two problem solving methods that al-Khwārizmī highlighted, *al-jabr* and *al-muqābala*, became associated in subsequent generations with the full range of possible stages in the solution of a problem in which an unknown quantity was to be found. Centuries later, in Latin translations of his work in Europe, the names of these methods were left untranslated and were given Latinized names "algebra" and "almucabola." Ultimately, only the first of the terms came to be used to refer to the procedures . . . and this is where algebra came from!

It is important to note that al-Khwārizmī was by no means the first to use these procedures to discover the unknown quantity in a problem. Problems were solved in similar ways for centuries prior to his time. The real advance here is what al-Khwārizmī did in the systematic study of these techniques in the earlier chapters of his book, a survey of which we take up next.

2 The Classification of Algebra Problems

After giving praise to God, to the Prophet Muhammad, and to Caliph al-Ma'mūn (in that order, of course), indicating that the latter "has encouraged me to compose a short work on calculation by completion and reduction," al-Khwārizmī opened *The Compendious Book on Calculation by Restoration and Reduction*–hereafter referred to by its more common and much shorter name, *Algebra*–as follows:

> When I considered what people generally want in calculating, I found that it always is a number. [. . .] I observed that the numbers which are required in calculating by Restoration and Reduction are of three kinds, namely, roots, squares, and simple numbers relative to neither root nor square.
>
> A root is any quantity which is to be multiplied by itself, consisting of units, or numbers ascending, or fractions descending.
>
> A square is the whole amount of the root multiplied by itself.
>
> A simple number is any number which may be pronounced without reference to root or square.
>
> A number belonging to one of these three classes may be equal to a number of another class; you may say, for instance, "squares are equal to roots," or "squares are equal to numbers," or "roots are equal to numbers."

Task 3 Consider the object of al-Khwārizmī's concern here, in the first sentence of the excerpt above. He calls it "what people generally want in calculating." In the context of the legacy problem we examined in the previous section, what was it that al-Khwārizmī wanted to calculate? Reflecting on this, what do you think al-Khwārizmī meant more generally by this phrase?

The rest of the section of source text above is concerned with the "three kinds" of "numbers which are required in calculating by Completion and Reduction, . . . namely, roots, squares, and simple numbers relative to neither root nor square." To help us make sense of what al-Khwārizmī meant by this, it will help to reflect on the terms he was using. In the same way that a plant grows up from its roots, or that someone's personal history is referred to as her roots, he used the term "root" here in a similar but mathematical sense to refer to a quantity from which some other quantity is derived.[18] Likewise, he used the word "square" in a very suggestive sense, one that pointedly made reference to a geometric square. Finally, al-Khwārizmī used the term "simple number" in order to qualify numbers of a special sort, namely those "without reference to root or square." That is, for al-Khwārizmī, there were numbers, and there were numbers which were "simple" because they were being considered separately from any reference to a "root" or a "square".

What has not been clarified yet by al-Khwārizmī here, but will soon be apparent, is that in his use of this terminology in problem solving, his "thing," the unknown quantity, was always to be identified with the "root." Thus, referring once again to the problem of the legacy which we examined in the previous section, when al-Khwārizmī spoke there of "three *dirhams* plus one-third of a *dirham* plus one-third of a thing," he would identify the "thing" as having the species of "root," and in the phrase "one-third of a thing," the number "one-third," being a number of "roots," would therefore not be "simple." On the other hand, the number in "three *dirhams* plus one-third of a *dirham*" was "simple" because it was not associated with a number of "roots" or "squares" but was "without reference to root or square." Note that a "simple number" was almost always identified as a number of *dirhams*.

This points out another feature of the context of problem solving that was part of the perspective of al-Khwārizmī and his contemporaries: *numbers were (almost always) assumed to be positive*. They were used to count with, or to measure out quantities of stuff, especially when they entered into problems to be solved. And although they might not have come directly from real-life problems, they were at least realistic in form. In particular, a number of *dirhams* would only be considered negative in the special context in which they were simultaneously being considered as part of a debt; and the root of a square could never be negative, since it represented the magnitude of a length. Thus, in what follows, you should consider all numbers in al-Khwārizmī's problems to be positive quantities.

Task 4

(a) Today, we would say that 7 is a (*square*) *root* of 49, or that −12 is a *root* of 144. What do we mean by this? That is, what is the relationship between a *root* and the number of which it is a *root*?

(b) When we say that 49 is the *square* of 7 or that 144 is the *square* of −12, what do we mean by this? And what relation does this idea have with a geometrical square? That is, what is the relation between the arithmetical operation of squaring and the geometrical square? Take note of how this interpretation is evidenced in the two examples given here; only one of them has a proper interpretation in the geometrical sense.

(c) The most subtle of these terms to get a handle on is "simple number." To help with this, consider the following short excerpt from later in the book (once again, circled numbers have been inserted into the text as reference marks):

> If the instance be, ① "ten *dirhams* and half a thing to be multiplied by half a *dirham*, minus five things," then you say: ② half a *dirham* by ten is five *dirhams*, positive; and ③ half a *dirham* by half a thing is a quarter of a thing, positive; and ④ minus five things by ten *dirhams* is fifty roots, negative. ⑤ This altogether makes five

[18] This use of the term root long predates al-Khwārizmī as well, and can be found in the works of ancient Greek mathematicians and geometers.

dirhams minus forty-nine things and three-quarters of a thing. After this, you ⑥ multiply five roots, negative, by half a root, positive: it is two squares and a half, negative. Therefore, ⑦ the product is five *dirhams*, minus two squares and a half, minus forty-nine roots and three-quarters of a root.

As in Task 2 above, take each of the seven portions of text in turn, and interpret them using modern algebraic symbols, where we use x to represent the "thing." Once you have finished this, identify what al-Khwārizmī has done here.

(d) In the seventh and final step of al-Khwārizmī's procedure in part (c) above, identify how many "squares," how many "roots," and how many "simple numbers" are being expressed. Why aren't all these numbers "simple numbers"?

In modern mathematics, the different kinds of expressions that al-Khwārizmī classified in his *Algebra* we now call *monomial* terms. Thus, in the symbolic expression

$$5 - 49.75x - 2.5x^2$$

(and to be clear, in modern mathematics, we would nearly always use only symbolic expressions!), there are three monomial terms: $5, -49.75x$, and $-2.5x^2$ whose sum is the given expression. More generally, then, a monomial is any symbolic term involving the unknown quantity x and which has the form ax^n, where a is a number (called the *coefficient* of the term), and n is a non-negative integer, called the *exponent* (or *power*) of the term. Notice that when $n = 0$, the monomial simplifies to $ax^0 = a$ (a "simple number"!), or what is often today called a *constant term*. When multiple monomial terms are summed together, as in the example at the beginning of the paragraph, the resulting expression is called a *binomial* (two terms), *trinomial* (three terms), or more generally, a *polynomial* expression (used to refer generally to any number of terms).

Refer back to the description of the various species of quantities that al-Khwārizmī enumerated at the beginning of his *Algebra* (p. 7 above). In the last sentence of that description, he identified the three kinds of equations which can be formed by setting some two of these three species of quantity equal to each other: "squares are equal to roots," "squares are equal to numbers," or "roots are equal to numbers."

Task 5

(a) Al-Khwārizmī provided the following three examples of such polynomial equations. Which of the three types listed just above corresponds to each example?

(i) the half of the square is equal to eighteen

(ii) five squares are equal to ten roots

(iii) four roots are equal to twenty

(b) Translate each of the three examples above into symbolic form and solve the equations, both using al-Khwārizmī's sense of the word (numbers must be positive) and in the broader modern sense. Be careful: the equations may have different numbers of solutions depending on this sense!

(c) The most general representation of the types of monomial described here will have the symbolic forms ax^2 (squares), bx (roots), and c (simple numbers). Use these to express symbolically the general form of the three types of al-Khwārizmī's equations. Solve each one for x, making sure to identify what conditions will have to be satisfied in order that the steps you employ are valid. (For instance, you may not divide by 0 or take the square root of a negative number if your answer is to be a real number.)

(d) The case "roots are equal to numbers" produces an equation in which the highest power of x that appears is the first power. Such equations are called *linear* equations in modern mathematics. Why?

As Task 5 shows, the three equation forms identified by al-Khwārizmī are rather straightforward to solve. His next task, however, was to consider more complicated problems that can be formed involving the three species.

3 Solving the Quadratic Equation

Once al-Khwārizmī had handled equations formed by using some pair of the three species, he next considered equations using all three types at once.

I found that these three kinds, namely roots, squares, and numbers, may be combined together, and thus three compound species arise; that is, "squares and roots equal to numbers"; "squares and numbers equal to roots"; "roots and numbers equal to squares".

Task 6

(a) Write out symbolic forms for the equations given by these "three compound species." Notice that the word "and" in each phrase is represented symbolically in the same way.

(b) In modern mathematics, all three "compound species" would be subsumed under a single form, namely,

$$ax^2 + bx + c = 0,$$

where none of a, b or c can be zero. Why did al-Khwārizmī require three different "species" to describe this single problem?

(c) Today we call the single equation form above (and any specific equation obtained from this by giving particular values to the three coefficients[19]) a *quadratic equation.* Look up the etymology of the word *quadratic*; why is it an appropriate term to use for these equations?

Al-Khwārizmī continued by giving examples of his "three compound species" and in each case he provided a "formula" that solved the corresponding problem, which for him meant identifying a value both for the "thing" (the "root") and for its "square." Once again, circled numbers are included in the text only for reference later; they are not part of the original source.

[19] This includes the possibility that either b or c equals 0, which al-Khwārizmī relegated to the cases "squares equal to roots" and "squares equal to numbers."

Roots and Squares are equal to Numbers: ① for instance, "one square, and ten roots of the same, amount to thirty-nine *dirhams*;" that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? ② The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. ③ Add this to thirty-nine; the sum is sixty-four. ④ Now take the root of this, which is eight, ⑤ and subtract from it half the number of the roots, which is five; ⑥ the remainder is three. This is the root of the square which you sought for; ⑦ the square itself is nine.

Task 7 Using modern algebraic symbolism, let us follow al-Khwārizmī's steps to set up the problem and produce a solution from the source text above. We will also be analyzing, by means of this same symbolism, how al-Khwārizmī "misses" an alternate solution to the problem.

(a) Formulate the problem symbolically in an equation (step ①). Note that this is not what al-Khwārizmī did! He made no reference to expressing his solution in terms of symbols of any sort. What you are doing here is a *modern treatment* modeled on what al-Khwārizmī reported in his solution. (We will comment on this again below.)

(b) At step ②, al-Khwārizmī performed some arithmetic associated with the equation you have formulated in part (a). You should tie this operation to your equation by modifying one side of the equation according to the instruction at step ③, making sure to modify the other side of your equation accordingly. Note that the number on one side of the resulting equation is a perfect square. Crucial for our solution method, however, is that *the opposite side of the equation is also a perfect square*! Factor this polynomial to verify that it is indeed a perfect square.

(c) At step ④, al-Khwārizmī performed another arithmetical operation; you should mimic this by performing the same operation on both sides of your last equation.

(d) Note that this last step has reduced the solution of the quadratic equation to that of *two* linear equations. Complete your solution process by executing steps ⑤ and ⑥ as indicated by al-Khwārizmī on one of the linear equations you obtained from part (c). Do the same for the other equation as well.

(e) Al-Khwārizmī only recognized one of the numbers that you found in part (d) as a solution to his problem. Why did he not consider the other one to also be a solution?

(f) For al-Khwārizmī, the original equation included both the "root" (which is the"thing") and its "square"; step ⑦ is needed to report the value of the latter of these. What are the two values of the square of the unknown?

The reason that al-Khwārizmī's *Algebra* enjoyed such acclaim was his systematic analysis of problem types and their solutions. Having given a prototypical example of a problem of the form "roots and squares are equal to numbers," one in which the number of "squares" was just one, he then presented a pair of similar problems in which the number of "squares" that appeared was greater than or less than one.

The solution is the same when two squares or three or more be specified; you reduce them to one single square, and in the same proportion you reduce also the roots and simple numbers which are connected therewith.

For instance, "two squares and ten roots are equal to forty-eight *dirhams*;" that is to say, what must be the amount of two squares which when summed up and added to ten times the root of one of them make up a sum of forty-eight *dirhams*? You must at first reduce the two squares to one; and you know that one square of the two is the moiety[20] of both. Then reduce everything mentioned in the statement to its half, and it will be the same as if the question had been, "a square and five roots of the same are equal to twenty-four *dirhams*."

Now halve the number of the roots; the moiety is two and a half. Multiply that by itself; the product is six and a quarter. Add this to twenty-four; the sum is thirty *dirhams* and a quarter. Take the root of this; it is five and a half. Subtract from this the moiety of the number of the roots, that is, two and a half; the remainder is three. This is the root of the square, and the square itself is nine.

The proceeding will be the same if the instance be, "half of a square and five roots are equal to twenty-eight dirhams"... Your first business should be complete your square, so that it amounts to one whole square. This you effect by doubling it. ...

Proceed in this manner whenever you meet with squares and roots that are equal to simple numbers, for it will always answer.

Task 8

(a) The first two paragraphs of the excerpt above present and begin to solve a second example of the form "roots and squares are equal to numbers." It appears that al-Khwārizmī replaces the original equation with another one. Explain what he is doing and why he does it.

(b) In the third paragraph, al-Khwārizmī uses the same formulaic means to solve the problem that was used to solve the earlier prototype example. Recount in your own words al-Khwārizmī's solution steps, and how a modern approach with symbolic notation would provide not one, but two solutions.

(c) In the fourth paragraph above, al-Khwārizmī proposed a third example whose full solution is not included in the excerpt here. Following the previous two examples, however, show how al-Khwārizmī would have solved this problem. Also show how a modern algebraic treatment would solve the problem.

The general form of the species "roots and squares are equal to numbers" which was analyzed in these sections of text from al-Khwārizmī is representable using modern notation as

$$ax^2 + bx = c.$$

[20]This is a lovely old word that simply means "middle," or "half," related to the French word *moitié*, also meaning "half."

In this formulation, the coefficients a, b, c, which normally stand for particular numbers. In al-Khwārizmī's earlier prototypical example, $a = 1, b = 10, c = 39$, while in his second example, from part (a) above, $a = 2, b = 10, c = 48$. The coefficients a, b, c are also called *parameters*; this term is used to emphasize that in the level of generality being applied to our analysis, the quantities a, b, c are fixed but unspecified. This is in contrast to the way in which the *unknown* quantity x is being treated. For while x is also unspecified in the problem, it cannot be freely chosen by the solver, as it is subject to the conditions of the equation. Indeed, these conditions are partly established by the choice of the parameters that appear in the equation. The unknown x is restricted by, and must be derived from, these conditions.

Next, al-Khwārizmī tackled the second of his three species. (Again, circled numbers have been added to the text as reference marks.)

Squares and Numbers are equal to Roots: ① for instance, "a square and twenty-one in numbers are equal to ten roots of the same square." That is to say, what must be the amount of a square which, when twenty-one *dirhams* are added to it, becomes equal to the equivalent of ten roots of that square?

Solution: ② Halve the number of the roots; the moiety is five. Multiply this by itself; the product is twenty-five. ③ Subtract from this the twenty-one which are connected with the square; the remainder is four. ④ Extract its root; it is two. ⑤ Subtract this from the moiety of the roots, which is five; the remainder is three. ⑥ This is the root of the square which you required, and the square is nine. Or, ⑦ you may add the root to the moiety of the roots; the sum is seven; this the root of the square which you sought for, and the square is forty-nine.

When you meet with an instance which refers you to this case, try its solution by addition, and if that does not serve, then subtraction certainly will. For in this case, both addition and subtraction may be employed, which will not answer in any other of the three cases in which the number of roots must be halved.

Task 9 As in Task 7 above, let us follow al-Khwārizmī's solution to his prototypical example for the species "squares and numbers are equal to roots."

(a) Formulate the problem symbolically (step ①).

(b) Perform the arithmetic steps (②, ③, ④, ⑤ and ⑥) which make up al-Khwārizmī's "formula" to produce one soluton from the values of the coefficients of the equation. Following steps ②, ③, ④ with ⑦ produces a second solution.

(c) Now, let's reproduce al-Khwārizmī's solutions by working in parallel symbolically. From the initial equation in part (a), subtract the linear term (the "number of roots") from both sides, and then subtract the constant term from both sides; this will serve to put the equation in the form of al-Khwārizmī's first species ("squares and roots are equal to numbers"). Now add the result of step ② to both sides of the equation; on the opposite side we will obtain the result of step ③.

(d) While on one side of the equation we now have the square number 4, on the other side is a trinomial expression which also happens to be a perfect square; factor this trinomial to represent it as the square of a binomial expression. Then perform the step indicated at ④, noting that we should obtain two different equations as a result, depending on which of the square roots of 4 we consider. Finally, solve for x in both equations, which reproduces the results of steps ⑤ and ⑥, or of step ⑦.

Task 10 Having worked Tasks 7, 8 and 9, you have seen how al-Khwārizmī's solutions to his problem schemes follow a recognizable pattern. Let us now apply his formulaic solution to the most general symbolic form of the quadratic equation,

$$ax^2 + bx + c = 0,$$

leaving the parameters unspecified.

(a) Begin by bringing the equation to the form of the species "squares and roots are equal to numbers." Reduce this to the case of "one whole square" by dividing the equation through by the coefficient of x^2.

(b) Next, halve the number of roots in the new equation, multiply the result of this by itself, and add this square to both sides of the last equation. This step ensures that there is a trinomial expression on one side of the equation which factors as a perfect square; factor the trinomial as a perfect square. On the other side of the equation is some number.

(c) Now take square roots on both sides of the equation. This should produce two new equations, both linear, depending only on which of the two square roots of the number is selected for the equation. In each equation, solve for x. You should obtain something equivalent to the following:

$$x = \pm\sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)} - \frac{b}{2a}. \tag{$*$}$$

(d) Finally, "rationalize" the expression by expressing all fractions with a common denominator. Show that the result produces the famous *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

It may come as a surprise to the reader that the use of symbols, like letters $a, b, c, \ldots, x, y, z$ for quantities and $+, -, =$, etc., for arithmetical operations and relationships, is nowhere to be found in the writings of "the Father of Algebra." Indeed, symbolism did not enter into algebraic procedures for another 700 years after al-Khwārizmī's publication of his *Algebra*! To modern sensibilities, the use of symbolism has become an integral part of the practice of algebraic problem solving. Instead, al-Khwārizmī is able to communicate precisely the same operations and relationships entirely in words; we call this *rhetorical algebra* to distinguish it from our now more familiar *symbolic algebra*.

In addition to the absence of symbolic notation, al-Khwārizmī was restricted by the prevailing understanding of what numbers were. As we have seen already, he was required to classify all quadratic equations into five types:

- squares equal to roots;
- squares equal to numbers;
- squares and roots equal to numbers;
- squares and numbers equal to roots;
- squares equal to roots and numbers.

Contemporary algebraic practice now makes do with only one type, namely

$$ax^2 + bx + c,$$

but this is because we moderns have extended the concept of number to include the possibility that the equation's coefficients need not be positive numbers only, but may take the value 0 (save that a may not be 0, else the equation reduces to a linear equation) or, indeed, even some negative value.

Still, al-Khwārizmī was perceptive enough to pick out a problematic case which might arise when trying to solve a problem of quadratic type:

And know that, when in a question belonging to this case you halved the number of the roots and multiplied the moiety by itself, if the product be less than the number of *dirhams* connected with the square, then the instance is impossible; but if the product be equal to the *dirhams* by themselves, then the root of the square is equal to the moiety of the roots alone, without either addition or subtraction.

Task 11 In light of the formula (stated at the very beginning of this project), which claims to produce the solutions to any equation of the form $(*)$, why would the first condition described by al-Khwārizmī here, that "the product [of half the number of roots by itself] be less than the number of *dirhams* connected with the square," make the solution "impossible"? And why would the second condition described here make "the root of the square equal to the moiety of the roots alone, without either addition or subtraction"?

Finally, al-Khwārizmī comes to the third and last of his problem species.

Roots and Numbers are equal to Squares: for instance, "three roots and four of simple numbers are equal to a square." Solution: Halve the roots; the moiety is one and a half. Multiply this by itself; the product is two and a quarter. Add this to the four; the sum is six and quarter. Extract its root; it is two and a half. Add this to the moiety of the roots, which was one and a half; the sum is four. This is the root of the square, and the square is sixteen.

Task 12 In the spirit of Tasks 7 and 9, formulate symbolically the example problem in this last bit of text, and explain how al-Khwārizmī obtains his solution. What is the "missing" solution, and why does al-Khwārizmī not find it?

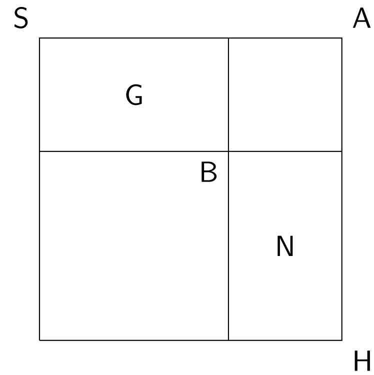
4 The Demonstration: Completing the Square

There is something a bit unsatisfying about al-Khwārizmī's solution method for quadratic equations. In the solutions he presented for all three compound species, he began by halving the number of the roots, multiplying this quantity by itself and then adding this quantity to the constant term of the equation (or to its negative, if the "simple number" is on the side of the equation opposite to the "squares"). This corresponds to steps ② and ③ of the solutions he gave for "roots and squares are equal to numbers" (p. 11) and "squares and numbers are equal to roots" (p. 13). While the computation of this number is critical for the solution of the problem, it is an odd choice of arithmetic to choose to perform. Why is this the quantity that is needed to effect the solution of the problem?

We get a peek into the reason for this by looking at al-Khwārizmī's justification of his solution. As a well-trained mathematician, he knew that a deductive proof, in the manner of the Greek geometers Euclid (c. 325–265 BCE) and Archimedes (287–212 BCE), would convince his readers that the methods he proposed were sound and unassailable.

> Demonstration of the Case "a Square and ten Roots are equal to thirty-nine *Dirhams*": The figure to explain this is a quadrate,[21] the sides of which are unknown. It represents the square, the which, or the root of which, you wish to know. . .
>
> We proceed from the quadrate AB, which represents the square. It is our next business to add to it the ten roots of the same. We halve for this purpose the ten, so that it becomes five, and construct two rectangles on two sides of the quadrate AB, namely G and N, the length of each of them being five, as the moiety of the ten roots, whilst the breadth of each is equal to a side of the quadrate AB. Then a quadrate remains opposite the corner of the quadrate AB. This is equal to five multiplied by five, this five being half of the number of the roots which we have added to each of the two sides of the first quadrate.
>
> Thus we know that the first quadrate, which is the square, and the two rectangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied by five, or twenty-five. This we add to thirty-nine in order to complete the great square SH. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrate. By subtracting from this the same quantity which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrate AB, which represents the square; it is the root of this square, and the square itself is nine. This is the figure:

[21] Another old word, used by Frederic Rosen in this nineteenth century translation [al Khwārizmī, 1831], which means "square or rectangular figure." Its use here is handy, for if we were to use the more common word "square" instead, we would likely be confused between whether the word signified the shape – which is what is meant in this instance – or a type of number which is the product of another number by itself – which is what the shape is meant to represent, as we will discover. In passing, I note that while the translation here is from Rosen, the superior diagram is taken from a different translated edition [Rāshid, 2009] of the Algebra.



Task 13 (a) Draw a copy of the diagram of the square that appears with al-Khwārizmī's Demonstration. Denote by C and D the other two corners of the "quadrate AB," C being the one on segment AS and D being the one on segment AH. Denote by E the unlabeled fourth corner of the "great quadrate SH." According to the explanation given in the text that accompany the problem "a square and ten roots are equal to thirty-nine *Dirhams*," identify next to your diagram the numerical values of the lengths of these line segments:

(i) AC

(ii) AD

(iii) CS

(iv) DH

(v) AS

(b) Also identify the numerical values of the areas of these regions in your diagram:

(i) quadrate AB

(ii) rectangle G

(iii) rectangle N

(iv) quadrate SH

(v) quadrate BE

(c) Which of the elements of the diagram above correspond respectively to the "square," the "roots" and the "simple number" of "*dirhams*" in the original problem? Which of them corresponds to the "thing," which we would represent as x?

Task 14 Consider another problem that al-Khwārizmī solved in the *Algebra*: "a square and five roots of the same are equal to twenty-four *dirhams*." Draw a diagram similar to the one above for this problem. Which segment or region in your diagram corresponds to the "thing" of the solution? Which is the "square"? the "five roots"? the "twenty-four *dirhams*"?

Task 15 (a) What does al-Khwārizmī mean when he said at the opening of this section that the "quadrate [AB], the sides of which are unknown, . . . represents the square"?

(b) Why is it geometrically necessary to "halve for this purpose" the number of "roots" of the equation?

(c) Why is the solving of quadratic equations known today as "completing the square"?

Task 16 Draw a diagram similar to the one presented by al-Khwārizmī above to accompany the quadratic equation $x^2 + 14x = 32$. Instead of labeling the diagram with letters for the points or regions, label the elements of the diagram that correspond to the quantities $x, x^2, 14$ and 32 that appear in the equation. Show how this helps to illustrate the process of completing the square that solves the equation.

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Notes to Instructors

Primary Source Project Content: Topics and Goals

This Primary Source Project (PSP) is designed to serve students who are learning algebraic methods for solving quadratic equations, a topic found in most high school and college algebra courses in the United States and in many precalculus courses as well. It will also be of value to future high school mathematics teachers who will be responsible for teaching algebra in their own classrooms, as well as instructors of higher algebra courses who are interested in conveying a sense of the early history of the theory of equations. Additionally, it is suitable for use in a general history of mathematics course as an introduction to the role of early Islamic-era mathematics in the development of algebra as a major branch of mathematics.

Its main goal is a simple one: to give the student a deep understanding of the method of completing the square, the universal procedure for solving quadratic equations, while acquainting the student with a sense of how algebraic problem solving was successfully carried out in the absence of symbolic notation, thereby conveying its importance in the modern development of the subject. Its purpose is to impart conceptual rather than procedural understanding.

Student Prerequisites

Nearly nothing in the way of special prerequisites are required by this PSP beyond what a typical high school student knows about solving algebraic equations. Indeed, the project may teach the student no mathematical techniques beyond what they come to the project already possessing.

Project Design and Task Commentary

The project opens with a brief section that informs the student about the goals of the PSP.

1 Algebra from *al-jabr*: The project moves on directly to introduce the person and times of Muḥammad ibn Mūsā al-Khwārizmī (ca. 780-850 CE), generally regarded as the "Father of Algebra," the author of an immensely influential work, *al-Kitāb al-mukhtaṣar fī hisāb al-jabr wal-muqābala* (*The Compendious Book on Calculation by Restoration and Reduction*), written around the year 825, probably in Baghdad. This influence extended not just within Arab-Islamic culture, but was very important in preparing European scholars to make advances in algebra and problem solving between the twelfth and sixteenth centuries. It is from this work that the project draws all its source texts, using two English translations, one from the nineteenth century [al Khwārizmī, 1831] by the orientalist Frederic Rosen (including some memorable and charming old words, like "moiety" and "quadrate"), and another produced by the eminent Egyptian historian of science, Roshdi Rashed (or Rushdī Rāshid) [Rāshid, 2009].

To acquaint the reader with al-Khwārizmī's style of algebraic problem solving, the project begins by analyzing a problem about dividing a legacy among a man's heirs; the problem leads to a linear equation and the student reviews al-Khwārizmī's solution. While the underlying arithmetical steps are straightforward, the student will likely be caught off guard by the total absence of symbolic notation, so this exercise is preparatory, meant to ease the student into working with this style of problem solving. Task 1 starts gently by asking students to reflect openly on their reaction to reading the text. A wide range of responses are possible here.

2 The Classification of Algebra Problems: We now turn to the beginning of the *Algebra*, in which al-Khwārizmī classifies the kinds of problems that algebra was currently capable of resolving, namely quadratic equations. He did this by defining "three classes" of number ("squares," "roots" and "simple numbers"), what today we would call monomial terms in a quadratic polynomial expression. Students come to see that while the terminology is foreign, al-Khwārizmī's mathematics is familiar. They learn that for al-Khwārizmī, there were five distinct classes of quadratic equation instead of one, since for him, all coefficients had to be positive numbers (see also Task 6). In Tasks 2 and 4, students use circled numbers embedded in the source text as markers to help them dissect its content and interpret the text for themselves.

3 Solving the Quadratic Equation: Students work through a handful of al-Khwārizmī's solutions to quadratic equations. Then, based on this experience, they are asked to derive the quadratic formula and identify the conditions under which there are two, one, or no real solutions. Again, in Tasks 7 and 9, circled numbers are used as markers to guide their textual analyses. Throughout the project, students are asked to compare al-Khwārizmī's non-symbolic approach with a modern version that employs standard algebraic symbolism. Task 10 is perhaps the most critical spot in the PSP, in which students derive the quadratic formula.

4 The Demonstration: Completing the Square: Students read through al-Khwārizmī's geometric demonstration of his method which – literally! – completed a square, illustrating how the corresponding algebraic procedure works.

Suggestions for Classroom Implementation

Despite the elementary nature of the mathematical ideas involved in this project, instructors are urged are to carefully work the student tasks here BEFORE you assign them to your students! Many of the tasks are not routine exercises and may pose challenges, even to those adept in algebra.

The project is designed to be completed in a week of classroom time, (either three 50-minute periods or two 75-minute periods), plus time in advance for the student to do some initial reading and time afterwards for them to write up their solutions to the last set of Tasks. If the goal is simply to provide students with a feel for early algebraic techniques, it may be possible to abbreviate the project further by omitting treatments of the second and third cases of the quadratic equation, "Squares and Numbers are equal to Roots" and/or "Roots and Numbers are equal to Squares," along with Tasks 9 and 12, in Section 3 above. Alternatively, consider using instead the mini-PSP *Completing the Square: From the Roots of Algebra*, which was designed to be a shorter, pared-down version of this project. See the section below, Connections to Other Primary Source Projects, for details about this mini-PSP.

Sample Implementation Schedule (based on a 50 minute class period)

Instructors should assign students to read through Section 1 of the project (pp. 1-5) and complete Task 1 before the first class. During the first period, students should work concertedly on Tasks 2 and 4. I recommend giving students a few minutes to work alone on each Task before reorienting them to work in small groups of 2-5 students. Task 3 can be handled in a brief discussion with the entire classroom, and Task 5 is best given for homework (a *very* short discussion about this can begin the second period).

For day 2, students should be assigned to read through all of Section 3 (pp. 8-14) and to complete Task 6. This period should be turned over to student work on Tasks 7, 9 and 10. Full write-ups, together with work on Task 8, should be assigned for homework.

For day 3, the rest of the project should be assigned to be read by students. The period can begin with a review of their answers to the previous day's homework, with special attention to Task

10, as the discussion for Task 11 follows on from this. The classroom might profit from reviewing Task 13 together, with reference to a single copy of the diagram (p. 15) on the board.

Instructors who teach in 75-minute periods might carry their first day through student work on Task 9, and begin the second period with Task 10.

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students. Estimates on the high end of the range assume most PSP work is completed by students working in small groups during class time.

Possible Modifications of the PSP

It might be possible to run a partial implementation of this PSP which would leave students to work on their own in advance on everything through Section 2 of the project, concentrating in the classroom on the material in Sections 3 and 4; this plan might accommodate a one-period implementation that focuses on the work of Tasks 7, 9, 10, 11 and 13. (But that seems rather ambitious!)

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Connections to Other Primary Source Projects

This project allows certain connections with other TRIUMPHS PSPs, both of which are freely available at the TRIUMPHS website (the URL is at the end of this project, below):

- The PSP titled *The Pythagorean Theorem and the Exigency of the Parallel Postulate*, by Jerry Lodder, presents a full demonstration of the Pythagorean Theorem as given by Euclid in Book I of his *Elements*, paying special attention to the dependence of the argumentation on the famous Parallel Postulate. Students who want to investigate the nature of geometrical demonstrations, of the type that al-Khwārizmī presented at the end of the current project, may be interested in investigating this foray into plane geometry.

- Students who are interested in other treatments of algebra that make use of no symbolism at all may find this PSP an enjoyable study: *Solving Systems of Linear Equations Using Ancient Chinese Methods*, by Mary Flagg. This project is also ideal for students interested in studying other non-Western mathematical works.

Recommendations for Further Reading

The history of the mathematics of the medieval Islamic world is fascinating, especially with regard to how it exploded into activity at roughly the time of al-Khwārizmī through the translation of scientific works from the surrounding civilizations, and how it influenced later developments of mathematics, especially at the time of the European Renaissance. Chapter 9 of [Lindberg, 1992] and Chapter 7 of [Katz and Parshall, 2014] tell this story rather well. The latter book is worth studying in its entirety to learn the full scope of the history of the development of algebra from ancient to modern times.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.