2017

Bolzano on Continuity and the Intermediate Value Theorem

Dave Ruch
ruch@msudenver.edu

Bolzano on Continuity and the Intermediate Value Theorem

David Ruch*

February 25, 2023

The foundations of calculus were not yet on firm ground in the early 1800s. Mathematicians such as Joseph-Louis Lagrange (1736–1813) made efforts to put limits and derivatives on a firmer logical foundation, but were not entirely successful.

Bernard Bolzano (1781–1848) was one of the great success stories of the foundations of analysis. He was a theologian with interests in mathematics and a contemporary of Gauss and Cauchy, but was not well known in mathematical circles. Despite his mathematical isolation in Prague, Bolzano was able to read works by Lagrange and others, and published work of his own.

This project investigates results from his important pamphlet *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*[1] [Bolzano, 1817]. We will read excerpts from this paper related to two major theorems that Bolzano studied in it. The first of these is the main theorem in Bolzano's Section 12, where he stated and proved a property of bounded sets. This inspired Weierstrass decades later to prove a version of the theorem nowadays called the Bolzano-Weierstrass Theorem.[2] The second major theorem, which appears in Section 15 of Bolzano's pamphlet, concerns continuous functions, and some version of this result is found in nearly every introductory calculus text. Naturally Bolzano's concept of continuity is vital for understanding both of these theorems, so we will first study it.

1 Bolzano's Definition of Continuity

Bolzano was very interested in logic, and he was dissatisfied with many contemporary attempts to prove theorems using methods he found inappropriate. In particular, he was interested in rigorously proving fundamental results that had often been considered obvious by other mathematicians. Here are some excerpts from the preface of his 1817 work in which he described his concerns.[3] As you read, remember that when Bolzano wrote his pamphlet, there were not yet precise and universally agreed upon definitions of limit or continuity.

---

*Department of Mathematical and Computer Sciences, Metropolitan State University of Denver, Denver, CO; `emailruch@msudenver.edu`.

[1]The title of Bolzano's pamphlet translates into English as *A Purely analytic Proof of the Theorem, that between any two Values that give opposite [sign] Results, there lies at least one real Root of the Equation*

[2]According to Kline, Bolzano's proof method "was used by Weierstrass in the 1860s, with due credit to Bolzano, to prove what is now called the Bolzano-Weierstrass theorem" [Kline, 1972, p. 953].

[3]All translations of Bolzano excerpts in this project were prepared by Michael P. Saclolo, St. Edward's University, 2023, with minor changes made by the project author for readability. The translations [Russ, 1980] and [Russ, 2004, pp. 251–278] were also consulted by the project author.

In the study of equations there are two propositions for which it could be said until recently that a fully correct proof is unknown. One is the proposition: *Between any two values of the unknown quantity that give opposite [sign] results, there must always lie at least one real root of the equation.*

. . . . . .

In the meantime, we find that mathematicians of high esteem are tackling this proposition and have already attempted several ways of proving it. To anyone who wants to be convinced of this, compare the various presentations of this proposition that, for instance, *Kästner, Clairaut, Lacroix, Metternich, Klügel, La Grange, Rösling*, and many others have given.

A thorough examination of these proof methods very readily shows that none of them could be considered sufficient.

I. The most common proof method relies upon a truth borrowed from geometry: Namely, *that each continuous line of a simple curve, whose ordinates are first positive then negative (or vice versa), must necessarily cut the x-axis*[4] *at a point somewhere between those ordinates.* There are no objections whatsoever to both the correctness of and the evidence for this geometric statement. But it is also just as obvious that it is an intolerable violation of good method to want to derive the truths of pure (or general) mathematics (i.e. Arithmetic,[5] Algebra, or Analysis) from considerations that belong to only an applied (or special) part of it, namely to Geometry.

Task 1 Rewrite Bolzano's Preface proposition ("between any two values ... at least one real root of the equation") in your own words with modern terminology. Sketch a diagram illustrating the proposition. What theorem from a first-year calculus course does this remind you of?

Task 2 Do you agree with Bolzano's philosophical criticism of using geometry to try to prove his Preface proposition ("between any two values ... at least one real root of the equation")? Explain why or why not. Start by restating Bolzano's criticism in your own words.

Later in his preface, Bolzano asserted a "proper expression" for defining continuity and gave an interesting example as a footnote. As you read his definition in the next excerpt, think about whether you agree with it, and how you could rewrite it with modern language.

[4] Bolzano actually used the German word "Abscissenlinie" (literally, "abscissae-line") here. Russ points out in [Russ, 2004, p.254n] that 'Abscissenlinie' suggests a geometric measuring line in contrast to the more abstract 'x-axis.'

[5] As was not uncommon in the nineteenth century, Bolzano's use of the word "Arithmetic" here referred to the mathematical discipline that is today called "number theory."

To express it properly, we say that a function $f(x)$ varies according to the laws of continuity for all values x that lie within or beyond certain bounds,* to mean that if x is any such value, the difference $f(x+\omega) - f(x)$ can be made smaller than any given quantity whenever ω can be taken to be as small as desired.

> * *Bolzano's footnote:* There are functions that are continuously variable for all values of their argument,[6] e.g., $\alpha x + \beta x$. But there are also others that vary according to the laws of continuity only within or beyond certain limiting values of their argument. For instance, $x + \sqrt{(1-x)(2-x)}$ varies continuously only for all values x that are $< +1$ or $> +2$, and not for those values that lie between $+1$ and $+2$.

Task 3 For a function $f : \mathbb{R} \to \mathbb{R}$, rephrase Bolzano's definition of continuity at x using modern ϵ-δ terminology and appropriate quantifiers.

Task 4 Use your definition in Task 3 to give a modern ϵ-δ proof of the continuity of the function $f(x) = 3x + 47$ at $x = 2$.

Task 5 Consider the function Bolzano discussed in his footnote.

(a) Sketch a graph of this function on the interval $[0, 3]$.

(b) Based on the Preface proposition that Bolzano was discussing, why is this an interesting example?

(c) How could you adjust the function to make it better fit the issues surrounding the Preface proposition?

Task 6 Adjust your continuity definition in Task 3 to include the notion of domain, so it applies to functions defined on an interval I within $\mathbb{R}$. Do you think Bolzano's footnote function should be continuous at $x = 1$ and at $x = 2$? Give an intuitive justification.

Task 7 Suppose a function h is continuous for all x in $[0, 4]$ and $h(3) = 6$. Show that there is a $\delta > 0$ for which $h(x) \geq 5$ for all $x \in (3 - \delta, 3 + \delta)$.

Task 8 Define $g(x) = 3 - 5x^2$ with domain $I = [4, 7]$. Show that g is continuous at an arbitrary $\alpha \in I$ using your continuity definition.

Bonus. For Task 8, change the domain of g to be $\mathbb{R}$. Show that g is continuous at an arbitrary $\alpha \in \mathbb{R}$. You may need to adjust your proof from Task 8.

---

[6] *Translation note:* Bolzano used the German word "Wurzel" here, which is generally translated as "root." Given the context of the footnote, however, it is clear that he was referring to what we would today call the independent variable, or argument, of a function.

Task 9 We define a function to be continuous on an interval if it is continuous at each point in the interval. Suppose that functions f and g are both continuous on an interval I. Prove that $f - 47g$ is also continuous on I, using your continuity definition.

For the next two tasks, the following properties of the sine and cosine functions will be useful:

$$\sin a - \sin b = 2\sin\left((a-b)/2\right)\cos\left((a+b)/2\right), \quad |\sin a| \le |a|$$
$$\cos a - \cos b = 2\sin\left((b-a)/2\right)\sin\left((a+b)/2\right), \quad |\sin a| \le 1, \quad |\cos a| \le 1 \quad \text{for } a, b \in \mathbb{R}$$

Task 10 Prove that $\sin x$ is continuous on $\mathbb{R}$.

Task 11 Prove that $\cos x$ is continuous on $\mathbb{R}$.

Task 12 Define $S(x) = x\sin(1/x)$ for $x \neq 0$. Find a value for $S(0)$ so that S will be continuous at $x = 0$. Prove your assertion.

In Section 3 of this project, we will return to Bolzano's proposition about equation roots that you examined in Task 1, and work through his proof of a related result. This material will involve continuous functions, but we will set continuity aside for now to study another important theorem from Bolzano's pamphlet.

2 Bolzano's Bounded Set Theorem

In this section we will leave continuity and study an important theorem from Bolzano about what we would today call bounded sets. The theory of sets had not been developed during Bolzano's era, so he did not use the same set theoretic language we might expect in a modern discussion of his ideas. As you read the next excerpt (taken from Sections 11 and 12 of Bolzano's pamphlet), think about how you could translate his ideas into set terminology.

§11

Prelude. In the studies of applied mathematics the case happens on occasion that one learns about a [nonnegative][7] variable quantity x, all of whose values less than a certain u have a particular property M, without at the same time learning the property no longer applies to those values greater than u. In such cases there can exist perhaps some u' that is $> u$, for which all values x below it have property M, in much the same way that applies to u. Indeed, this property can perhaps apply to all x without exception. If on the other hand one learns that M does not apply to all x at all, then from the combination of these two statements one will now be justified to conclude that there is a certain value U that is the largest of which it is true that all smaller x have property M. The following theorem demonstrates this.

[7] Bolzano intended to discuss only $x \geq 0$ in this note and his Section 12 theorem statement. The term "nonnegative" has been included in this project for clarity.

§12

Theorem. If property M does not apply to all values of a [nonnegative] variable quantity x, but rather to all such that are less than a certain u, then there is always a quantity U that is the largest among them for which it can be asserted that all smaller x have property M.

Let's look at some examples of this concept that Bolzano was discussing.

Task 13 Let M be the property "$x^2 < 3$" applied to the set $\{x \in \mathbb{R} : x \geq 0\}$.

(a) Find rational numbers u, u' such that the property M applies to all smaller nonnegative values for this example and $u < u'$. (The values of u, u' are not unique.) What is the value of "the quantity U that is the largest among them for which it can be asserted that all smaller x have property M" for this example?

(b) Let S_M be the set of ω values for which all ω' values satisfying $0 \leq \omega' < \omega$ possess property M. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property M?

(c) Does U possess property M?

Task 14 Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$, and let $\alpha \in \mathbb{R}$ be arbitrary. Let M be the property "$f(\alpha + \omega) \leq f(\alpha) + 2$" applied to the set $\{\omega \in \mathbb{R} : \omega \geq 0\}$.

(a) Find rational numbers u, u' such that the property M applies to all smaller nonnegative values for this example and $u < u'$. Are these values unique? What is the value of the greatest such quantity U for this example?

(b) Let S_M be the set of ω values for which all ω' values satisfying $0 \leq \omega' < \omega$ possess property M. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property M?

(c) Does U possess property M?

Task 15 Rewrite Bolzano's theorem from his Section 12 using modern terminology and set notation.

We will refer to the theorem you stated in Task 15 as **Bolzano's Bounded Set Theorem**. In Section 12 of his pamphlet, Bolzano went on to give a proof of his Bounded Set Theorem based on a Cauchy sequence-like convergence assumption for infinite series. While the proof is correct given that assumption, we will omit it for this project.[8] Instead, we next look at Section 15 of [Bolzano, 1817], to see how he used both his definition of continuity and his Bounded Set Theorem to prove his main result on the solution of certain equations involving continuous functions.

[8] You can explore the details of that proof in the related project "Investigations into Bolzano's Bounded Set Theorem" (also by Dave Ruch), available at `https://digitalcommons.ursinus.edu/triumphs_analysis/14/`.

3 An Application of Bolzano's Bounded Set Theorem

We are now ready to work through Bolzano's main result, given in the excerpt below.[9] He broke his proof into three parts, and we will pause after each part to do tasks that will help unpack his proof and rephrase it with modern language.[10]

§15

Theorem. If two functions of x, $f(x)$ and $\varphi(x)$, vary according to the laws of continuity either for all values of x or for all those that lie between α and β, and if further, $f(\alpha) < \varphi(\alpha)$ and $f(\beta) > \varphi(\beta)$, then each time there is a certain value x lying between α and β for which $f(x) = \varphi(x)$.

Proof.

I. 1. First assume that both α and β are positive and that (because it is of no consequence) β is the larger of the two, and so let $\beta = \alpha + i$, where i indicates some positive value. As $f(\alpha) < \varphi(\alpha)$, so is $f(\alpha + \omega) < \varphi(\alpha + \omega)$, where ω indicates a positive value that can be as small as one likes. For since $f(x)$ and $\varphi(x)$ vary continuously for all x lying between α and β, and $\alpha + \omega$ lies between α and β as long as it can be taken that $\omega < i$, then $f(\alpha + \omega) - f(\alpha)$ and $\varphi(\alpha + \omega) - \varphi(\alpha)$ can become as small as one wants them to be, if ω is taken to be small enough. Therefore, if Ω and Ω' also denote quantities that can be as small as one wants, we can set $f(\alpha + \omega) - f(\alpha) = \Omega$ and $\varphi(\alpha + \omega) - \varphi(\alpha) - \Omega'$. Therefore,

$$\varphi(\alpha + \omega) - f(\alpha + \omega) = \varphi(\alpha) - f(\alpha) + \Omega' - \Omega.$$

But by assumption $\varphi(\alpha) - f(\alpha)$ equals some positive constant A. Thus,

$$\varphi(\alpha + \omega) - f(\alpha + \omega) = A + \Omega' - \Omega,$$

which remains positive when one lets Ω and Ω' be small enough, i.e., when ω is given a very small value and further for all smaller values. Thus we can claim that the two functions $f(\alpha+\omega)$ and $\varphi(\alpha+\omega)$ are in a relationship of a smaller value to a larger one relative to each other for all ω that are smaller than a particular one. Denote this property of the variable ω by M; then we can say that all ω that are smaller than a particular one have property M. Nevertheless, that this property M does not apply to all values of ω is clear, in particular not to the value $\omega = i$, because according to the assumption $f(\alpha + i) = f(\beta)$ is no longer $<$ but rather $> \varphi(\alpha + i) = f(\beta)$. From there, according to Theorem §12, there must be a value U that is the largest among them for which it can be claimed that all ω that are $<$ U possess property M.[11]

[9] This excerpt and all others that appear in this section of the project are taken from Section 15 of Bolzano's pamphlet.

[10] Throughout his theorem statement and proof below, Bolzano wrote fx where we would write $f(x)$, and similarly deleted argument parentheses for other functions and variables. The translator has inserted these parentheses to reduce distractions for the modern reader.

[11] This is the theorem that you re-wrote in modern terminology in Task 15.

Task 16 Sketch a diagram with graphs of f and φ that illustrates the theorem statement and label α, β and A. For an arbitrary ω possessing property M, label Ω' and Ω. Also draw an ω number line and label key values i, U, and the set of values ω possessing property M.

Task 17 Bolzano stated that ω, Ω and Ω' can be made "as small as one wants." Explain the dependencies between these quantities. Use ϵ-δ terminology to clarify what is going on.

Task 18 Rewrite Bolzano's claim in the first two sentences of I.1 using modern terminology and call this Lemma 1.

Task 19 Convert Bolzano's argument in the first part of I.1[12] into a proof of Lemma 1 with your modern definition of continuity.

Task 20 Rewrite with symbols Bolzano's definition of property M in the context of Part I.1. Then rephrase his statement that "all ω that are smaller than a particular one have the property M" using set notation, and name this set S_M.

Task 21 As an example, consider the functions $f(x) = 4 + (x-2)(x-4)(x-6)$ and $\varphi(x) = 4$ with $\alpha = 1$ and $\beta = 7$. Informally find the set S_M and the value of U for this example.

Task 22 We can summarize the results of Part I.1 of Bolzano's proof by stating some facts about U. First, Bolzano showed that the hypotheses of his §12 theorem are met, so that the quantity U exists. What additional facts does the conclusion of his §12 theorem tell us about U? In particular, explain why U must be positive.

Now proceed to Bolzano's Part I.2 of his proof.

2. And this U must lie between 0 and i. First it cannot be $= i$, as this would mean that each $f(\alpha+\omega) < \varphi(\alpha+\omega)$ whenever $\omega < i$, however close it may come to the value i. But in the same way that we just proved that the result $f(\alpha+\omega) < \varphi(\alpha+\omega)$ follows from the assumption $f(\alpha) < \varphi(\alpha)$, as long as ω is taken to be small enough, so does it unfold that the result $f(\alpha+i-\omega) > \varphi(\omega+i-\omega)$ follows from the assumption $f(\alpha+i) > \varphi(\alpha+i)$, as long as ω is taken to be small enough. So it is not true that the two functions $f(x)$ and $\varphi(x)$ are in a relationship of a smaller value to a larger one for all values of x that are $< \alpha + i$. Secondly, even less so can it be that $\mathsf{U} > i$, because otherwise i is also one of the values of ω that are $< \mathsf{U}$, and therefore it must be that $f(\alpha+i) < \varphi(\alpha+i)$ as well, which the assumption of the theorem downright contradicts. Thus, U surely lies between 0 and i as it is positive, and consequently, $\alpha + \mathsf{U}$ lies between α and β.

[12] Bolzano's argument for why this lemma is true ends with the sentence that starts "So we can claim that . . . ."

Task 23 Rewrite Bolzano's claim that "the result $f(\alpha + i - \omega) > \varphi(\omega + i - \omega)$ follows from the assumption $f(\alpha+i) > \varphi(\alpha+i)$, as long as ω is taken to be small enough" using modern terminology and call this Lemma 2.

Task 24 We can summarize this part of Bolzano's plan as the claims "$0 < \mathsf{U} < i$" (so that $\alpha < \alpha + \mathsf{U} < \beta$), followed by his proof of the claim that $\mathsf{U} < i$. Rewrite his proof using your own words and modern terms, referencing the set S_M and Bolzano's Bounded Set Theorem from Task 15.

Now read Part I.3 of Bolzano's proof.

3. Now begs the question of how do the functions $f(x)$ and $\varphi(x)$ compare to each other for the value $x = \alpha + U$. First of all, it cannot be that $f(\alpha+U) < \varphi(\alpha+U)$, since this would also give $f(\alpha+u+\omega) < \varphi(\alpha+U+\omega)$, if one takes ω to be small enough; and consequently $\alpha+U$ would not be the largest value for which it can be claimed that all values x below it have property M. Secondly, neither can it be that $f(\alpha+u) > \varphi(\alpha+U)$, because this would also give $f(\alpha+u-\omega) > \varphi(\alpha+U-\omega)$, as long as ω is taken to be small enough; and so it would go against the supposition that property M is not true for all x below $\alpha+U$. Thus, there is nothing else left but $f(\alpha+U) = \varphi(\alpha+U)$; and consequently it has been proven that there is a value x between α and β, namely $\alpha+U$, for which $f(x) = \varphi(x)$.

Task 25 Adjust your Lemmas 1 & 2 to give modern justifications of the first two claims in this excerpt.

Task 26 What property of the real numbers justifies the statement "there is nothing else left but $f(\alpha+U) = \varphi(\alpha+U)$"?

Task 27 At the beginning of part I.1 of the proof, Bolzano made the assumption "that α and β are both positive." Can you find a place in the proof where he used this assumption?

Bolzano continued in Section 15 to address the cases α and β are both negative, one is zero, and of opposite sign. We will omit these proofs, as they are not terribly enlightening.

Now that we have completed our journey with Bolzano through his proof, let's return to his preface proposition that you examined in Task 1.

Task 28 Consider Bolzano's proposition from his preface that: "between any two values of the unknown quantity that give opposite [sign] results, there must always lie at least one real root of the equation."

(a) In Task 1, you restated this proposition using modern terminology. Look back at your answer to that task with Bolzano's §15 theorem in mind. Do the hypotheses of his theorem apply to your restatement? If not, modify your restatement of the proposition as needed so that they do.

(b) Use Bolzano's theorem to prove your restated version of Bolzano's Preface proposition.

Task 29 Use Bolzano's theorem to prove the following result from a standard introductory Calculus text, which is typically referred to as the "Intermediate Value Theorem."

> Consider an interval $I = [a, b]$ in the real numbers $\mathbb{R}$ and a continuous function $f : I \to \mathbb{R}$. If $f(a) < L < f(b)$ then there is a $c \in (a, b)$ such that $f(c) = L$.

4 Conclusion

As the final task of the last section illustrates, Bolzano's §15 theorem is a generalized version of the Intermediate Value Theorem that is featured in a standard first-year Calculus textbook. Today, various versions of the Intermediate Value Theorem serve as powerful tools in numerical analysis and other areas of mathematics. The Bounded Set Theorem that Bolzano used to prove his version of the Intermediate Value Theorem was a highly original idea, and is closely linked to what are nowadays called the least upper bound and greatest lower bound existence properties of the real numbers. If you have studied these completeness properties, you might enjoy the following task.

Task 30 Let S be a nonempty subset of $\mathbb{R}$ such that $s > 47$ for all $s \in S$. Use Bolzano's Bounded Set Theorem to prove that S has a greatest lower bound.

Caution: S is a subset of the interval $(47, \infty)$, but don't assume $S = (47, \infty)$.

It is interesting to note that Augustin-Louis Cauchy (1789–1857), a key player in building the theory of calculus, also proved a version of Bolzano's preface proposition, probably a few years later than Bolzano.[13] Although Cauchy used a very different method of proof,[14] his approach also depended on the completeness property of the real numbers. Given the strong influence of the nineteenth century approach to reshaping calculus that was adopted by Bolzano, Cauchy and their contemporaries in the nineteenth century, it is no coincidence then that some version of the completeness property continues to lie at the foundation of the proofs of the Intermediate Value Theorem that are found in today's analysis textbooks.

References

B. Bolzano. *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege (A Purely analytic Proof of the Theorem, that between any two Values that give opposite [sign] Results, there lies at least one real Root of the Equation).* Gottlieb Haase, Prague, 1817.

[13]Bolzano's proof was published in 1817, but wasn't widely known when it first appeared. Cauchy's proof appeared in a note on the numerical solution of equations that was included at the end of his textbook *Cours d'analyse (Course on analysis)*, published in 1821. [Lützen, 2003, p. 175]

[14]To learn how Cauchy's approach to the Intermediate Value Theorem compares to that of Bolzano (without actually reading Cauchy himself!), see the overviews of Cauchy's proof found in [Lützen, 2003, pp. 175–176] and [Grabiner, 2005, pp. 69–74].

J. Grabiner. *The Origins of Cauchy's Rigorous Calculus.* Dover, Mineola, NY, 2005. Originally published in 1981 by The MIT Press, Cambridge, Massachusetts.

H. N. Jahnke, editor. *History of Analysis.* American Mathematical Society, Providence, RI, 2003.

M. Kline. *Mathematical Thought from Ancient to Modern Times.* Oxford University Press, New York, NY, 1972.

J. Lützen. The Foundation of Analysis in the 19th Century. In H. N. Jahnke, editor, *A History of Analysis*, pages 155–196. American Mathematical Society, Providence, RI, 2003.

S. B. Russ. A Translation of Bolzano's Paper on the Intermediate Value Theorem. *Historia Mathematica*, 5:156–185, 1980.

S. B. Russ. *The Mathematical Works of Bernard Bolzano.* Oxford University Press, New York, NY, 2004.

Notes to Instructors

PSP Content: Topics and Goals

This PSP is designed to introduce continuity and the Intermediate Value Theorem (IVT) for a course in Real Analysis. Specifically, its content goals are to:

1. Develop a modern continuity definition with quantifiers based on Bolzano's definition.
2. Develop facility with the modern continuity definition by applying it to various functions.
3. Analyze Bolzano's Bounded Set Property and rewrite it in modern terminology.
4. Modernize Bolzano's proof of his Intermediate Value Theorem.
5. Apply Bolzano's Intermediate Value Theorem to obtain two other formulations of the Intermediate Value Theorem.

Student Prerequisites

This project is written with the assumption that students have done a rigorous study of quantifiers and limits of real-valued functions. The project also assumes that students have seen the least upper bound property for bounded sets of real numbers, but the project could be used to introduce this concept, with instructor supplements.

PSP Design and Task Commentary

The first section of this project contains excerpts from Bolzano introducing his version of the IVT and his definition of continuity. Most of the tasks in this section focus on developing the definition of continuity for a function defined on an interval, the appropriate setting for a discussion of the IVT. Getting a correct definition of continuity in Task 3 is crucial before going much further; a class discussion of Task 3 and the next problem can be helpful after students work on them for awhile or in groups.

Bolzano's choice of $x+\sqrt{(1-x)(2-x)}$ in his footnote is mildly perplexing, as it does not change signs in its domain $[1,2]$. Indeed, the first set of students using the project were rather critical of Bolzano's footnote function. They inspired Task 5. In Bolzano's defense, he discussed the function $x+\sqrt{(x-2)(x+1)}$, which lacks a root between -1 and 2, later in his very lengthy preface.

Task 7 foreshadows a crucial result in the next section, namely writing a modern ϵ-δ proof of Bolzano's assertion: "As $f(\alpha)<\varphi(\alpha)$, so is $f(\alpha+\omega)<\varphi(\alpha+\omega)$, where ω indicates a positive value that can be as small as one likes." This is difficult for some students, and they may need a hint/reminder that THEY get to choose ϵ if f is known to be continuous.

The final group of Section 1 tasks, 8–12, are standard problems to sharpen skills in working with continuity. Instructors may sample the set for classroom examples or homework problems. However, they are not needed for the flow of Bolzano's discussion.

Section 2 is written with the assumption that students have seen the least upper bound property for bounded sets of real numbers. The theorem in Bolzano's Section 12 basically asserts this property for a special class of bounded sets, but in a form students (and the PSP author!) have not seen before. It is a bit tricky to unravel and put into modern set notation. The first two tasks should help ease this process for students. Task 14 should help them with Bolzano's Section 15, where the theorem is applied.

The third section focuses on Bolzano's proof of his IVT. Part I.1 of the IVT proof in his Section 15 is crucial and contains some subtleties. It is worth taking time to make sure students understand this part of the proof. In addition the symbols Ω', Ω may cause confusion for some students. It is not completely clear from Bolzano's writing whether the set of all ω with the property M in this part of his proof should be the set $\{\omega : f(\alpha+\omega') < \varphi(\alpha+\omega') \text{ for all } \omega' \in [0,\omega)\}$ or the set $\{\omega : f(\alpha+\omega) < \varphi(\alpha+\omega)\}$. Task 21 illustrates the difference. An opportunity for discussion and careful reading![15]

In his original paper, Bolzano began the proof of the Section 15 theorem with the following:

> We must remember that in this theorem, the values of the functions $f(x)$ and $\varphi(x)$ are to be compared with each other merely according to their absolute values, i.e. without regard to sign, or as if they were not quantities of opposite signs at all. But it is dependent upon the signs of α and β.

Bolzano then split his proof into several cases I–IV, beginning with case $0 < \alpha < \beta$. Here is S. B. Russ' explanation [Russ, 1980, p. 183]:

> Bolzano always used inequality signs to apply only to the magnitude of quantities and not to their position on a number line. This was common practice at the time as there was still no standard symbol for modulus. Thus in Bolzano's usage $x > -1$ meant the range we should now describe as $x < -1$. For example, in Section 2 he stated that the general term of a geometric progression ae^r becomes "ever larger when $e > \pm 1$" and "ever smaller when $e < \pm 1$." And in Section 15.IV he described the range of values of x between α and β when α is negative and β positive as "all values of x which if negative are $< \alpha$, and if positive are $< \beta$."

These comments by Bolzano have been excluded from the project because they seem likely to cause considerable confusion with little payoff to most students of analysis. Instructors could bring up these issues in a class discussion of Task 27.

The statement and proof of Lemma 2 in the task set following Part I.2 of the IVT proof in Section 15 of Bolzano's pamphlet may seem a bit repetitive. However, it should clarify things for some students and serve as proof writing practice, especially if Lemma 1 is done in class.

[15] Note that in either case, the value of U produced by Bolzano's §12 theorem is the least upper bound of the first of these two options; that is, the set $S_M = \{\omega : f(\alpha+\omega') < \varphi(\alpha+\omega') \text{ for all } \omega' \in [0,\omega)\}$.

Suggestions for Classroom Implementation

This is roughly a one or two week project under the following methodology (basically David Pengelley's "A, B, C" method described on his website https://www.math.nmsu.edu/~davidp/):

1. Students do some advanced reading and light preparatory tasks before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the tasks or your grading load will get out of hand! Some instructor have students write questions or summaries based on the reading.

2. Class time is largely dedicated to students working in groups on the project—reading the material and working tasks. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the tasks they missed. This is usually a good incentive not to miss class!

3. Some tasks are assigned for students to do and write up outside of class. Careful grading of these tasks is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project. If students have already studied continuity in a rigorous fashion, then the first section should move very quickly and many tasks can safely be skipped.

Sample Implementation Schedule (based on a 50-minute class period)

Full implementation of the project can be accomplished in 3–4 class days, as outlined below.

For Class 1, students read through the introductory material and the beginning of Section 1, and do Tasks 1–3 before the first class. After discussing their results at the beginning of Class 1, students work on and discuss Tasks 4 and 7. Homework practice with the definition of continuity could be some subset of Tasks 5, 6, 8–12. However, none of these are essential for the following material.

As preparation for Class 2, students read the first Bolzano excerpt in Section 2 and do Task 13. After discussing their results at the beginning of Class 2, students work on and discuss Tasks 14 and 15. Students then begin Section 3 by reading the first part of Bolzano's IVT proof and doing Task 16, which is essentially drawing diagrams for the proof.

As preparation for Class 3, students do Tasks 17, 18. After discussing their results at the beginning of Class 3, students work on and discuss Tasks 19–22.

As preparation for Class 4, students read Bolzano's Part I.2 proof excerpt and do Task 23. After discussing their results at the beginning of Class 4, students work on and discuss Tasks 24–30. Some of these tasks can be given as homework, as time permits.

If students have already studied continuity in a rigorous fashion, then the first section should move very quickly and many of its tasks can safely be skipped, thereby shortening the implementation time required.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be completed in at most 2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from `https://digitalcommons.ursinus.edu/triumphs\_topology`; all other listed projects are available at `https://digitalcommons.ursinus.edu/triumphs\_analysis`.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Heine Barnett)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)
  Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)
  A second version of this project suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *Understanding Compactness: Early Work, Uniform Continuity to the Heine-Borel Theorem* (Naveen Somasunderam)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Mean Value Theorem*(David Ruch)
- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Heine Barnett)
- *Euler's Rediscovery of e** (David Ruch; sequence convergence, series & sequence expressions for e)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

Recommendations for Further Reading

The translations in [Russ, 1980] and [Russ, 2004] also include interesting background on Bolzano as well as commentary on some of the subtleties of Bolzano's work. The articles in [Jahnke, 2003] give some perspective on other works in analysis during Bolzano's era.

Acknowledgments

The development of this student project has been partially supported by the TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) Program with funding from the National Science Foundation's Improving Undergraduate STEM Education Program under Grant Nos. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily represent the views of the National Science Foundation.

For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.