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# Argand's Development of the Complex Plane

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## 1 Introduction

Complex numbers are a puzzling concept for today's student of mathematics. This is not entirely surprising, as complex numbers were not immediately embraced by mathematicians either. Complex numbers showed up somewhat sporadically in works such as those of Cardano, Tartaglia, Bombelli and Wallis. It wasn't until Caspar Wessel (1745–1818), a Norwegian map surveyor, that we first see a systematic and full theory of complex numbersof complex numbers Wessel's 1797 paper *Om Directionens analytiske Betegning, et Forsog, anvendt fornemmelig til plane og sphaeriske Polygoners Oplosning (On the Analytical Representation of Direction. An Attempt Applied Chiefly to Solving Plane and Spherical Polygons)* [Wessel, 1999]. It was the first to develop the geometry of complex numbers, though it was unfortunately not given much attention until about a century after it was written.. However, this work was not noticed by the mathematical community until almost a century later.

Meanwhile Jean-Robert Argand (1768–1822), a book keeper in Paris and amateur mathematician, developed his own geometric treatment of the complex numbers. He published these in 1806 at his own expense, and his name was not even on the cover! As a result, his work surfaced in an interesting way. Jean-Louis Legendre sent a copy of the book to François Français, whose brother Jacques Français used this work to give his own geometric representation of the complex numbers based on Argand's ideas. At the end of his paper, he credited an unknown mathematician and asked for this individual to come forward. Argand did, and ultimately Jacques Français and Argand argued with another mathematician, François-Joseph Servois, over the geometric versus algebraic approach.

In modern times, both are considered important. In this project, we'll follow the treatment of Argand in his essay *Imaginary quantities: their geometrical interpretation* ([Argand, 2010, 1874]), as translated by A.S. Hardy.

## 2 Are imaginary numbers real?

One of the very interesting themes we see in this work is that besides the goal of developing the algebra and geometry of imaginary numbers, Argand had the additional but just as important goal of convincing his readers of not only the appropriateness of defining imaginary numbers, but the

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very realness of imaginary numbers. Argand defended the real existence of imaginary numbers over a series of letters published in Gergonne's Journal in the early 1800s. Along with Jacques Français, they argued against the view of François-Joseph Servois.

One of Argand's arguments is by analogy. He began his essay *Imaginary Quantities: their geometrical interpretation* with the following:

Lat $a$ be any arbitrary quantity. If to this quantity another equal to it be added, we may express the resulting sum by $2a$. If we repeat this operation, the result will be $3a$, and so on. We thus obtain the series $a, 2a, 3a \ldots$, each term of which is derived from the preceding by the same operation, capable of indefinite repetition. Let us consider the series in reverse order, namely, $\ldots,\ 4a,\ 3a,\ 2a,\ a$. As before, each term of this new series may be regarded as derived from the preceding by an operation which is the reverse of the former; yet, between these series there is this difference: the first may be indefinitely extended, but the second cannot. After the term $a$, we should obtain $0$, but beyond this point the quantity $a$ must be of such a nature as to permit our operating on zero as we did on the other terms $\ldots, 4a, 3a, 2a, a, 0$, cannot be extended beyond $0$; for while we may take $1$ gram from $3, 2$, or $1$ gram, we cannot take it from $0$. Hence the terms following zero exist only in the imagination; they may therefore be called imaginary.([Argand, 2010, p. 17])

**Task 1** What is the modern terminology for these terms that "extend beyond 0?" Why do you think Argand is calling them imaginary?

But instead of a series of weights, let us consider them as acting in a pan $A$ of a balance containing weights in the other pan also; and for the purpose of illustration, let us suppose the distance passed over by the arms of the balance is proportional to the weight added or withdrawn, which indeed would be the case if a spring were adjusted to the axis. If the addition of the weight $n$ to the pan $A$ moves the extremity of the arm $A$ a distance $n'$, the addition of the weights $2n, 3n, 4n, \ldots$ will cause this same extremity to move over the distances $2n', 3n', 4n', \ldots$ which may be taken as measures of the weight in the pan $A$: this weight is zero when the pans are balanced. By placing the weights $n, 2n, 3n, \ldots$ in the pan $A$, we may obtain the results $n', 2n', 3n', \ldots$, or, by starting with $3n'$ and withdrawing the weights, the result $2n', n', 0$. But these results may be reached not only by taking the weights out of the pan $A$, but also by adding them to the pan $B$. Now the addition of weights to the pan $B$ can be continued indefinitely; and in so doing we shall obtain results expressed by $-n', -2n', -3n', \ldots$, and these terms, called *negative*, will express quantities as real as did the positive ones. We, therefore, see that when two terms, numerically equal, have opposite signs, as $3n', -3n'$, they designate the different positions of the balance arms, such that the extremity indicating the weight is in both cases equally distant from the point $0$. This distance may be considered apart from *direction*, and be then called *absolute*.

⋮

These ideas are very simple; yet it is not so easy, as it at first seems, to set them forth clearly and to give them the generality which their application requires. Moreover, the difficulty of the subject will not be questioned if we remember that the exact sciences had been cultivated for many centuries, and had made great progress before either a true conception of negative quantities was reached or a general method for their use had been devised. Moreover, it was not our intention to endeavor to state these principles more rigorously or more clearly than they are to be found in the works which deal with this subject; but simply to make two remarks on negative quantities. First, that whether a negative quantity is real or imaginary depends upon the kind of magnitude measured; and second, when we compare two quantities which are of a kind yielding negative values, the idea involved in the ratio is complex, including $1^\circ$ a relation dependent upon number, considered *absolutely* and $2^\circ$ a relation of *direction*, or of the sense in which they are estimated, a relation either of identity or opposition. ([Argand, 2010, p. 18])

**Task 2** Re-read the first paragraph of the above excerpt carefully. As you read, draw a scale that illustrates the ideas that Argand is expressing.

### 2.1 The existence of complex numbers

Argand used the idea of a "mean proportional" as a means to approach complex numbers. The mean proportional, also known as the geometric mean, between two numbers is the square root of their product. This comes from the more formal definition of the mean proportional between $a$ and $b$ as that $x$ satisfying $a : x :: x : b$.

**Task 3** How would you write $a : x :: x : b$ in modern notation? (Hint: this is often read as "$a$ is to $x$ as $x$ is to $b$".)

**Task 4** What is the mean proportional between 1 and 9? Between 9 and 16? Between $a$ and $b$?

We may also view the geometric mean in terms of triangles. Using the same example as above with 1 and 9, we seek the value $x$ which forms a triangle right with the base of length $1 + 9 = 10$. This is illustrated in Figure 1 below:

We will utilize this idea in the discussion after Task 10.

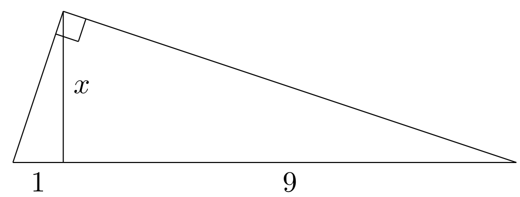



Figure 1

If now, setting aside the ratio of absolute magnitude, we consider the different possible relations of direction, we shall find them reducible to those expressed in the two following proportionals:

$$+1\colon\ +1 :: -1\colon\ -1,$$

$$+1\colon\ -1 :: -1\colon\ +1.$$

([Argand, 2010, p. 22])

**Task 5** Explain why Argand is considering "ratio of directions." Why are there only two of them?

The above two proportionals are the only ones that can be constructed with the values $+1$ and $-1$. But now Argand considers a hypothetical $x$ which would satisfy a third proportional.

Taken directly and by inversion, these proportionals show that the signs of the means are alike or different when those of the extremes are so. Now let it be required to find the geometrical mean between two quantities of different signs, that is, to find the value of $x$ in the proportional

$$+1\colon\ +x :: +x\colon\ -1,$$

([Argand, 2010, p. 23])

**Task 6** Explain what Argand means when he writes that "the signs of the means are alike or different when those of the extremes are so."

**Task 7** Determine the value of $x$ that satisfies

$$+1\colon\ +x :: +x\colon\ -1,$$

Wow. Here we have the natural emergence of $\sqrt{-1}$. Notice how it arises organically through this mean proportional.

Argand now makes an adept comparison to negative numbers and initial obstacles that mathematicians faced in defining and coming to accept them.

Here we encounter a difficulty, as when we wished to continue the decreasing arithmetical progression beyond zero, for $x$ cannot be made equal to any quantity; but, as before, the quantity which was imaginary, when applied to certain magnitudes, became real when to the idea of *absolute number* we added that of *direction*, may it not be possible to treat this quantity, which is regarded imaginary, because we cannot assign it a place in the scale of positive and negative quantities with the same success? ([Argand, 2010, p. 23])

**Task 8** What does this mean? Explain in your own words the difficulty he is expressing.

### 2.2 The geometry of complex numbers

After having developed the notion of the geometric mean and $\sqrt{-1}$, Argand now turns to how to view this value geometrically.

On reflection this has seemed possible, provided we can devise a kind of quantity to which we may apply the idea of direction, so that having chosen two opposite directions, one for positive and one for negative values, there shall exist a third – such that the positive direction shall stand in the same relation to it that the latter does to the negative. ([Argand, 2010, p. 23-24])

**Task 9** Draw the situation that Argand is describing in the above quote.

If we now assume a fixed point $K$ (Fig. 2) and the line $KA$ be taken as positive unity, and we also regard its direction, from $K$ to $A$, one writes $\overline{KA}$ to distinguish it from the line $KA$ as simply an absolute distance,

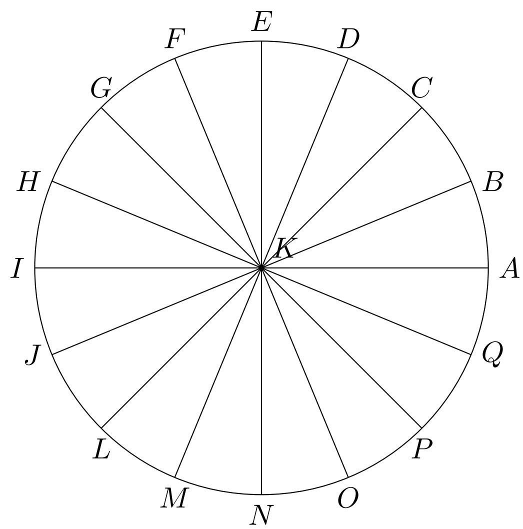



Figure 2

([Argand, 2010, p. 24])

...negative unity will be $\overline{KI}$, the vinculum[1] having the same meaning as before, and the condition to be satisfied will be met by $KE$, perpendicular to above and with a direction from $K$ to $E$, expressed in like manner by $\overline{KE}$. For the direction of $\overline{KA}$ is to that of $\overline{KE}$ as the latter is to that direction of $\overline{KI}$. Moreover we see that this same condition is equally met by $\overline{KN}$ as well as by $\overline{KE}$, these two last quantities being related to each other as $+1$ and $1$. They are, therefore, what is ordinarily expressed by $+\sqrt{-1}$, and $-\sqrt{-1}$. ([Argand, 2010, p. 25])

**Task 10** Explain the difference between $KA$ and $\overline{KA}$.

**Task 11** Draw $\overline{KI}$, $\overline{KE}$, and $\overline{KA}$.

We give a possible interpretation of what Argand has in mind. The points $K$ and $A$ create a length which is considered "unit" or "+1." Argand then constructs the negative unit $KI$ in the opposite direction and finally, $KE$ perpendicular to $KA$. Recall the discussion preceding Task 5 of the mean proportional and its interpretation in terms of triangles. If we draw the line from $A$ to $E$ and $I$ to $E$, we obtain the following triangle:

[1]Authors' note: This is the technical name of the bar over the symbols $KI$. It is also the name of the the fraction $\frac{a}{b}$ as well as many others bars in mathematical contexts.

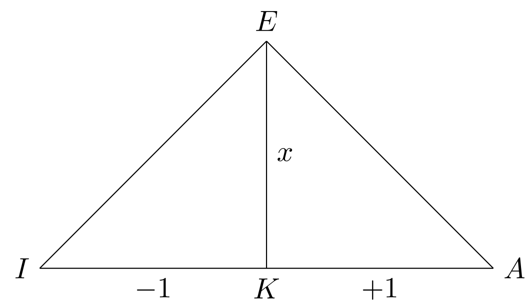



Figure 3

Computing the mean proportional, we obtain

$$+1 \colon x :: x \colon -1,$$

or $x = \sqrt{-1}$.

Argand now looks at other mean proportionals.

In an analogous manner we may insert other mean proportionals between the quantities just considered. Thus to construct the mean proportional between $\overline{KA}$ and $\overline{KE}$, the line CKL must be drawn so as to bisect the angle AKE (see Fig 2), and the required mean will be $\overline{KC}$ or $\overline{KL}$. So the line GKP gives in like manner the means between $\overline{KE}$ and $\overline{KI}$ or between $\overline{KA}$ and $\overline{KN}$. We shall obtain in the same way $\overline{KB}, \overline{KD}, \overline{KF}, \overline{KH}, \overline{KJ}, \overline{KM}, \overline{KO}, \overline{KQ}$ as means between $\overline{KA}$ and $\overline{KC}, \overline{KC}$ and $\overline{KE}, \ldots$ and so on. ([Argand, 2010, p. 25-26])

**Task 12** Draw the mean proportionals and angles noted above.

We may use the same idea of the mean proportional to compute other values. For example, the line $KC$ bisects the 90° angle made by $KA$ and $KE$. The mean proportional $x$ then satisfies

$$1 \colon x :: x \colon \sqrt{-1},$$

or $x = \sqrt[4]{-1}$.

**Task 13** Compute the mean proportional for $\overline{KQ}$ and $\overline{KA}$.

Similarly we might insert a greater number of mean proportionals between two given quantities, and the number of constructions involved in the solution would be equal to the number of ratios of the required series. Thus, for example, to construct two means $\overline{KP}, \overline{KQ}$, between $\overline{KA}$ and $\overline{KB}$, we should have the three ratios:

$$\overline{KA} : \overline{KP} :: \overline{KP} : \overline{KQ} :: \overline{KQ} : \overline{KB},$$

and necessarily,

$$\text{angle}(\overline{AKP}) = \text{angle}(\overline{PKQ}) = \text{angle}(\overline{QKB},$$

the vinculum indicating that these angles are similarly situated with respect to the bases $AK, PK, QK$. Now this may be effected in three ways, namely, by trisecting $i)$ the angle $AKB$; $ii)$ the angle $AKB$ increased by $360°$ $iii)$ the angle $AKB$ increased by twice $360°$ ...([Argand, 2010, p. 26])

**Task 14** Given angle $AKB$, draw three pictures that illustrate Argand's three possibilities for angles satisfying $\text{angle}(\overline{AKP}) = \text{angle}(\overline{PKQ}) = \text{angle}(\overline{QKB})$

...From these reflections it follows that we may generalize the meaning of expressions of the form $\overline{AB}, \overline{CD}, \overline{KP}, \ldots$, every such one representing a line of a certain length, parallel to a certain direction, the latter taken definitely in one of the two opposite senses which this direction presents, with any point as an origin; these lines themselves being capable of representing magnitudes of another kind.

As they are to be the subject of the following investigations, it is proper to give some special designation. They will be called *lines having direction* or, more simply, *directed lines*. They will be thus distinguished from *absolute* lines whose length is considered without regard to direction. ([Argand, 2010, p. 29-30])

**Task 15** How would you specify the length and direction of $\overline{KA}$? $\overline{KE}$? What term do we use today for directed lines?

Applying the terms of common usage to different varieties of directed lines which arise in connection with a primitive unit $\overline{KA}$, it is seen that every line parallel to the primitive direction is expressed by a real number, that those perpendicular to it are expressed by imaginary numbers or by the form $\pm a\sqrt{-1}$ and, finally, that those having other directions are of the form $\pm a \pm b\sqrt{-1}$, and are composed of a real part and an imaginary part.([Argand, 2010, p. 30-31])

We shall soon see how this leads naturally to coordinate axes. For now, we return to Argand, who continues to argue for the very existence of imaginary numbers.

But these lines are quantities quite as real as the positive unit; they are derived from it by the association of the idea of direction with that of magnitude, and are in this respect like the negative line, which has no imaginary signification. The terms *real* and *imaginary* do not therefore accord with the above exposition. It is needless to remark that the expressions *impossible* and *absurd*, sometimes met with, are still less appropriate. The use of these terms in the exact sciences in any other sense than that of *not true* is perhaps surprising. An absurd quantity would be one whose existence involved the truth of a false proposition; as, for example, the quantity $x$, satisfying at once $x = 2, x = 3$, whence $2 = 3$. The admission of such a quantity into the calculus would entail consequences as contradictory as $2 = 3$; but the results obtained from the use of the so-called imaginaries are in all respects conformable to those derived from reasonings in which only real quantities appear. ([Argand, 2010, p. 31-32])

**Task 16** Using Argand's analogies (or ones of your own), convince a non-mathematically inclined friend of the reality of imaginary numbers.

## 3 The Complex Plane

In this section, we use Argand's ideas about the geometry of complex numbers to develop the modern understanding of the complex plane.

It is to be observed that while there exists an infinite variety of directed lines, practically they are all referred, as will be shortly shown, to $\overline{KA}, \overline{KC}, \overline{KB}, \overline{KD}$ the position unit being $\overline{KA}$ the negative $\overline{KC}$ and the means $\overline{KB}$ and $\overline{KD}$.

([Argand, 2010, p. 33])

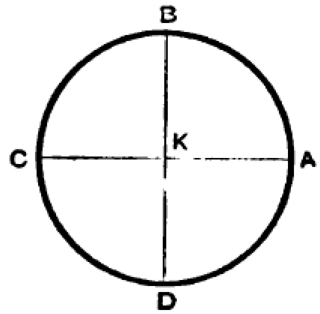
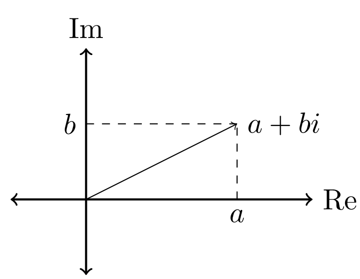



Figure 4

**Task 17** What does Argand mean by "there exists an infinite variety of directed lines"?

It is furthermore convenient to classify any two opposite directions under one head, to which we shall apply the term *order*. The primitive $\overline{KA}$ with its negative $\overline{KC}$ we shall designate as the *prime order*, and the means $\overline{KB}$ and $\overline{KD}$ as the *medial order*. We shall speak of a *prime quantity* or *medial quantity* when we refer to one of a *prime* or *medial order*, respectively. ([Argand, 2010, p. 33])

This leads to a natural place for us to introduce modern notation and terminology to the reader. Today, we refer to the axis through $KA$ as the **real axis** while the axis along $KB$ is called the **imaginary axis**. In this coordinate system, the plane becomes the **complex plane**. When a **complex number** is written as $a+bi$ for $a, b$ real numbers and $i = \sqrt{-1}$, we say that it is expressed in **rectangular coordinates**. The number $a$ is called the **real part** while $b$ is called the **imaginary part**. We let $\mathbb{C}$ stand for the set of all complex numbers. For $z = a + bi \in \mathbb{C}$ we write $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$ to denote the real and imaginary parts, respectively.




**Task 18** (a) What is the relation between $KA$, the prime order, the real axis, and the complex number $+1$?

(b) What is the relation between $KB$, the medial order, the imaginary axis, and the complex number $i$?

### 3.1 Addition in the complex plane

We are now to examine the various ways in which directed lines are combined by addition and multiplication, and to determine the resulting constructions. Suppose, first, that we have to add to the positive prime line $\overline{KP}$, the line $\overline{KQ}$, also a positive prime. The construction would not differ from that of finding the sum of the absolute lines $KP, KQ$; it consists in laying off the distance $PR = KQ$ on the prolongation of $KP$. We then have $\overline{KP} + \overline{KQ} = \overline{KP} + \overline{PR} = \overline{KR}$. To add a negative prime line $\overline{QK}$ to another $\overline{PK}$, the construction is the same, but in the opposite direction. ([Argand, 2010, p. 37-38])

**Task 19**

(a) Why do you think Argand introduces $PR$ as a renaming of $KQ$?

(b) Write down the corresponding equation for the addition $\overline{QK} + \overline{PK}$.

In general, if we are to add two lines of the same direction $\overline{AB}, \overline{AC}$, we take in this direction, $PQ = AB, QR = AC$, and we have $\overline{PQ} + \overline{QR} = \overline{AB} + \overline{AC} = \overline{PR}$. If we are to add to the positive line $\overline{KP}$ the negative $\overline{QK}$, we take the a distance $PS = QK$ in the negative direction from $P$, and obtain $\overline{KP} + \overline{QK} = \overline{KS} = \overline{QP}$. The same course is pursued for any other order.([Argand, 2010, p. 38])

**Task 20** Draw a carefully chosen selection of pictures to illustrate Argand's rules for addition of lines in the same direction.

Argand now tells us how to add lines going in different directions.

Now the principle underlying these constructions is that we regard $P$, the final point of $\overline{KP}$, as the initial point of the line to be added, and that we take respectively for the initial and final points of the sum, the initial point of $\overline{KP}$ and the final point of the added line. Applying this same principle to lines of other orders, we conclude that $K, P, R$, being any points whatever, we always have $\overline{KP} + \overline{PR} = \overline{KR}$. ([Argand, 2010, p. 38-39])

**Task 21** (a) Draw multiple pictures to illustrate Argand's rules for addition of lines that go in different directions. Use a variety of examples to illustrate the wide variety of possibilities.

(b) Does it matter which line is considered the "line to be added"? That is, does it matter in what order one adds two lines? Explain graphically.

Let's now look at addition using modern notation.

**Task 22** (a) Write an algebraic formula for the sum of two complex numbers $z = a+bi$ and $w = c+di$.

(b) Show algebraically that addition of complex numbers is commutative, that is, show that $z + w = w + z$ for all complex numbers $z$ and $w$.

(c) Show both algebraically and geometrically that addition of complex numbers is associative. That is, show that $(z + w) + u = z + (w + u)$ for all complex numbers $z$, $w$, and $u$.

While Argand doesn't seem to directly address a zero element in the complex numbers, modern treatments make it essential. We let 0 be the degenerate directed line segment that coincides with the origin. This is essentially what Argand calls $K$. This would be the line segment with length 0 no direction. This will also more readily allow us to discuss additive inverses.

**Task 23** (a) Show that 0 is the additive identity in the complex numbers. That is, show that $z + 0 = z = 0 + z$ for all complex numbers $z$.

(b) Show geometrically that $\overline{KA} + \overline{AK} = 0$.

The additive inverse of a complex number $z$ is a complex number $w$ such that $z+w = 0 = w+z$. The above task shows that $\overline{KA}$ and $\overline{AK}$ are additive inverses of each other. If you have taken a course in abstract algebra, you may have proven that in rings the additive inverse of an element is unique. As such, it makes sense to denote the additive inverse of a complex number $z$ by $-z$, which we read 'the opposite of $z$,' 'the additive inverse of $z$,' or sometimes, lazily, just 'negative $z$'.

**Task 24** (a) What is the additive inverse of $z = a + bi$?

(b) What is the additive inverse of $z = 0$? Of $z = 1$? Of $z = i$?

Having defined additive identity and additive inverse, we can now introduce the concept of subtraction. As is common in number systems, subtraction is defined in terms of addition. In particular, we define subtraction of complex numbers $z$ and $w$ as follows: $z - w := z + (-w)$. That is, $z - w$ is defined to be $z$ plus the additive inverse of $w$.

**Task 25** (a) Is subtraction of complex numbers commutative? Show or give a counter-example using specific complex numbers.

(b) Is subtraction of complex numbers associative? Show or give a counter-example using specific complex numbers.

The **modulus** of a complex number $z$ is simply its length, denoted $|z|$.

**Task 26** (a) Find the following

(i) $|0|$

(ii) $|KA|$

(iii) $|i|$

(iv) $|3+4i|$

(v) $|3-4i|$

(vi) $|a+bi|$

(vii) $|a-bi|$

(b) Describe all of the complex numbers of length 1? Of length 2? of length $n$?

(c) Look back at Figure 4. In terms of the complex numbers, how would you describe the circle of points there?

### 3.2 Multiplication in the complex plane

The reader may at this point recognize that we are defining operations for a number system. It makes sense for us to now move to multiplication, as Argand did. As before, we present his treatment, and then translate it to modern terms.

> Let us now pass to the multiplication of directed lines, and let us first construct the product $\overline{KB} \times \overline{KC}$, the factors being units but not prime units.

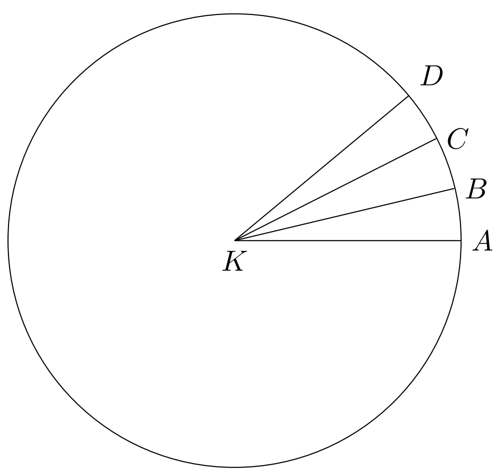



Figure 5

> Construct the angle $\overline{CKD} = \overline{AKB}$. From what was said in [i) in Argond quote directly preceding Task 14], we have $\overline{KA} : \overline{KB} :: \overline{KC} : \overline{KD}$, whence $\overline{KA} \times \overline{KD} = \overline{KB} \times \overline{KC}$; but $\overline{KA} = +1$, hence $\overline{KB} \times \overline{KC} = \overline{KD}$. Therefore, to construct the product of two directed radii, lay off, from the origin of arcs, the sum of the arcs corresponding to each radius, and the extremity of the arc thus laid off will determine the position of the radius of the product... ([Argand, 2010, p. 40-41])

Here Argand tells us how to multiply two complex numbers that are each units; that is, have length 1.

**Task 27** Geometrically illustrate how to perform the following multiplication on the unit circle:

1. $1 \cdot 1$
2. $-1 \cdot i$
3. $\overline{KB} \cdot \overline{KD}$ (Figure 5)

If the factors are not units, they can be put under the form $m.\overline{KB}, n.\overline{KC}, \ldots, m$ and $n$ being coefficients or positive prime lines, and the product would be $(mn\ldots.).\overline{KB}.\overline{KC}\ldots.).\overline{KP}$. Now, the product of the positive prime line $mn\ldots$ by the radius $\overline{KP}$ is this very line, drawn in the direction of this radius. Division is the inverse of this operation, and its explanation in detail is unnecessary.([Argand, 2010, p. 41])

**Task 28** Use Argand's definition of multiplication of complex numbers to draw the product $2\overline{KD} \cdot \overline{KB}$ (Figure 5).

Recall that in a general number system, an element $u$ is a multiplicative identity if $au = a$ for all elements $a$.

**Task 29** In the complex numbers, is there a multiplicative identity? If so, what is it? Explain.

Recall that in a general number system, an element $a$ is the multiplicative inverse of $b$ if $ab = 1 = ba$, where 1 is the multiplicative identity. In the complex numbers, we say that $w$ is the multiplicative inverse, often called just inverse, of $z$ if $zw = 1 = wz$. Following established notation, we write $w = \frac{1}{z}$ or $z^{-1}$.

**Task 30** Find the multiplicative inverse of $1, i, -i$, and $-1$.

### 3.3 de Moivre's theorem

After having established much of the basics of the geometry of the complex plane, Argand proved some applications.

One of the immediate consequences that is still frequently used to this day is de Moivre's (1667–1754) Theorem, which states that for an angle $\theta$ and positive integer $n$, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$. We follow Argand's development.

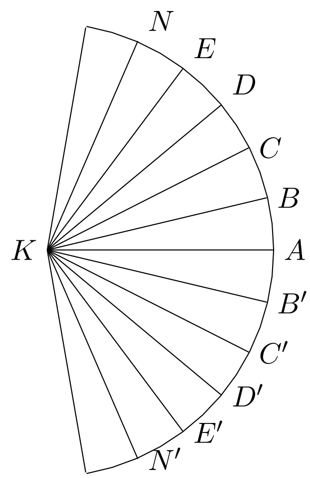



Figure 6

If $AB$, $BC$, ..., $EN$ are equal arcs, $n$ in number, and we make $\overline{KB} = u$, we shall have $\overline{KC} = u^2$, $\overline{KD} = u^3, \ldots, \overline{KN} = u^n$.([Argand, 2010, p. 42])

**Task 31**

(a) While he does not explicitly seem to note it, Argand is making an assumption about the modulus of the arc. What assumption is he making? Why is this result not true without this assumption?

(b) Use the definition of multiplication of directed lines to verify the claim that $\overline{KN} = u^n$.

Now let $AB, BC, \ldots, EN$ be equal arcs, $n$, in number; then $\overline{KN} = \overline{KB}^n$; but $\overline{KN} = \overline{Kv} + \overline{vN}$ and $\overline{KB} = \overline{K\beta} + \overline{\beta B}$; hence

$$\overline{Kv} + \overline{vN} = (\overline{K\beta} + \overline{\beta B})^n.$$

Let the arc $AB = a$, and, therefore, $AN = na$; then $\overline{K\beta} = \cos a, \overline{Kv} = \cos na, \overline{\beta B} = \sqrt{-1}\sin a, \overline{vN} = \sqrt{-1}\sin na$; and the above equation becomes

$$\cos(na) + \sqrt{-1}\sin(na) = (\cos a + \sqrt{-1}\sin a)^n.$$

This theorem, expressed in the ordinary notation by

$$\cos(na) \pm \sqrt{-1}\sin(na) = (\cos a \pm \sqrt{-1}\sin a)^n,$$

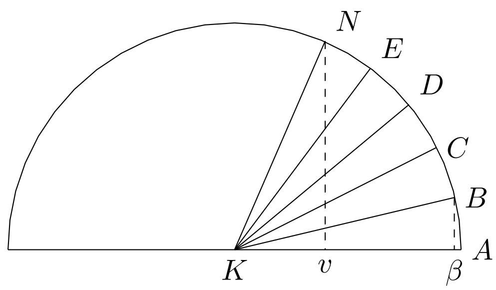



Figure 7

is a fundamental one in the theory of circular functions; among its uses is the expansion of $\sin x$ and $\cos x$ into series ([Argand, 2010, p. 45])[2]

**Task 32** Let us verify each part of Argand's claim.

(a) Use Task 30 to show that $\overline{KN} = \overline{KB}^n$.

(b) Verify that $\overline{KN} = \overline{Kv} + \overline{vN}$ and $\overline{KB} = \overline{K\beta} + \overline{\beta B}$.

(c) Use Argand's definition of cos and sin to show that
$\overline{K\beta} = \cos a, \overline{Kv} = \cos na, \overline{\beta B} = \sqrt{-1}\sin a$, and $\overline{vN} = \sqrt{-1}\sin na$.

(d) Conclude de Moivre's Theorem.

### 3.4 Polar and rectangular coordinates

The form $\cos a + \sqrt{-1}\sin a$ of a complex number from de Moivre's theorem is a special case of the more general form of a complex number using polar coordinates: Let $r$ be a real number and $0 \leq \theta \leq 2\pi$ any angle. The complex number $z := r(\cos(\theta) + i\sin(\theta))$ is called **polar form** of the complex number $z$.

We have already seen that $a + bi$ for $a, b$ real numbers and $i = \sqrt{-1}$ is rectangular coordinates of a complex number. Argand has established that complex number can be written in both polar and rectangular form. However, Argand does not provide a formula that translates from rectangular to polar coordinates or vice-versa. For example, suppose we are given the line $3 + 7i$ in rectangular coordinates. What is its length and angle? In other words, what are the polar coordinates for this line? We will work through exercises to compute this.

**Task 33** Let $a + bi = A(\cos\theta + i\sin\theta)$ and consider the following diagram:

(a) Use the diagram and the Pythagorean theorem to find $A$ in terms of $a$ and $b$.

[2] We have replaced Argand's use of the archaic symbol $\sim$ with $\sqrt{-1}$ to avoid confusion.

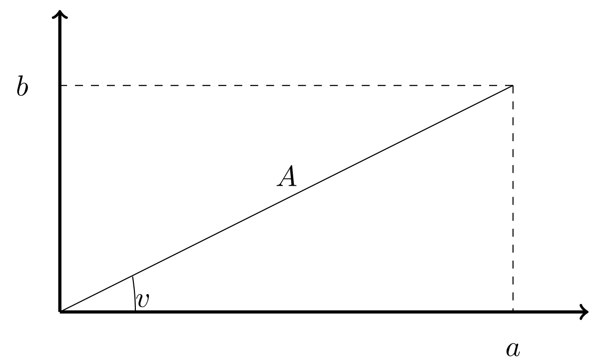



Figure 8

(b) Use the diagram and the definition of $\tan(\theta)$ to compute $\theta$ in terms of $a$ and $b$.

(c) Write $a + bi$ in the form $r(\cos(\theta) + i\sin(\theta))$.

We now connect Argand's definition of multiplication to this new form.

**Task 34** Use Argand's definition of multiplication from Section 3.2 to compute the product of $zw$ when $z = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $w = r_2(\cos(\theta_2) + i\sin(\theta_2))$.

For our final step, we show that multiplication of two complex numbers in rectangular form is what we would expect using the distributive property; that is, because Argand only defines multiplication in polar form, we must use his definition of multiplication in polar form to derive a formula for multiplication in rectangular form. Thus we cannot assume any particular formula for multiplying complex numbers in rectangular form.

**Task 35** (a) Let $z = a + bi = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $w = c + di = r_2(\cos(\theta_2) + i\sin(\theta_2))$. Use the sum and difference formulas for sine and cosine to show that if $zw = (ac-bd)+(ad-bc)i$.

(b) Compute $(a + bi)(c + di)$ as if the distributive property holds.

(c) What can you conclude? Write a formula for multiplication in each of polar and rectangular form.

Let's practice.

**Task 36** (a) Use the formula for rectangular multiplication to compute the following products

(i) Compute $(3 + 4i)(6 + 5i)$.

(ii) Compute $(4 - 2i)(1 + i)$.

(iii) Compute $(1 - 2i)(-2 - 3i)$.

(iv) Compute $(a + bi)(a - bi)$.

(b) For each of the above, estimate graphically what the product will be based on plotting the rectangular coordinates and using Argand's definition of multiplication.

**Task 37** Think through the following both in terms of the algebraic formula for multiplication of complex numbers in rectangular coordinates as well as the geometric definition provided by Argand.

(a) What happens to a complex number when it is multiplied by a number with imaginary part 0? By a number with real part 0? How does this arise algebraically? Geometrically? Discuss in terms of rotations.

(b) What happens to a complex number when it is multiplied by a number with no real part (a number with no imaginary part)? How does this arise algebraically? Geometrically? Discuss in terms of rotations.

### 3.5 Division, Conjugate, and Modulus

**Task 38** (a) Find a complex number $z$ that satisfies $3z = 1$. Explain this graphically.

(b) Find a complex number $z$ that satisfies $\frac{3}{5}z = 1$. Explain this graphically.

(c) Find a complex number $z$ that satisfies $zi = 1$. Explain this graphically.

(d) Find a complex number $z$ that satisfies $(3 + 2i)z = 1$.

We now naturally turn to division of lines. Notice that Argand defines division in terms of multiplication.

Division is the inverse of this operation, and its explanation in detail is unnecessary.([Argand, 2010, p. 41])

Argand leaves a lot for us to fill in here. For complex numbers $x$ and $y$, we define $x$ divided by $y$, written $\frac{x}{y}$ to be the product of $x$ with the inverse of $y$. That is, $\frac{x}{y} = x \cdot \frac{1}{y}$.

**Task 39** (a) Show that $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$.

(b) Find a formula for $\frac{a+bi}{c+di}$ with no imaginary part in the denominator.

A couple of very important concepts arose in the above computation. In addition to the complex number $c + di$, we saw that the related complex number $c - di$ and their product, the real number $c^2+d^2$ appeared more than once. In general, if $a+bi$ is a complex number, its **(complex) conjugate** or **conjugate** is written $\overline{a + bi} = a - bi$. Furthermore, the **absolute value, length, magnitude** or **modulus** of $a + bi$ is written by $|a + bi| = \sqrt{a^2 + b^2}$.

**Task 40** In the following, let $z = a + bi$ and $w = c + di$. Prove that

(a) $\overline{\overline{z}} = z$.

(b) $\overline{z \pm w} = \overline{z} \pm \overline{w}$.

(c) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

(d) $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$

(e) $|z| = |\overline{z}|$

(f) $|z \cdot w| = |z| \cdot |w|$

(g) $|-z| = |z|$

(h) $|z+w|^2 = (z+w)\overline{(z+w)}$

$\boxed{\boxed{\textbf{Task 41}}}$ Let $z = a + bi$. Prove that $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

## References

J.R. Argand. *Essai sur une manire de reprsenter des quantits imaginaires dans les constructions gomtriques*. Gauthier Villars, Paris, 1874. 2e dition.

J.R. Argand. *Imaginary Quantities: Their Geometrical Interpretation*. Kessinger Publishing, LLC, 2010. translated by A.S. Hardy.

J. B. Conway. *Functions of one complex variable*, volume 11 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1978.

A.P. Knott. The history of vectors and matrices. *Mathematics in School*, 7(5):32–34, Nov. 1978.

D. E. Smith. *A source book in mathematics*. 2 vols. Dover Publications, Inc., New York, 1959.

C. Wessel. On the analytical representation of direction. An attempt applied chiefly to solving plane and spherical polygons. *Mat. Medd. Danske Vid. Selsk.*, 46(1):103–143, 1999. Caspar Wessel: On the analytical representation of direction, translated by Flemming Dumhus.

# Notes to Instructors

## PSP Content: Topics and Goals

This Primary Source Project draws on the work of Jean-Robert Argand (1768–1822), a book-keeper in Paris and amateur mathematician, following closely his essay *Imaginary quantities: their geometrical interpretation* ([Argand, 2010]), as translated by A.S.Hardy.

The goal of this project is to introduce the basics of complex numbers from a geometric perspective, including important and fundamental algebraic connections and de Moivre's Theorem. There are many ways that instructors could use this project in their classroom. It could be used in the first couple of weeks of an undergraduate (or possibly graduate) introduction to complex analysis course, a part of a capstone course, or even a special project in a trigonometry or pre-calculus course that introduces complex numbers. The latter, of course, would require more support from the instructor.

## Student Prerequisites

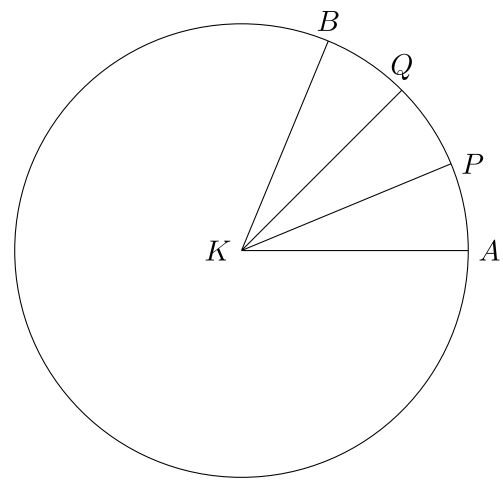
Strictly speaking, there are really no prerequisites beyond some basic trigonometry. However, as is often the case, the prior mathematical background of the students will significantly impact pacing and amount of support needed from the instructor.

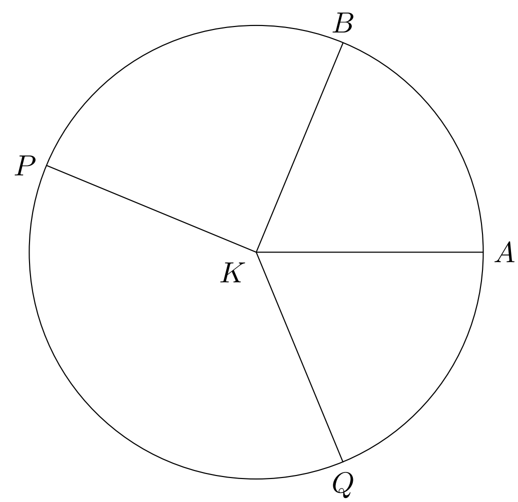
In order to be effective and set the tone for the rest of the course, there must be student buy-in for the project. Some students will feel that the project is a waste of time because you are developing material about the complex plane that "we already know." Yet this, of course, is precisely the point. We have sometimes found that "pulling out the trust card" can help quell student concerns, especially if you have a good rapport with your students. Perhaps let the students know that they are going to do something very different from what they are used to and that it might seem frustrating at first. Letting the students know that you know that the project might be frustrating is a great way to show that you desire to see them succeed.

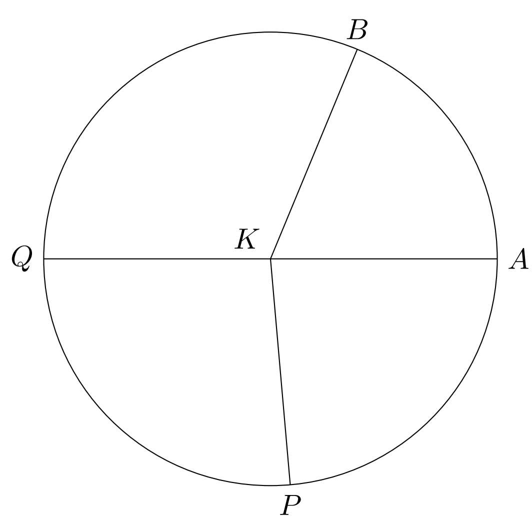
## PSP Design, and Task Commentary

Tasks 5 and 7 are great for discussion questions in class, either in small groups or whole class discussions. They would perhaps be more challenging to assign as homework problems for students to submit.

We share the solution to Task 13 here. Much more commentary and discussion of the PSP as a whole and in parts is given in Section 3.5. The solution to Task 13 is given below:




LaTeX code of this entire PSP is available from the authors and can be used to facilitate preparation of advanced preparation or reading guides, 'in-class worksheets' based on tasks included in the project, homework assignments, and slide decks. The PSP itself can also be modified by instructors as desired to better suit their goals for the course. The authors would be thrilled to receive a copy of any modifications or supplementary material produced that you're willing to share with them.

## Sample Implementation Schedule (based on a 50-minute class period)

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students.

- Pre-assignment: Have students read Section 1 introducing both the work of Argand and the project. In addition, have students read the beginning of Section 2, stopping at Section 2.1. The students should do Tasks 1 and 2 for homework.

- Day 1 (Sections 1 and 2.1): Begin class by discussing the assignment for the day. Argand's thought experiments are quite interesting, and hopefully students will be intrigued. This could easily move into a more philosophical discussion about the existence of imaginary numbers and the existence of numbers in general. Depending on the interest of the instructor, this could be a very fruitful direction to pursue. If the instructor is less interested in leading a philosophical discussion, the instructor can move the discussion in the direction of what Argand is trying to do in the reading for the day. It seems that a reasonable interpretation of Argand's words is that he is trying to convince the reader of the reasonableness of the existence of imaginary numbers. Again, if the instructor would rather not get into the philosophical issues here, the instructor can simply tell the students that this is Argand's assertion and move on.

  The rest of Section 2 is devoted to constructing the number $i$ and its relationship to the real numbers from a geometric perspective. It is obtained by studying the "mean proportional," a concept that most students have not seen before. There are tasks devoted to defining and exploring this concept (which students will work on the next day). To begin this process, have students read and work through Tasks 3-6 in groups for 10-15 minutes. The value that Task 6 is looking for is $x = i$, and in fact, the PSP gives the answer to Task 6 in the very next line. We have constructed the PSP so that there is a natural page break after Task 6, so the instructor could choose to give pages 1-4 to the students, withholding the rest of the PSP for the time being. So the point for now is that in groups, students should come to the definition of $i$ on their own. After students have worked in groups for a while, the rest of the class can be devoted to discuss how $i$ has naturally emerged, as well as read together and discuss the meaning behind Argand's words in the quote preceding Section 2.2.

- Day 2 (Section 2.2): The goals for today are to develop some of the basics of the geometry of complex numbers. This includes developing a complex number as a vector and studying the angles. Begin class by having students work in groups or individually for a few minutes. Have them read the excerpt at the beginning of the section and come up with a picture to answer Task 8. It can then be an interesting class discussion to have a representative from each group (or a few groups volunteer) to draw their picture on the board, comparing and contrasting. Given that many of the students "already know the answer," all groups may draw the same picture; namely, the $x - y$ plane. If desired, push back on the "obvious" answer a little. Is this the only possible solution? Can you explain exactly how you see this picture in his words? That is, justify every part of your picture with some of Argand's words. After this discussion, have students return to their groups to read and work through Tasks 9-12. You may want to remind them of the mean proportional discussed yesterday and that they will be working through this idea in these tasks. Then you can have a class discussion about what the mean proposition is and why it seems like a useful concept. In particular, after the diagram after Task 11, the mean proportional is of this picture is determined to be $i$. The connection with the derivation of $i$ from rations made yesterday should be made. Finally, students should spend the rest of the class working on Tasks 13-15. Whatever is not finished in class can be done for homework.

- Day 3 (Section 3.1): The goal for the rest of the project is to complete Section 3, accomplished over three class periods. This section bridges much of the thought of Argand with the modern viewpoint, language, and notation. Have students work through the beginning of this section (until Section 3.1) in groups, completing the tasks. As a class, regroup to discuss student findings. Here the goal is to help students start to see the connection between Argand and the modern viewpoint. This connection will be further emphasized and explored in Section 3.1. The remainder of the day should be devoted to Section 3.1. There are many tasks here, and the instructor has some options. One option is to have the students work on this section in groups for the rest of the class period, while you go around the room listening to group discussion, answering questions, and inserting yourself where appropriate. There are opportunities for whole class discussion. Much like the discussion of Task 8 suggested for Day 2, there can be a class discussion of Task 20 and what are some reasonable pictures to convey what Argand has in mind for adding two complex numbers. The students should be cautioned against adding these complex numbers algebraically, but rather, to add them in a way that is true to Argand's words. It is then the job of Task 21 to make the connection between the geometric addition of Argand and the modern, algebraic definition. Certain tasks or all of them may then be assigned for homework.

- Day 4 (Sections 3.2 and 3.3): Day 4 breaks nicely into two equal parts: the first half being devoted to constructing complex multiplication, and the second to deriving de Moivre's theorem. If the group work has been going well so far, the instructor can have students work in groups on Section 3.2 for most of the first half, stopping 20 minutes in for a 5 minute class discussion on possibly Task 27. This task asks students to give the geometric explanation of complex multiplication, and it is a good idea to make sure that students see this conceptually. The last 25 minutes can then be devoted to students working in groups on Section 3.3. If the group work hasn't been going as well or the students seem to want a break from the group work, try working some of the PSP as a class. The instructor or a student volunteer can read a quote out loud to the class, and the class can begin a group discussion on how to do the follow up task. Listening to some one read a quote out loud, if done thoughtfully, is actually a much more enjoyable experience than one might think. It is certainly worth trying if the instructor is searching for something new to try.

- Day 5 (Sections 3.4 and 3.5): It is worth beginning this final day by reviewing de Moivre's theorem with the class as a whole. Make sure they understand the statement of the theorem and that it is one that they should probably commit to memory, as it will most likely be used in the future. Like yesterday's class, this class can be divided into two parts. The first being devoted to Section 3.4 and the second to Section 3.5. As before, there are many, many tasks in this section and the instructor should take a look at them beforehand to see which ones to assign and which ones to skip.

### Connections to other Primary Source Projects

A very similar project, written by the same authors, titled "An Introduction to the Algebra of Complex Numbers and the Geometry in the Complex Plane," develops complex numbers through the work of the Norweigan surveyor Caspar Wessel.

The PSP entitled "The Logarithm of $-1$", written by Dominic Klyve, treats the logarithm function on negative and complex numbers. While historically that work came before that of Wessel's, in the context of a modern complex analysis course, implementation of that PSP would generally come later in the term than the current one.

Finally, the current project connects loosely with Danny Otero's PSP entitled "A Genetic Context for Understanding the Trigonometric Functions."

Project titles along with links are given below.

- *The Logarithm of -1*
  `https://digitalcommons.ursinus.edu/triumphs_complex/1/`
- *A Genetic Context for Understanding the Trigonometric Functions*
  `https://digitalcommons.ursinus.edu/triumphs_precalc/1/`
- *An Introduction to the Algebra of Complex Numbers and the Geometry in the Complex Plane*
  `https://digitalcommons.ursinus.edu/triumphs_complex/2/`

The most recent versions of these PSPs can be found onthe TRIUMPHS website.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.