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Dual Perspectives on Desargues’ Theorem

Carl Lienert∗

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Girard Desargues (1591 - 1661) is often credited with being one of the founders of projective geometry. Desargues was an engineer and much of his work was motivated by application: stone cutting, placement of sun dials, or in the case of his most famous theorem, a technique for perspective drawing. The connection between projective geometry and perspective drawing is fascinating, but our focus is on the mathematics leading to and from Desargues’ Theorem.

The statement of Desargues’ Theorem is present in many modern geometry texts but a proof often is missing. We will work through his proof as he would have, that is, from a Euclidean school of thought. Desargues may well have led the way to modern geometry but he did so from a thoroughly classical background; he was, as we will see extremely well versed in classical Greek results. Along the way you will be asked to think carefully about various mathematical statements. To help with this you will make geometric constructions. I would encourage you to try at least a few of these in the same way as Desargues would have: with a pencil and a straight-edge. But, in order that you can more easily experiment with these constructions, take advantage of some modern technology, like the application GeoGebra, which will create well drawn and dynamic diagrams without the difficulties that arise in handling drawing instruments.

There may be a few parts of the proof of Desargues’ Theorem that seem unsatisfactory; look out for these. In section 4, we’ll use ideas of Jean-Victor Poncelet (1788 - 1867) and Joseph Diez Gergonne (1771 - 1859) both to resolve potential difficulties with Desargues’ proof and to provide an introduction to two main ideas of modern projective geometry.

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1 Desargues’ Theorem

It’s unlikely that anyone would stumble on Desargues’ own proof of the theorem that goes by his name. The statement of the theorem is central to modern projective geometry, so much so that it is sometimes simply taken as an axiom. Perhaps the most succinct statement of the theorem is: Two triangles are in perspective from a point if and only they are in perspective from a line. We’ll start with a more prescriptive modern statement:

The following is a modern statement of Desargues’ Theorem [Casse, 2006, p. 15-16].

Let $ABC$, $A'B'C'$ be two triangles with distinct vertices, such that the lines $AA'$, $BB'$, and $CC'$ are concurrent (intersect simultaneously) in the point $V$. Then the points of intersection of the pairs of lines:

- $BC$ and $B'C'$,
- $AB$ and $A'B'$, and
- $AC$ and $A'C'$

are collinear.

[Task 1] Use a straight-edge to carefully draw, by hand, a diagram that illustrates the theorem. Try to draw several pictures that illustrate different possible configurations.

[Task 2] Make a construction that illustrates the theorem in GeoGebra. Describe what happens when you drag the vertices of the triangles around, as well as the point $V$. What happens in extreme situations? For example, if the point $V$ is inside one of the triangles, or if the point $V$ is far from either of the triangles, or if the edges of the triangles are parallel, etc., what do you observe?

Think carefully about the order of the steps by which you make the construction. Suggestion: start with the point $V$ and three lines through $V$. You’ll want to make lines as opposed to line segments.

[Task 3] Are there cases when the theorem statement is problematic? (To be fair, we have taken Casse’s statement a bit out of context.) Can you modify the conclusion of the theorem to avoid this issue?
Desargues originally published his proof in a pamphlet on perspective drawing. This pamphlet was effectively lost. In fact, much of Desargues' work was lost or ignored. He had at least two things going against him. One was his own rather odd choice of terminology, which earned him a fair amount of criticism. Second, and perhaps more significantly, he was contemporary with René Descartes (1596 - 1650). Descartes, and many mathematicians of that time, advocated an analytic, coordinatized approach to mathematics and were dismissive of Desargues' synthetic approach.

The following proof of Desargues' Theorem is taken from Manière universelle de M. Desargues, pour pratiquer la perspective par petit-pied, comme le Géométral..., a book on perspective drawing written by a follower of Desargues, Abraham Bosse (1604 - 1676) [Bosse 1648]. Bosse was an engraver and defender of Desargues. He included this proof in an appendix, but it is widely believed to be the work of Desargues, not Bosse. Our English translations of Desargues are from The Geometrical Work of Girard Desargues by Field and Gray [Field and Gray 1987].

Read the following statement and proof of Desargues' Theorem and answer the questions that follow. Keep in mind that you may not, yet, be able to justify all the steps of Desargues' argument.

When straight lines $HDa$, $HEb$, $cED$, $lga$, $lfb$, $Hlk$, $DgK$, $EfK$, $[cab]$ which either lie in different planes or in the same one, cut one another in any order and at any angle in such points [as those implied in the lettering]; the points $c$, $f$, $g$, lie on a straight line $cfg$.

For, whatever form the figure takes, in every case; if the straight lines lie in different planes, the lines $abc$, $lga$, $lfb$, lie in a plane; the lines $DEc$, $DgK$, $EfK$ lie in another; and the points $c$, $f$, $g$ lie in each of these two planes; consequently they lie on a straight line $cfg$.

And if the same straight lines all lie in the same plane,

$$\frac{gD}{gK} = \frac{aD}{aH} \frac{lH}{lK},$$

and

$$\frac{fK}{fE} = \frac{lK}{lH} \frac{bH}{bE},$$

and

$$\frac{aD}{aH} = \frac{cD}{cE} \frac{bE}{bH},$$

therefore

$$\frac{cD}{cE} = \frac{gD}{gK} \frac{fK}{fE}.$$  

Consequently, $c$, $g$, $f$ lie on a straight line.

---

1 Euclid’s geometry is an example of synthetic geometry; the focus is on the logical results of an axiomatic system. The predominate geometry in the undergraduate curriculum is analytic. The focus is on the use of coordinates (developed by Descartes) and algebra.
1.1 Thinking about the statement of Desargues’ Theorem

The first paragraph gives the statement of Desargues’ Theorem. Use GeoGebra to make the construction that Desargues describes. (Hint: read the next task before you start.) The paragraph is recopied here for convenience:

When straight lines $HDa$, $HEb$, $cED$, $lga$, $lfb$, $Hlk$, $DgK$, $EfK$, $[cab]$ which either lie in different planes or in the same one, cut one another in any order and at any angle in such points [as those implied in the lettering]; the points $c$, $f$, $g$, lie on a straight line $cfg$.

The translator corrected one error in the text: the line $cab$ (which appears in brackets in the first line of the excerpt) was left off in the original text. There is one other typographical error. Have you found it? If not, look for it. Hint: how many points are in the diagrams you’ve drawn so far?

Desargues’ Theorem is about triangles, but he gives his statement in terms of nine lines. Try to identify the triangles; give the labels of the vertices. This may be difficult right now; we’ll revisit this later to make sure you found the “correct” triangles.

What role do the lines that are not the edges of triangles play in this configuration?

Which point plays the role of $V$ from the modern statement made by Casse (p. 2)?

1.2 Proof of Desargues’ Theorem, Different Plane Case

Desargues quickly dispensed with the case where the lines do NOT all lie in the same plane. Perhaps it is interesting that the higher dimensional case is easier.

Desargues claimed the lines $abc$, $lga$, $lfb$ lie in the same plane. Why is this true? What about the other three lines?

Does this observation give some insight into Desargues’ use of upper case vs. lower case letters?

If you didn’t identify the two triangles earlier (in Task 6), you might go back and look again.

Finally, Desargues observed that $c$, $f$, and $g$ lie in a straight line. Desargues’ argument is pretty clear, but elaborate on it, using your own words.
1.3 Proof of Desargues’ Theorem, Case of One Plane

There are, of course, many different ways to place the points of the theorem. Just so that we all have the same configuration in mind, let’s use the following arrangement of points and lines.

![Diagram of points and lines](image)

**Task 12** The first three lines of Desargues’ argument

\[
\begin{align*}
\frac{gD}{gK} &= \frac{aD}{aH} \frac{bH}{bE} \\
\frac{fK}{fE} &= \frac{lK}{lH} \frac{bH}{bE} \\
\frac{aD}{aH} &= \frac{cD}{cE} \frac{bE}{bH}
\end{align*}
\]

are perhaps a bit mysterious. The relevant line segments are emphasized in the diagrams below. We’ll prove these 3 equations in Section 2, but for now try to find and explain a pattern in these equations.
\[
\frac{gD}{gK} = \frac{aD \perp H}{aH \perp K}
\]
\[
\frac{fK}{fE} = \frac{1K \ bH}{1H \ bE}
\]
\[
\frac{aD}{aH} = \frac{cD \ bE}{cE \ bH}
\]
Task 13. The last line of the argument is more straightforward. Show how the last line follows from the first three lines.

\[
\frac{cD}{cE} = \frac{gD}{gK} \frac{fK}{fE}
\]

Task 14. The final conclusion is

“Consequently, \(c, g, f\) lie on a straight line.”

Find a connection between the pattern you found in Task 12 and this conclusion. Again, we’ll provide a proof a little later.
2 Menelaus’ Theorem

Desargues used what is today known as Menelaus’ Theorem to prove the three equations in (1). Menelaus' Theorem was perhaps so well known to mathematicians of Desargues' time that he would have felt no need to cite a theorem. Menelaus of Alexandria (circa 70 - 140 CE) was a Greek mathematician who lived in Alexandria. His work included the study of spherical geometry and contributed to the development of trigonometry. Menelaus states and proves in Sphaerica the spherical version of the plane version of the theorem that goes by his name. It is unclear whether Menelaus, in fact, proved the plane version of the theorem. We provide two proofs of Menelaus' Theorem below. The first is from Desargues’ most well know work, Brouillon project d’une atteinte aux evenemens des rencontres du Cone avec un Plan (1639). The primary focus of this work was a study of conic sections. Desargues presented Menelaus’ Theorem as a tool. The original French can be found in L’Œuvre Mathématique de G. Desargues [Taton, 1988] and an English translation can be found in [Field and Gray, 1987]. The second proof is from Ptolemy’s The Almagest [Toomer, 1984]. Claudius Ptolemy (circa 85 - 165 CE) was a contemporary of Menelaus; his writings on spherical geometry, trigonometry and astronomy extended and consolidated the work of Alexandrian mathematicians and survived the test of time better than other mathematicians' writings.

Desargues’ proof illustrates his, often criticized, choice of notation and terminology. Perhaps, this is a lesson for us that good communication is as important as the result. Ptolemy’s proof is much easier to follow. You only need to work through one of the proofs to continue the project.

2.1 Desargues’ View of Menelaus’ Theorem

We’ll start with Desargues’ statement and proof of Menelaus’ Theorem, complete with his unusual choice of terminology.

The proposition which follows, set out at length, with its proof, is the same as that at the top of page 3, and which is mentioned as having been stated differently by Ptolemy.

When in a straight line $H, D, G$ which is a trunk through three points $H, D, G$, which are knots, there pass three straight lines which are branches springing from the trunk, the lines $HKh, D4h, G4K$, then the ratio of any shoot $Dh$ of any of these branches $D4h$, contained between its knot, $D$, and either of the two other branches, [say] $HKh$, to its partner the shoot $D4$, contained between the same knot $D$ and the third of the branches $G4K$, is the same as the ratio compounded of the ratios between the two shoots of each of the other two branches, in suitable order, that is, the ratio compounded of the ratio of, say, the shoot $Hh$, to the shoot $HK$ and the ratio of the shoot $GK$ to the shoot $G4$.

For, if through the point $K$, the butt of the ordinance between the two other shoots $Hh, G4$, we draw a straight line $Kf$, parallel to the trunk $HDG$, to cut the branch $D4h$ in the point $f$, then take the shoot $Df$ as intermediate between the two shoots $Dh$ and $D4$, and note that
$Kf$ and $HG$ are parallel, then the ratio of the shoot $Dh$ to the shoot $D4$ is the same as the ratio compounded of the ratios of the shoot $Dh$ to the shoot $Df$ (or of the shoot $Hh$ to the shoot $HK$) and that of the shoot $Df$ to the shoot $D4$ (or of the shoot $GK$ to the shoot $G4$).

There are several things to note in this statement, when two of the three branches are parallel to one another, when two of the knots on the trunk coincide, and what follows from this, in which the understanding is at a loss.

The converse of this proposition, which may be correctly stated as concluding that the three points lie on a straight line, is also true.

2.1.1 Statement of Menelaus’ Theorem

The first main paragraph is Desargues’ statement of Menelaus’ Theorem. It illustrates the odd choice of terminology for which Desargues’ received a fair amount of criticism.

Task 15 Draw a diagram that represents the text of the statement. Your diagram should include all the points and lines mentioned in the first main paragraph of the excerpt.
**Task 16** Here is the diagram Desargues provided. Compare your diagram to Desargues’. It’s possible that yours is different, but make sure the appropriate points are collinear, and that the points of intersection are correct.

![Diagram](image)

**Task 17** Desargues described a relationship between three ratios. Given that “compounded of” means “multiplied by”, what is this relationship? You should express one ratio as equal to the product of two ratios.

**Task 18** Empirically check in GeoGebra that you have the equations correct. Desargues would have been opposed to measuring; but you can use the “measure” tool in GeoGebra and calculate the ratios of segment lengths.²

**Task 19** Desargues’ statement concerning the combination of proper ratios was a bit vague: he said to take them “in suitable order.” Try to formulate a more precise “rule” for which ratios of segments are the ones that Desargues selected to combine. For example, why did Desargues use the ratio $GK$ to $G4$ and not $G4$ to $GK$ in the example that he gave in the first paragraph of the preceding excerpt?

**2.1.2 Desargues’ Proof of Menelaus’ Theorem**

**Task 20** Translate into modern notation, and justify this claim: “then the ratio of the shoot $Dh$ to the shoot $D4$ is the same as the ratio compounded of the ratios of the shoot $Dh$ to the shoot $Df$ and

²Why do you think Desargues would have been opposed to measuring?
that of the shoot $Df$ to the shoot $D4$.”

The rest, and perhaps the most interesting part, of Desargues’ proof is in the parenthetical comments: “$Dh$ to the shoot $Df$ (or of the shoot $Hh$ to the shoot $HK$) and that of the shoot $Df$ to the shoot $D4$ (or of the shoot $GK$ to the shoot $G4$).” Desargues claimed that

$$\frac{Dh}{Df} = \frac{Hh}{HK}$$

and that

$$\frac{Df}{D4} = \frac{GK}{G4}.$$

These two statements can be proved using similar triangles and standard results about transversals to parallel lines. However, Desargues probably used a result of Euclid which is much less known today. From Euclid’s *Elements* Proposition 2, Book VI [Heath, 1956]:

If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.

As is typical with Euclid, it helps to begin to read the proof to understand the statement. Euclid started:

For let $DE$ be drawn parallel to $BC$, one of the sides of the triangle $ABC$; I say that, as $BD$ is to $DA$, so is $CE$ to $EA$.

**Task 21** Draw a diagram that illustrates Euclid’s proposition.
Task 22

Now use Euclid’s proposition to prove the two equations from Desargues:

\[ \frac{Dh}{Df} = \frac{Hh}{HK} \]

and that

\[ \frac{Df}{D4} = \frac{GK}{G4}. \]

You will need to think about triangles as Desargues would have. He would have thought of a triangle as a triple of points\(^3\) The “sides” of the triangle for Desargues would be lines, not line segments.

2.2 Ptolemy’s proof of Menelaus’ Theorem

This excerpt is from Toomer’s translation of Ptolemy’s *The Almagest* [Toomer, 1984]. The notation

\[ GA : AE = (GD : DZ).(ZB : BE) \]

is equivalent to

\[ \frac{GA}{AE} = \frac{GD}{DZ} \cdot \frac{ZB}{BE}. \]

Let two straight lines, \(BE\) and \(GD\), which are drawn to meet two straight lines, \(AB\) and \(AG\), cut each other at point \(Z\).

I say that

\[ GA : AE = (GD : DZ).(ZB : BE). \]

\(^3\)Think of a triangle as a set of 3 non-collinear points; the “sides” of the triangle are a consequence of the 3 points.
[Proof:] Let $EH$ be drawn through $E$ parallel to $GD$. Then, since $GD$ and $EH$ are parallel,

$$GA : AE = GD : EH.$$ 

If we bring $ZD$ in [as auxiliary],


$\therefore GA : AE = (GD : DZ).(DZ : HE).$

But $DZ : HE = ZB : BE$ ($EH$ parallel to $ZD$).

$\therefore GA : AE = (GD : DZ).(ZB : BE).$

Q.E.D.

Task 23
First, highlight the line segments in the statement of the theorem,

$$GA : AE = (GD : DZ).(ZB : BE),$$

in the diagram provided. Compare this to the diagrams of Task 12. Give the correspondence of line segments for the first of these diagrams.

Task 24
Justify each of the steps in Ptolemy’s proof:

1. $GA : AE = GD : EH$
2. $GD : EH = (GD : DZ).(DZ : HE)$ (hint: use algebra)
3. $GA : AE = (GD : DZ).(DZ : HE)$
4. $DZ : HE = ZB : BE$
5. $GA : AE = (GD : DZ).(ZB : BE)$

2.3 Back to Desargues’ Theorem

Task 25
Now, go back and use Menelaus’ Theorem to prove the three equations [1].

Task 26
Use the converse of Menelaus’ Theorem to prove the final conclusion of Desargues’ Theorem: [2].

Task 27
(Optional) Prove the converse of Menelaus’ Theorem.
3 Converse of Desargues’ Theorem

3.1 Statement of converse

Before we consider the converse of Desargues’ Theorem, it would be a good idea to reread the “for-
ward” statement. (p. 3)

Desargues’ statement of the converse is below. To make sense of the statement we need to think
very carefully about what Desargues’ intended to prove. Remember, Desargues’ did not explicitly
make a statement about triangles, but rather about a configuration of nine lines. His conclusion was
that three of the points $c, f$, and $g$ are necessarily collinear. So for the converse we start with the
assumption that $c, f$ and $g$ are collinear.

And, conversely, if the straight lines $abc$, $HDa$ $HEb$, $DEc$, $HK$, $DKg$, $KEf$ meet one
another in any manner and at any angles, in points such as those [that are given], the lines
lying either in different planes or in the same one; the lines $agl$, $bfl$ will always meet at a
butt $l$ which lies on the line $HK$.

Desargues gave a relationship between seven of the nine lines in the statement on the converse.
We need to confirm that he didn’t place any unjustified dependencies in this configuration.

Task 28

Make a GeoGebra construction which reflects the converse of Desargues’ Theorem as follows:

(a) Draw any two triangles. (This is for our convenience.)
(b) Label two of the vertices of one triangle $a$ and $b$, leave the third vertex unlabeled.
(c) Label the vertices of the second triangle $D$, $E$, and $K$.
(d) Construct lines $ab$ and $DE$. This forces the placement of point $c$.
(e) Construct line $KE$ and place a point $f$ on this line.
(f) Use the hypothesis that $c, f$, and $g$ are collinear: draw a line through $c$ and $f$.
(g) Construct line $DK$ which forces the placement of point $g$ since $c, f$, and $g$ are collinear.
   (Notice, at this point you have lines through some of the sides of the triangles.)
(h) Construct lines $Da$ and $Eb$ which forces the placement of point $H$.
(i) Construct the line $HK$.

Check that you have all seven lines shown in the first part of the statement of the converse.
You should also have the line on which $c, f$, and $g$ lie. This construction should not restrict
your two triangles! Drag the vertices of the triangles around. Moving one vertex should not
affect the location of the others.
3.2 Proof of the Converse of Desargue’s Theorem, All Lines in One Plane Case

Let’s take Desargues argument out of order and first address the case when all these lines lie in the same plane. His argument was that there is only one place to put the remaining point \( l \) and hence the configuration given in the forward direction holds. Desargues writes:

\[
\text{∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞}
\]

And if the same straight lines all lie in one plane; if we draw through the point \( a \) the line \( agl \) to meet the line \( HK \), and then draw the line \( lb \), it has just been proved that this line meets the line \( EK \) in a point such as \( f \) which is collinear with the points \( c \) and \( g \), which is to say that the line \([lb]\) passes through \( f \), and consequently that the two lines \( ag, bf \) meet at a butt \( l \), on the line \( HK \).

\[
\text{∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞}
\]

Let’s do exactly what Desargues described:

**Task 29** Continue your construction from above:

(a) Construct the line \( ag \) which forces the placement of point \( l \) since we also want \( l \) to be on the line \( HK \).

(b) This point \( l \) is meant to be the third vertex of the first triangle (which you left unlabeled earlier).

(c) Construct the line \( lb \).

(d) Desargues claimed “it has just been shown” that \( lb \) meets \( EK \) at \( f \) which is collinear with \( c \) and \( g \). What did he mean by this? Where was this shown?

(e) You can confirm that the placement of \( l \) depends on the the other 5 vertices in the configuration. If you drag any of the vertices \( a, b, D, E, \) or \( K \) they should move the placement of \( l \) but, again, not the placement of any other vertex.

3.3 Proof of the Converse of Desargues’ Theorem, Lines in Different Planes Case

Now, let’s consider the case where the given lines lie in different planes.

\[
\text{∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞}
\]

For if the straight lines lie in different planes, one of these planes is \( HKgDag \); another is \( HKfEbf \); and another \( cbagf \): and the straight lines \( HlK, bfl, agl \), are the lines of intersection of these three planes; therefore they all meet at the butt \( l \).

\[
\text{∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞∞}
\]
Let's make sure Desargues didn’t cheat. Three points should determine a plane, so for example when he listed $H$, $K$, and $g$ in the first plane, he should have been able to justify why the other points, $D$ and $a$, must be on this plane. Do this.

Make the similar confirmation for the other two planes.

Two different planes intersect in a line, so for example the first two planes intersect in a line determined by $H$ and $K$. What are the common points that similarly determine a line of intersection for the second and third planes? The first and third planes?

Next, Desargues implicitly claimed there is a unique point of intersection, $l$, of these three lines. Is it possible that, in fact, these three planes intersect in a line rather than a point?

### 3.4 The elephant in the room

At the end of Desargues’ proof of the converse of his theorem, he addressed the issue of parallel lines in the configuration.

**For lines in different planes:**

And if, again, the same lines lie in different planes, if through points on them, $H$, $D$, $E$, $K$ there pass other straight lines $Hh$, $Dd$, $Ee$, $Kk$ which all meet at some butt at an indeterminate distance, or, to put it another way, are parallel to the one another; and these lines meet one of the planes $cbagfl$, in points such as $h$, $d$, $e$, $k$; the points $h$, $l$, $k$ lie on a straight line; the points $h$, $d$, $a$ lie on one; the points $h$, $e$, $b$ lie on one; the points $k$, $g$, $d$ lie on one; the points $k$, $f$, $e$ lie on one; and the points $c$, $e$, $d$ lie on one. For by this construction the straight lines $Hh$, $Kk$, $Hlk$ all lie in a plane; the lines $abc$, $bfl$, $klh$ lie in another; and the points $h$, $l$, $k$ lie in each of two planes. Consequently they lie on a straight line; and similarly for every other set of three points [in the proposition].

**For lines in the same plane:**

And all these straight lines lie in a plane, $cbagfl$, and each of them is divided by parallel lines through the points $H$, $D$, $E$, $K$ in the same way as the corresponding line in the three-dimensional figure. So the figure which these parallel lines have defined in the plane $hdabedgflk$ corresponds straight line for straight line; point for point; and ratio for ratio; the three-dimensional figure $abcEHIkgf$. And one can discuss their properties in the same way in the one figure as in the other, and so do without the solid figure, by using instead the figure in the plane.
Write one comment and one question about each of these excerpts.

(optional) Work through the details of these two excerpts. Provide figures, GeoGebra plots, and explanations.

Aside from the proof of the converse of Desargues’ Theorem, Desargues snuck in an interesting comment:

“...[lines] which all meet at some butt at an indeterminate distance, or, to put it another way, are parallel to the one another....”

Explain how this comment contradicts Euclidean geometry.

4 Poncelet and Gergonne

Two hundred years after Desargues there was a revived interest in synthetic geometry. We’ll consider two important concepts in projective geometry that were further developed during that period. The first is that two distinct lines always have a point of intersection. This idea is implicit in perspective art that predates Desargues, is hinted at by Desargues (just above), and would eventually become a key axiom in modern projective geometry. The second is the idea of duality. Again this is an idea that predates Desargues. For example, if you replace the vertices of a Platonic solid with faces, and the faces with vertices you obtain a (possibly different) Platonic solid, called its dual. Poncelet and Gergonne are often credited with founding modern projective geometry and so we will look at these two ideas in their writings. Jean-Victor Poncelet (1788 - 1867) and Joseph-Diez Gergonne (1771 - 1859) both contributed to the development of synthetic geometry, whose revival was largely inspired by a teacher of Poncelet, Gaspard Monge (1746 - 1818). Poncelet and Gergonne collaborated and published together, but ultimately fell out over who deserved credit for developing the idea of duality in projective geometry.

Keep in mind as your read the excerpts below that Poncelet and Gergonne were key players in the development of projective geometry; but they were neither the first, nor the definitive, players.

4.1 Lines always meet

We’ll start with a thought experiment of Poncelet. He wrote a lengthy treatise on conic sections, *Traité des propriétés projectives des figures*; it is this work that earned Poncelet credit for the advent of modern projective geometry. Here we consider an early part of his development [Poncelet, 1822, p. 18, articles 34 and 35]. Poncelet’s main focus, the connection between conic...
sections (like the ellipse) and projective transformations, would take us along a lengthy detour; instead we shall focus on his consideration of what happens when the vertex $A$ of a triangle recedes away from the other two vertices:

Let $ABC$ be any triangle in the plane . . . . Suppose the sides $AB$ and $AC$ become parallel, the vertex $A$ moves to infinity...

Gergonne expressed the same idea in a preface to a paper on duality [Gergonne, 1825-26, p. 217], and, here, without any particular context:

Frequently in the future, we will find ourselves concluding that two lines in the same plane meet in a point. But, this necessarily implies this point may well be infinitely far away.

Euclid defines parallel lines as:

Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Explain the conflict that Poncelet and Gergonne created. In particular, what is the conflict when Poncelet used the word “parallel” in his description of what happens to the point $A$?
Task 38 Suggest a modification to Euclid’s definition that would resolve this conflict.

Poncelet later explored several cases of points *passing to infinity* (again in the context of conic sections) and made the following general conclusion [Poncelet 1822, p. 50, article 96]:

In general, when a line moves continually, but otherwise in any manner, towards infinity, without leaving the plane to which it belongs, the line necessarily becomes of indeterminate direction with respect to other objects in the plane, which at the least implies, due to the principle of continuity: *all the points at infinity in a plane can be considered ideally to be along a unique line, situated itself at infinity*.[6]

Task 39 Mathematicians before Poncelet – Desargues and Pascal among others – used the idea of points at infinity to some extent. They were however, unclear about how to treat *all* the points at infinity. What is Poncelet’s solution?

One of the problematic cases of Desargues’ Theorem is if one pair of sides of the two triangles are parallel. Desargues’ Theorem says the points of intersection of corresponding sides all lie on a line. From a Euclidean perspective this doesn’t make sense if a pair of sides are parallel. One can modify the statement (perhaps you did this in Task 3) and just make a conclusion about the points of intersection of the non-parallel sides. However, if *any* two lines have a point of intersection, we can resolve the issue differently:

Task 40 Explain the case of Desargues’ Theorem where corresponding sides of two triangles are parallel. These two lines will meet at a point at infinity. Is it still true that the three points of intersection all meet on the *same* line?

4.2 You say point, I say line.

Poncelet extended an idea of duality considered by others including Philippe de La Hire (1640 - 1718) and Charles Julien Brianchon (1783 - 1864). Relative to a conic section, to each point (called a pole) corresponds a line (called a polar) and vice versa: to each line (polar) corresponds a point (pole). We’ll look at Poncelet’s construction of these dual pairs: [Poncelet 1822, p. 105, article 194]

If one inscribes on a conic section a series of chords: \(AB, A'B', A''B''\ldots\), all directed toward the same point \(P\) chosen at will in the plane of the conic, we will have:

[6]The emphasis is Poncelet’s.
1° All the points $C, C' \ldots$ which are, relative to these chords, the 4th harmonic conjugate of point $P$, will be located on the same line, the polar of point $P$.

2° All the points of intersection $L$ and $M, L'$ and $M', \ldots$ of new chords which join pairwise, the endpoints belonging to different pairs form the first, will again be located on the polar line of $P$.

3° In the same manner all the points of intersection, $T, T' \ldots$ of the pairs of tangents at the endpoints of each chord $AB, A' B' \ldots$ will be located on the polar line of $P$.

4° Reciprocally if, different points $T, T', \ldots$ of an arbitrary line in the plane of the conic, one constructs pairs of tangents to the conic, the corresponding chords of contact $AB, A' B' \ldots$ will intersect in a unique point, $P$, the pole of the line.
In items 1, 2, and 3, Poncelet described a construction to find the polar of a point, $P$. If, as in his figure, this point $P$ is located outside the conic section all three produce the same line. The third generalizes to the case when $P$ is inside the conic so let’s focus on the method in item 3.

**Task 41** Use GeoGebra to reconstruct Poncelet’s figure:

1. Instead of using chords, place a point $P$ exterior to a conic section (you can use an ellipse as Poncelet did or experiment with other conic sections), and draw three lines from the point $P$ through the conic section. Label the points of intersection between the lines and the conic section appropriately. You don’t need to construct any of the points labeled with $L$s, $M$s, or $C$s.

2. Start the construction of the polar by constructing pairs of tangents to each of the chords $AB, \ldots$. Connect the intersections of the pairs of tangents.

3. Drag the point $P$ around, both outside the conic where you first placed it, and inside the conic section and notice what happens. What happens to the polar? What happens to the polar when the point $P$ lies on the conic section?

4. Why is important to draw three lines from the point $P$? What does this show that only drawing two lines from $P$ would not? (Once we are convinced this construction is well defined, we need only use two lines.)

**Task 42** Now let’s construct a pole (point) given a line.

1. Start a new GeoGebra sketch with a conic section and a line. I suggest placing the line so it does not intersect the conic. Place two points on the line (don’t use the points GeoGebra created when you drew the line) and construct pairs of tangents to the conic. Find the intersection of the resulting chords. This is the pole of the line.

2. Drag the original line around so that it intersects the conic. What happens to the pole? You may notice that in some cases part of your construction vanishes! Why? How can you resolve this problem so that to each line corresponds a pole?

**Task 43** If you start with a point, construct its polar, then pick any point on the polar and construct its pole, what happens?

We’ll illustrate this idea of duality with a theorem from Pascal. Blaise Pascal (1623 - 1662) was contemporary with Desargues and was aware of his work. He published the following theorem at the age of 16 and, as with Desargues, much of his work has been lost. We present a modern version of
his theorem; it’s an interesting exercise to compare Pascal’s presentation with the form it typically
takes today. We encourage the interested reader to read the original as translated in [Smith, 1959,
p. 326 - 330].

Pascal’s Hexagon Theorem: If ABCDEF is a hexagon inscribed in a conic then the intersection of
pairs of opposite (extended) sides are collinear. That is: \( AB \cap DE \), \( BC \cap EF \), and \( CD \cap FA \) are
collinear.

[Task 44] Make a GeoGebra construction that illustrates Pascal’s Hexagon Theorem. It is easiest to
construct and see on an ellipse, but a hyperbole or a parabola will work as well.

[Task 45] Look for the dual to Pascal’s theorem:
1. For each side (polar) of the hexagon, construct the dual point (pole).
2. The sides of the original hexagon intersect, so the dual operation is to join the dual points.
   Do this: construct lines between the poles you constructed.
3. The six lines you constructed appear to be tangent to the conic. Why should this, in fact,
   be true? (Hint: see Task 41.)
4. The conclusion of Pascal’s statement is that three points are collinear. State the dual
   conclusion and illustrate this conclusion in your GeoGebra construction.

[Task 46] State the theorem you discovered.

Charles-Julien Brianchon (1783 - 1864), a contemporary of Poncelet and also a student of Monge,
first discovered this theorem. A translation of Brianchon’s work can also be found in [Smith, 1959,
p. 331 - 336].

Gergonne illustrates this idea of duality in the following two column comparison [Gergonne, 1825-26,
p. 219].

If two triangles, in the same plane, are such
that the lines which connect corresponding
vertices all pass through the same point; the
points of intersection of corresponding sides
will all three lie on the same line.

If two triangles, in the same plane, are such
that the points of intersection of correspond-
ing sides all three lie on the same line; the lines
which connect corresponding vertices will all
pass through the same point.

[Task 47] These two statements should be familiar. What are they saying?
Task 48 We already know there is a duality between point and line. Read Gergonne's two columns word for word and make note of all the dual correspondences:

\[
\begin{align*}
\text{line } & \leftrightarrow \text{ point } \\
\text{connect } & \leftrightarrow \text{ ??? } \\
\text{etc.}
\end{align*}
\]

Task 49 One last task: go back to the modern statement of Desargues' Theorem at the beginning of the project.

1. Carefully state its converse.
2. Use the duality idea of Gergonne and Poncelet to state the dual of Desargues’ Theorem.
References


Notes to Instructors

PSP Content: Topics and Goals

This project is meant for a Modern Geometry course or an Introduction to Proofs course. The primary goal of the project is to understand how Desargues proved the theorem in projective geometry which today goes by his name. The theorem says that two triangles are perspective from a point if and only if they are perspective from a line. You can find this elegant theorem in nearly any geometry textbook, but it often shows up in rather interesting ways. Sometimes it appears in the exercises, sometimes it appears as a theorem without proof given, sometimes it even appears as an axiom. Rarely is a classical Euclidean proof presented. Desargues is often credited with helping to found projective geometry, but his proof of this theorem is very much in a Euclidean spirit.

There are two secondary goals for this PSP: to provide an introduction to modern projective geometry and to provide a framework for students to practice “doing” mathematics. After the proof of Desargues Theorem is (almost) completed, excerpts from Poncelet and Gergonne are used to introduce two key ideas of projective geometry: that all lines intersect, and the principle of duality. We do not present proofs of these facts by Poncelet or Gergonne but rather try to understand their ideas and then apply them to Desargues’ Theorem.

The first part of the project could be used to develop further practice with Euclidean-type proofs. The second part of the project could be used as an introduction to projective geometry.

There is a fair amount of structure in the development of the project to provide guidance for students new to reading, understanding, developing, and writing proofs. Students experiment, are asked to make observations, and ultimately produce proofs.

Student Prerequisites

While not absolutely necessary, it would help if students have some prior experience with geometric proofs with a Euclidean flavor.

If the instructor desires more information about the work of Desargues, [Field and Gray, 1987] is an excellent resource.

PSP Design, and Task Commentary

Throughout this project, students are encouraged to make and experiment with GeoGebra constructions (or with similar geometric construction software). Optionally, GeoGebra constructions can be provided to the students by the instructor ready for them to experiment with. The author’s constructions are available at https://ggbm.at/yts7n7d8. However, I strongly recommend that students (and instructors) make the constructions themselves. While I am not an expert with GeoGebra I do have a few suggestions:

- Use the downloadable version.
• Place points before constructing lines. Don’t use the line tool in “open” space. The line tool in open space will automatically create two points. If you then use these points as part of the construction, you risk creating inappropriate and unintended dependencies.

• In general, lines are better to use than line segments, but sometimes line segments make things easier to see. Alternatively, the polygon tool can be used to highlight triangles.

• Hide points and lines you aren’t using. The downloadable version of Geogebra seems to facilitate this much better than the online version.

• Name points in your constructions as you go along to match the labels given in the text of the project.

• Make good use of color coding of elements in your constructions. They aid visibility.

• Make sure to tell your students to use the “place on object” or “place on intersection” tools. I’ve seen students simply place a “free” point where they think the object or intersection is.

• You can customize a template so that students only see and use the tools they should be using. I also suggest making at least one construction by hand with a pencil and straightedge – for the cultural experience, if nothing else.

4.3 Section Overview

Section 1 Desargues’ Theorem is introduced in this section. In particular, we consider the statement:

If two triangles are perspective from a point, they are perspective from a line. First, students spend some time figuring out what the statement says, and then, understanding the general idea of the proof. The key part of the proof uses Menelaus’ Theorem three times, and its converse once. These parts are left unresolved until Section 2.

Task 5: Uppercase $K$ and lowercase $k$ should both be labeled $K$.

Task 13 might be done in some Euclidean fashion, but I suggest taking an algebraic approach. Tasks 12 and 14 can be challenging. The key is to focus on one segment (Desargues’ trunk) from which three other segments (branches) emanate. Any reasonable observation may later help students make the connection to Menelaus’ Theorem.

If you want to include Desargues’ drawing here it is:
Section 2 This section takes a detour to consider Menelaus’ Theorem. I have included both Desargues’ proof and Ptolemy’s proof. Students only need to work through one in order to continue.

Here is an annotated version of Desargues’ diagram:

![Diagram of Desargues' Theorem](image)

Task 17: \( \frac{Dh}{Dh} = \frac{Hh}{HK} \cdot \frac{GK}{GA} \)

Task 19: The best answer I’ve seen to this question is in *Axiomatic Geometry* by John M. Lee [Lee, 2013, p. 221]. This is also the only place I’ve seen the converse to Menelaus’ Theorem proved.

Proposition 2 from Book VI of Euclid’s *Elements* appears here. I suggest using it as a result and not diverting further from the main goal by spending time to prove it.

Tasks 25 and 26 are straightforward if you assume Task 27. I’ve chosen to leave Task 27 without guidance just to have one completely open-ended and more challenging task. For a proof see Lee’s book [Lee, 2013, p. 221]. If you assign Task 27, you might want to hold back sharing the page of References with the students so as not to tempt them from consulting Lee for the full proof.

Section 3 I’ve provided more guidance for the construction here because the proof, in my opinion, is awkward. If you compare the verbosity of the proof of the converse to the proof of the statement, you might suspect Desargues was also unhappy with the proof. At any rate, it’s odd because the 7 dependencies given at the beginning are not limited to the intersection points of corresponding sides of triangles. Rather, they include some dependencies related to being perspective from a point. At any rate, if you agree that the proof is awkward, you can use that
to your advantage to emphasize the value of the principle of duality later.

The key step in the proof comes in part (d) of Task 28.

I have left the last part of the proof of the converse, section 3.4, brief. I feel like I would be pressing my luck with my students at this point, and it’s better to leave it to the interested student to work it out, or wait for the next section to gain a deeper understanding of what’s going on here. Again, the difficulty of this section emphasizes the power of working in a projective geometry rather than a Euclidean one.

Do notice that Task 36 is a lead-in to the next section.

Section 4 The source texts in this section do not provide theorems and proofs; rather they are ideas of Poncelet and Gergonne which would become key ideas in modern projective geometry. Poncelet’s ideas are concerned with conic sections and their projective transformations, which would be too much of a detour from our main goal. Instead, the ideas are presented, and then students are asked to apply them to Desargues’ Theorem.

The construction of poles and polars could be omitted. It’s the correspondences point↔line, connect↔intersect, etc. that are important to help see that the converse and the dual of Desargues’ Theorem are the same result. That said, the construction of poles and polars is super cool! And seeing the idea of duality in Task 44 - 46 is satisfying. This author thanks Victor Katz for the suggestion.

Suggestions for Classroom Implementation
I would deliver the project either via group work or classroom discussion, where the students provide as much of the guidance as I can get them to. Whichever method works seems to depend on the dynamics of the particular set of students. I think there are enough “easy” tasks along the way that each day could end with one or two of the next tasks given as homework. The sample schedule below often assigns the next construction as that day’s homework.

\LaTeX{} code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or ‘in-class worksheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Possible Modifications of the PSP
The converses of both Desargues’ Theorem and Menelaus’ Theorem could be left out/taken as results. But don’t leave out the statement that the converse of Desargues’ Theorem is dual to itself.

GeoGebra constructions could be provided instead of having students make them themselves. This can save time, but students seem to enjoy these constructions and, in my opinion, the act of making
the constructions is valuable.

To save several days you could present Menelaus' Theorem as a canned result and skip Section 2 altogether.

**Sample Implementation Schedule (based on a 50 minute class period)**

I have a hard enough time predicting how much time *my* students will spend on a given task. Nonetheless, here are my suggestions for how to plan the project:

Day 0  Assign Task 1.

Day 1  Introduce students to GeoGebra. Have them recreate the modern picture first. Make sure their GeoGebra constructions do the right thing. (The first few times I did this myself, I inadvertently created incorrect dependencies.) Then have them construct Desargues’ configuration. Make sure you’ve done this yourself first! Whichever tasks are not completed up through Task 8, should be assigned as homework.

Day 1b  It is vital that the students understand the building of Desargues’ configuration. If they didn’t get it in one day’s class, be willing to spend another day on it.

Day 2  Perform tasks 9 - 14, either by student group work or by instructor-led activity, depending on the size and dynamics of the class. Homework: Read Menelaus’ Theorem and Task 15. (Don’t let the students see page 12 yet, which shows the answer to Task 15!)

Day 3 and 4  (Desargues’ proof option) Tasks 16 - 22. Get as far as you can on day 3. Assign one or two tasks as homework and finish on day 4. Homework: Tasks 25 and 26 for presentation on the next day.


Day 6  First check student work on Task 28 and answer questions about this. Then have students do Tasks 29 - 33. Homework: Task 36 and read through Task 40

Day 7  Discussion: Tasks 36 - 40. Perhaps introduce the axioms of projective geometry. Homework: Task 41

Day 8  Check on Task 41. Have students do Tasks 32 - 43 in class. Homework: Task 44 - 46 (you might assign some student to start with an ellipse, while others start with a hyperbole; in any case, get them to use different configurations).

Day 9  Students present their constructions for Tasks 44 and 45, and their statement for Task 46. If time permits, have students work Tasks 47 - 48 in class. It will help if you introduce the idea of triangle (3 points and the lines they determine) vs. trilateral (3 lines and the points they determine) and some notation: $ABC$ for a triangle, $abc$ its dual. Homework: Task 49.

Day 10  Just make sure they have understood the point of doing Task 49.
Connections to other Primary Source Projects

Jerry Lodder’s geometry projects, *The Exigency of the Parallel Postulate and the Pythagorean Theorem*, and *The Failure of the Parallel Postulate and Distance in Hyperbolic Geometry* are excellent. Both are available at the TRIUMPHS website listed at the very end of this project. In particular, the first project would provide students with ample experience with Euclidean theorems and their proofs. The second project, of course, would provide yet another view of the issues concerning the parallel postulate.

Recommendations for Further Reading

For further information concerning the work of Desargues [Field and Gray, 1987] is excellent. It includes historical background, translations and commentary on Desargues’ main works.

Kline’s book contains two nice chapters on Projective Geometry and are worth reading.

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