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# An Introduction to the Algebra of Complex Numbers and the Geometry in the Complex Plane


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# An Introduction to the Algebra of Complex Numbers and the Geometry in the Complex Plane

Nicholas A. Scoville\*, Diana White†

January 11, 2019

## 1 Introduction

Complex numbers are a puzzling concept for today's student of mathematics. This is not entirely surprising, as complex numbers were not immediately embraced by mathematicians either. Complex numbers showed up somewhat sporadically in works such as those of Cardano, Tartaglia, Bombelli and Wallis. However, it wasn't until Caspar Wessel (1745-1818), a Norwegian map surveyor, that we first see a systematic and full theory of complex numbers. This project studies the basic definitions, as well as geometric and algebraic properties, of complex numbers via Wessel's 1797 paper *Om Directionens analytiske Betegning, et Forsog, anvendt fornemmelig til plane og sphaeriske Polygoners Oplosning (On the Analytical Representation of Direction. An Attempt Applied Chiefly to Solving Plane and Spherical Polygons)* [Wessel, 1999]. It was the first to develop the geometry of complex numbers, though it was unfortunately not given much attention until about a century after it was written. As Michael Crowe writes [Crowe, 1994]:

Caspar Wessel...lays out for the first time the geometrical representation of complex numbers. His goal was not only to justify complex numbers, but also to investigate how we may represent direction analytically... Wessel publish[ed] for the first time the now standard geometrical interpretation of complex numbers as entities that can be added, subtracted, multiplied, and divided.

## 2 Geometric basics

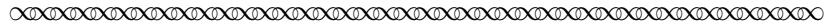
As mentioned above, we will begin to work through Caspar Wessel's 1797 paper *On the Analytical Representation of Direction...* [Wessel, 1999]. All quotes are taken from this paper, unless otherwise noted. One may also consult an earlier English translation of Wessel in [Smith, 1959, Volume 1, pp. 55-66].

Wessel begins by stating his purpose in writing this paper.

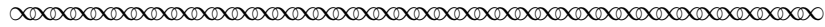
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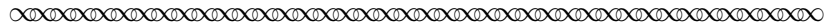


The present attempt deals with the question of how to represent direction analytically, or, how one ought to express straight lines; if from a single equation in one unknown line and some given lines one is able to find an expression representing both the length and the direction of the unknown line.



**Task 1**

Recall that Caspar was a map surveyor. Describe Wessel's goal in your own words, and explain why such goals would be something that a map surveyor would be interesting in learning.



There are homogeneous quantities which will only increase or decrease one another, like increments or decrements, when associated to the same subject.

There are other quantities that in the same situations may change one another in numerous other ways. Straight lines are of this kind.

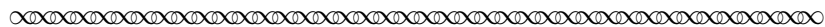
Thus the distance of a point from a plane can change in numerous ways, when the point describes a more or less inclined straight line outside the plane. If this line is *perpendicular*, i.e., the path of the point makes a right angle with the axis of the plane, then the point remains in a parallel to the plane, and its motion has no influence on the distance from the plane.

If the described line is *indirect*, i.e., it makes a skew angle with the axis of the plane, then it contributes a smaller segment than its own length to the extension or reduction of distance, and may increase or decrease the distance in infinitely many ways.

If the line is *direct*, i.e., collinear with the distance, it adds or subtracts to the distance by its full length; in the first case it is positive, otherwise negative.

Thus, all the straight lines described by a point are, with respect to their effect on the distance of the point from a plane outside the lines either *direct*, *indirect*, or *perpendicular*, depending on whether they add or subtract all of, part of, or none of their own length.

Since a quantity is called absolute if given by itself and not relative to another quantity, then in the previous definition the distance may be called the absolute line, and the contribution of the relative to the extension or reduction of the absolute may be called the effect of the relative.

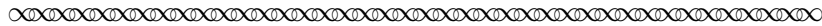


**Task 2**

- (a) What do you think Wessel is referring to in the above excerpt?
- (b) Write down one comment and one question that you have.

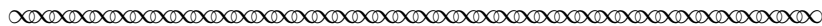
## 2.1 Addition of lines

Immediately after the above quote, Wessel writes



§1

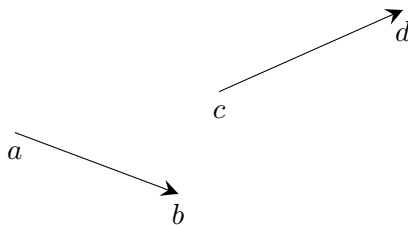
Two straight lines are added together, when one joins them together so that one begins where the other ends, and next one draws a straight line from the first to the last point of the joined lines, and takes this to be their sum.



**Task 3** You may have learned the difference between what today we would call lines, line segments, and rays. Which of these, if any, does Wessel mean by “straight line”?

In the following exercise, you will add two straight lines together.

**Task 4** Consider the two lines in the picture below.



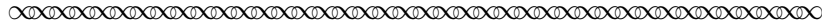
Denote the lines by their endpoints so that the first line is  $ab$  and the second line is  $cd$ .

- (a) Following Wessel, add line  $cd$  to line  $ab$ . That is, redraw line  $cd$  by moving point  $c$  onto point  $b$  and then draw the straight line from point  $a$  to point  $d$ .

Now let's do this in the other order.

- (b) Add line  $ab$  to line  $cd$ .
- (c) What do you observe about adding line  $cd$  to line  $ab$  compared to adding line  $ab$  to line  $cd$ ?

Wessel continues by discussing the geometry of these lines.

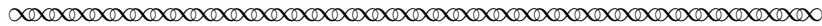


If, for example, a point moves forward 3 feet and then backwards 2 feet, then the sum of these two paths is not the first 3 and the last 2 feet together, but the sum of one foot forward, because this path has the same effect as the other two paths.

Similarly, when one side of a triangle extends from  $a$  to  $b$ , and the other from  $b$  to  $c$ , then the third from  $a$  to  $c$  must be called the sum, and should be denoted  $ab + bc$ , so that  $ac$  and  $ab + bc$  have the same meaning, or  $ac = ab + bc = -ba + bc$ , if  $ba$  is the opposite of  $ab$ . If the added lines are direct then the definition agrees completely with the usual one. If they are not direct it does not disagree with the analogy to call a straight line the sum of two other joined lines, in so far as it has the same effect of the other two. The meaning I have given to the  $+$  sign is not so unusual either; for example, in the expression

$$ab + \frac{ba}{2} = \frac{1}{2}ab$$

the term  $\frac{ba}{2}$  is no part in the sum. Thus one may write  $ab + bc = ac$  without thinking of  $bc$  as part of  $ac$ ;  $ab + bc$  is only the sign representing  $ac$ .

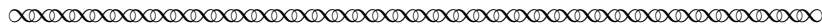


**Task 5** Be an active reader! Go back through the last excerpt and draw diagrams to explain what Wessel means in each paragraph. Ensure that you address what Wessel means by opposite and what he means by the plus sign. What does he mean by “is no part in the sum”?

Let’s take a look at addition of straight lines in more detail.

**Task 6** Let’s consider what we mean in this context by 0. Physically, Wessel would say that if a point moves 4 feet forward and then 4 feet backward, the the sum of this action is the same as no movement at all. Let’s call this 0. Consider the line from  $a$  to  $b$ , which Wessel would denote  $ab$ . What happens if you add  $ba$ , the line from  $b$  to  $a$  to this line? That is, what is  $ab + ba$ ?

The prior task shows that all straight lines have what today we would call an additive inverse. That is, given any straight line, there is another straight line that when added to it (in either order), results in no line at all, or what we will call 0.

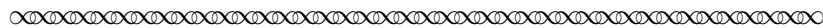


## §2

When more than two straight lines are to be added the same rule is followed; they are joined together so that the last point of the first is joined to the first one of the second, the last point of the second to the first point of the third, etc. Finally a straight line is drawn from the beginning of the first to the end of the last, and this is called the sum of all of them.

Which line one chooses to be the first, and which the second, the third, etc. is immaterial, because wherever a point describes a straight line within three mutually orthogonal planes, the segment has the same effect on the distance of the point from each of the three planes; consequently, one of several added lines contributes the same to the position of the last point of the sum, whether it is the first, the last, or has any other number among the addends; thus the order in the addition of straight lines is immaterial, and the sum always remains the same, because the initial point is assumed to be given, and the last point always attains the same position.

Hence, in this case one may also denote the sum by inserting the sign  $+$  between the lines to be added. For instance, when in a quadrilateral the first side is drawn from  $a$  to  $b$ , the second from  $b$  to  $c$ , the third from  $c$  to  $d$ , and the fourth from  $a$  to  $d$ : then one can write  $ad = ab + bc + cd$ .

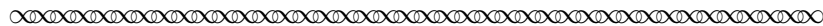


**Task 7**

- (a) Draw three lines in the plane, and illustrate how to add them.
- (b) Now add those same three lines in another order. Compare the result of the second sum with the first.
- (c) Let's break it down more to see why these are the same. Start with two straight lines that both go in the same direction. Explain why it does not matter what order you add them in. That is, explain why if  $ab$  and  $cd$  are two straight lines going in the same direction, then  $ab + cd = cd + ab$ .
- (d) To argue that  $ab + cd = cd + ab$  for all straight lines  $ab$  and  $cd$ , we introduce an orthogonal set of axes.

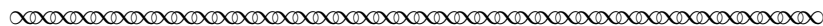
Draw two orthogonal rays (using the modern term) that emanate from a common point. Draw a third line emanating from the intersection of those rays. Consider this line as the sum of two lines, one of which goes in the same direction as one of the orthogonal rays, the other in the direction of the other ray i.e., break this line down into two orthogonal components that sum to the original line. Do the same for a second line. Notice that to add these lines, you add the components of each in the two orthogonal directions. Appeal to the earlier observation that adding components in each of these two orthogonal directions are independent of order.

- (e) We have worked in just the plane, but the argument extends naturally into three dimensions (though easily illustrating it on two-dimensional paper becomes more complicated!). Now draw three lines in three dimensions and illustrate how to add them.



§3

If the sum of several lengths, widths, and heights = 0, then the sum of the lengths, that of the widths, and that of the heights, each sum separately = 0.

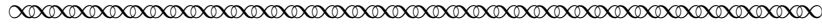


**Task 8**

Explain what Wessel means here, using several illustrations and an argument analogous to that used in the excerpt preceding Task 7.

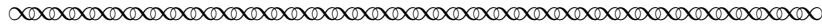
## 2.2 Multiplication of lines

Now that we know a little bit about adding lines, the next thing we wish to do is to understand how to multiply lines.



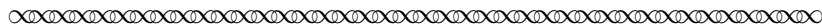
### §4

The product of two straight lines should in every respect be formed from the one factor, in the same way as the other factor is formed from the positive or absolute unit line that is set  $= 1$ ,



Wessel will explain in more detail below what he means by this, but before we move on, we must explain his use of the term “positive unit line.” This is a choice of both a positive direction and a unit length. Suppose you find yourself literally floating in a sea of blackness, without any sense of direction or size of scale. In order to gain some sense of normalcy, you might pick a direction to be “north.” Relative to the direction you chose for north, you would then be able to determine south, west, and east. Furthermore, if you happen to come upon a thin steel rod, you could declare that it had a length of 1. Anything else you came across could be measured in terms of how many rods long it is. Of course, there is nothing special or absolute about your choice of north and length, but once you make those choices, you should in theory be able to develop a coherent system of length and direction, assuming you are consistent. In a similar manner, Wessel is developing a theory which holds for any choice of direction being decreed positive and length chosen as unit.

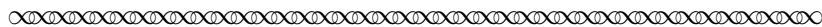
We continue with Wessel’s definition of multiplication of straight lines.



First, the factors must have such directions, that they can both be included in the same plane as the positive unit.

Next, concerning the length of the product, it must be to the one factor as the other is to the unit; and

Finally, if the positive unit, the factors, and the product are given a common initial point, then the product, with respect to the direction must lie in the plane of the unit and the factors, and the product must deviate as many degrees from the one factor, and to the same side, as the other factor deviates from the unit, so that the directional angle of the product or its deviation from the positive unit is the sum of the directional angles of the factors.



**Task 9** We’ll unpack each of Wessel’s three properties in turn.

- (a) What does it mean for both factors to have directions such “that they can both be included in the same plane as the positive unit”? Give an example of lines that do not have this property.

- (b) The second property gives the length of the product. Let  $s$  and  $t$  be the lengths of the factors and denote by  $x$  the length of the product. Write down an algebraic equation relating these three values. (Note that the length of the unit is 1).
- (c) The third property gives the angle of the product. Let  $v$  and  $w$  be the angles of the factors from the positive unit and  $y$  the angle of the product, all measured from the positive unit. Write down an algebraic equation relating these three values.
- (d) Refer to the lines in Task 4. Draw a picture of the product of these lines.

Notice that in order to multiply straight lines, they have to lie in the same plane as the unit line, as well as originate from the same common point that the unit line originates from. This is very different from addition of lines, where we have no such constraints.

Once again, we see that if we know the length of the product line and the direction of the product line, then we know the product line.

Now let's look at some of the same properties for multiplication that we did for addition – identity, inverse, commutativity, associativity.

**Task 10** Let  $+1$  denote the positive unit line.

- (a) What is  $+1$  times any straight line emanating from the same point as  $+1$ ?
- (b) What is the product of any straight line emanating from the same point as  $+1$  with  $+1$ ?
- (c) What do the previous two observations tell you about the existence of a multiplicative identity for lines?

**Task 11** We now consider inverses of straight lines under Wessel's definition of multiplication. Recall that in a number system, the inverse of an element  $x$  is an element  $y$  such that  $x \cdot y = y \cdot x = 1$ , where  $\cdot$  represents the operation and 1 the multiplicative identity.

- (a) What is the inverse of  $+1$ ? What about  $-1$ ? Explain each geometrically using Wessel's definition of multiplication.
- (b) Draw a set of perpendicular axes. Display the positive unit along the first axis. Now draw any other straight line emanating from the same point as the unit (i.e., from the intersection of the axes). Does this line have a multiplicative inverse? That is, does there exist another line emanating from the same point, that when multiplied by this line gives a result of the absolute unit line (i.e.,  $+1$ )? Explain this geometrically using the definition of multiplication of straight lines.
- (c) What do these observations say about the multiplicative inverses of lines?

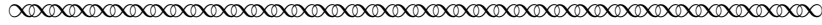
Finally, we deal with commutativity of multiplication of lines. Recall that an operation  $\cdot$  is commutative if for all  $x$  and  $y$ ,  $x \cdot y = y \cdot x$ .

**Task 12** Explain why multiplication of straight lines is commutative.



### 3 The complex plane

Wessel next develops what we would today call the complex plane.

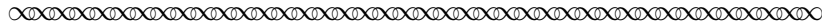


#### §5

Let  $+1$  denote the positive, rectilinear unit, and  $+\epsilon$  a certain different unit, perpendicular to the positive unit, and with the same initial point; then the directional angle of  $+1$  is  $0^\circ$ , of  $-1$  is  $180^\circ$ , of  $+\epsilon$  is  $90^\circ$ , and of  $-\epsilon$  is  $-90^\circ$  or  $270^\circ$ ; and according to the rule that the directional angle of the product is the sum of those of the two factors, one gets

$$\begin{array}{lll} (+1) \cdot (+1) = +1 & (+1) \cdot (-1) = -1 & (-1) \cdot (-1) = +1 \\ (+1) \cdot (+\epsilon) = +\epsilon & (+1) \cdot (-\epsilon) = -\epsilon & (-1) \cdot (+\epsilon) = -\epsilon \\ (-1) \cdot (-\epsilon) = +\epsilon & (+\epsilon) \cdot (+\epsilon) = -1 & (+\epsilon) \cdot (-\epsilon) = +1 \\ (-\epsilon) \cdot (-\epsilon) = -1. & & \end{array}$$

From this it follows that  $\epsilon$  becomes  $\sqrt{-1}$ , and the deviation of the product is determined so that not a single one of the usual rules of operation is violated.



Wow. Here we have the natural emergence of  $\sqrt{-1}$ . Notice how it arises naturally as that “straight line” that when multiplied by itself gives  $-1$ .

Today, we refer to the axis through  $+1$  as the **real axis** while the axis along  $\epsilon$  is called the **imaginary axis**. In this coordinate system, the plane becomes the **complex plane**.

Let’s now verify the many multiplication facts that Wessel lists above. But be careful! These may look obvious. Of course, “one times one is one” – duh! But wait! Recall that  $+1$  denotes the “positive, rectilinear unit”. So it’s a line segment, and  $-1$  denotes its opposite. You’ll need to carefully use the definition of multiplication of lines to verify these.

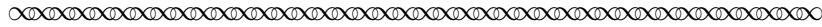
**Task 13**

- (a) Use the definition of multiplication of straight lines from Section 2.3 to explain each of the 10 different products expressed above.
- (b) Explain why  $(-\epsilon) \cdot (-\epsilon) = -1$  implies that  $\epsilon = \sqrt{-1}$

Isn’t that crazy? The root of the algebraic equation  $x^2 + 1 = 0$  that you learned in high school, pops up quite naturally in this geometric setting. In Section 3.3 we’ll see another way in which we can see  $\sqrt{-1}$  naturally arises.

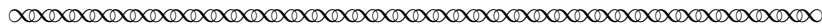
### 3.1 Trig functions and their properties

Now that we have the basic set up and the algebra of lines established, we formulate a way to work with points on a circle. As you probably know, no discussion of circles will be complete without trigonometric functions, and Wessel begins by defining the cosine function. However, when reading through his words and working through the exercise, it is important that you don't impose certain properties on cosine by assuming facts you know about cosine but which Wessel does not mention. Use those – and only those – properties that Wessel specifies.



#### §6

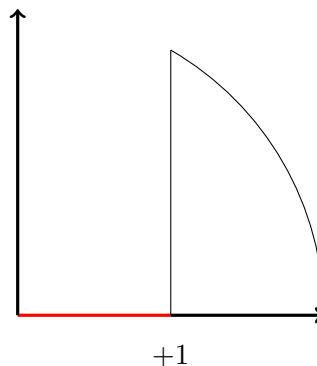
Cosine to a circular arc that begins at the endpoint of its radius  $+1$ , is the piece of  $+1$  or the opposite radius, starting from the center and ending at the perpendicular from the last point of the arc. Sine to the same arc is drawn perpendicular to cosine from its last point to the last point of the arc.



What kind of mathematical objects are sine and cosine according to Wessel? Lines? Points? Numbers? Equations? Vectors? From this excerpt, they seem to be viewed as lines or possibly vectors. However, as we continue to read from Wessel, we will see that the precise nature of sine and cosine is not so clear.

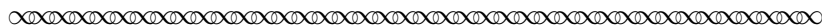
#### Task 14

(a) Explain how the following picture is an illustration of Wessel's definition of cosine.

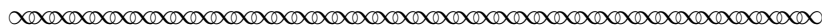


(b) Draw and label a similar picture illustrating sine.

Wessel continues:



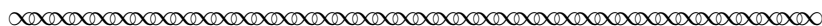
From §5 it follows that sine to a right angle is  $= \sqrt{-1}$ . Let us put  $\sqrt{-1} = \epsilon$ ; let  $v$  denote any angle, and  $\sin v$  a straight line of length sine of the angle  $v$ , but positive when the measure of the angle ends in the first semi-circular circumference, and negative when it ends in the second semi-circular circumference; then it follows from §§4 and 5, that  $\epsilon \sin v$  expresses the sine of the angle  $v$  in direction as well as length.



Note that it is unclear from the above if sine is a number or vector.

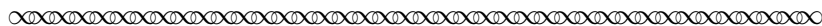
**Task 15** Use the results in §5 to show that sine of a right angle is  $\sqrt{-1}$ .

Again, in the following paragraph, pay attention to Wessel's use of sine as both a number and a vector.



### §7

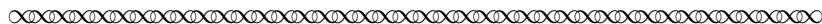
The radius that starts at the center and deviates the angle  $v$  from the absolute or positive unit is, according to §§1 and 6, equal to  $\cos v + \epsilon \sin v$ . But the product of two factors, of which one deviates from the angle  $v$  from the unit, and the other the angle  $u$  from the unit, must itself deviate the angle  $v + u$  from the unit, according to §4. So, when the segment  $\cos v + \epsilon \sin v$  is multiplied by the segment  $\cos u + \epsilon \sin u$ , then the product becomes a straight line, whose directional angle is  $v + u$ . Therefore, following §§1 and 6, the product may be denoted  $\cos(v + u) + \epsilon \sin(v + u)$ .



**Task 16** Wessel points out that multiplying two straight lines that end on the unit circle amounts to adding their angles. Why is this?

**Task 17** Explain why §§1 and 6 imply that  $(\cos v + \epsilon \sin v)(\cos u + \epsilon \sin u) = \cos(v + u) + \epsilon \sin(v + u)$ . Illustrate with several examples involving different choices of angles  $u, v$ .

The next excerpt from Wessel establishes a trigonometric identity. Take note that in Wessel's argument, he does not assume distributivity of the values in the multiplication.



§8

This product  $(\cos v + \epsilon \sin v)(\cos u + \epsilon \sin u)$  or  $\cos(v + u) + \epsilon \sin(v + u)$  may be expressed in still another way. . .

Thus,

$$(\cos v + \epsilon \sin v)(\cos u + \epsilon \sin u) = \cos v \cdot \cos u - \sin v \cdot \sin u + \epsilon(\cos v \cdot \sin u + \cos u \cdot \sin v)$$

which follows from the well-known trigonometric formulas

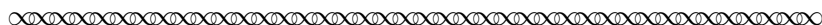
$$\cos(v + u) = \cos v \cdot \cos u - \sin v \cdot \sin u$$

and

$$\sin(v + u) = \cos v \cdot \sin u + \cos u \cdot \sin v$$

.

These two formulas may be proved precisely and without much trouble for all cases whether both of the angles  $v$  and  $u$ , or just one of them, are positive, negative, greater than or less than a right angle. Consequently, the theorems that one derives from them become valid in general.

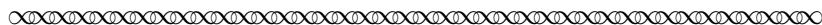


**Task 18**

- (a) Use Wessel's definition of sine and cosine above to prove the two "well-known trigonometric formulas." Do this, as Wessel suggests, by considering cases.
- (b) Verify the formula

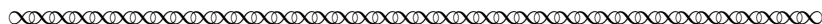
$$(\cos v + \epsilon \sin v)(\cos u + \epsilon \sin u) = \cos v \cdot \cos u - \sin v \cdot \sin u + \epsilon(\cos v \cdot \sin u + \cos u \cdot \sin v).$$

So far, Wessel has developed the trigonometric functions for arcs along a circle of radius 1, or of unit length. In the following excerpt, he will show how to use trigonometry to study a circle of arbitrary radius. Multiplying an expression of the form  $\cos v + \epsilon \sin v$  by a positive number  $r$  does change the direction, but rather scales the vector.



§9

According to §7  $\cos v + \epsilon \sin v$  is a radius of a circle and of length = 1; its deviation from  $\cos 0^\circ$  is the angle  $v$ ; from this it follows that  $r \cos v + r\epsilon \sin v$  denotes a straight line whose length is  $r$ , and whose directional angle is  $= v$ , for when the smaller sides of a right triangle are increased  $r$  times, then so is the hypotenuse, and the angles are unchanged; but the sum of the smaller sides is, according to §1, equal to the hypotenuse, that is,  $r \cos v + r\epsilon \sin v = r(\cos v + \epsilon \sin v)$ . So, this is a general expression for a straight line coplanar with the lines  $\cos 0^\circ$  and  $\epsilon \sin 90^\circ$ , of length  $r$  and deviating from  $\cos 0^\circ$  by  $v$  degrees.



**Task 19**

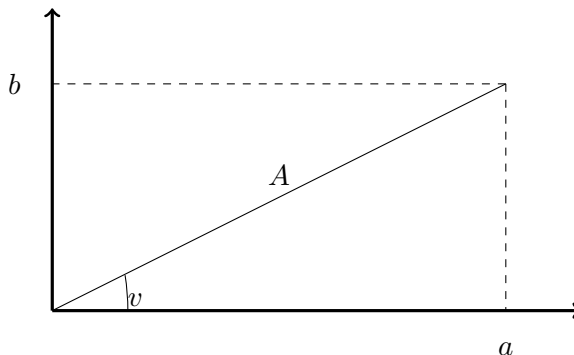
Use the definition of multiplication to explain why “ $r \cos v + r\epsilon \sin v$  denotes a straight line whose length is  $r$ , and whose directional angle is  $= v$ ”.

**3.2 Polar and rectangular coordinates**

Now is a good time to establish a relationship between the two forms of a line. We have seen that when a **complex number** is written as  $a + \epsilon b$  for  $a, b$  real numbers and  $\epsilon = \sqrt{-1}$ , we say that it is expressed in **rectangular coordinates**. The number  $a$  is called the **real part** while  $b$  is called the **imaginary part**. In §9, Wessel has established that any line  $a + \epsilon b$  can be written as  $A(\cos v + \epsilon \sin v)$  for some positive  $A$  and angle  $v$  (this is a fact that Wessel will refer back to several times). The form  $A(\cos v + \epsilon \sin v)$  expresses the complex number or line in **polar coordinates**. However, Wessel does not provide a formula that translates from rectangular to polar coordinates or vice-versa. For example, suppose we are given the line  $3 + \epsilon 7$  in rectangular coordinates. What is its length and angle? In other words, what are the polar coordinates for this line? We will work through exercises to compute this.

**Task 20**

Let  $a + \epsilon b = A(\cos v + \epsilon \sin v)$  and consider the following diagram:



- (a) Use the diagram and the Pythagorean theorem to find  $A$  in terms of  $a$  and  $b$ .
- (b) Use the diagram and the definition of  $\tan(v)$  to compute  $v$  in terms of  $a$  and  $b$ .

### 3.3 Another way to see $\sqrt{-1}$

Recall from Section 2.2 that Wessel gives a rule for how two lines should be multiplied. The length of the product should be the product of the lengths and the angle of the product should be the sum of the angles. Writing this algebraically, this means that

$$A(\cos v + \epsilon \sin v) \cdot B(\cos u + \epsilon \sin u) = AB(\cos(u + v) + \epsilon \sin(u + v)).$$

We can define  $\epsilon$  to be the value that makes the above equation true. In the next task, you will deduce the value of  $\epsilon$ .

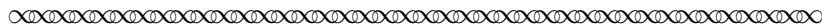
#### Task 21

- (a) Multiply out the expression  $A(\cos v + \epsilon \sin v) \cdot B(\cos u + \epsilon \sin u)$ , treating  $\epsilon$  as an unknown.
- (b) Use the sum of angles formula from §8 to simplify your expression in part 1.
- (c) Setting your expression in part 2 equal to  $AB(\cos(u + v) + \epsilon \sin(u + v))$ , determine the value of  $\epsilon$  that will make your equation true.

Note that  $\epsilon$  is not unique in the sense that it is determined by choice of orientation of the plane.

## 4 Algebraic manipulations of geometric objects

Next, Wessel next provides a key result – that the distributive property of multiplication over addition holds for lines. Notice that this will allow us to perform easy algebraic computations.



### §10

Let  $a, b, c, d$  denote direct line segments of any lengths whatsoever, positive or negative, and assume that the two indirect lines  $a + \epsilon b$  and  $c + \epsilon d$  are coplanar with the absolute unit; then their product can be found, even when their deviation from the absolute unit is unknown; all one needs to do is multiply each of the added lines in the one sum with each of those whose sum is the second factor; adding up all these products one gets the required product, its length as well as its direction, namely

$$(a + \epsilon b)(c + \epsilon d) = ac - bd + \epsilon(ad + bc)$$

Proof: Let the line  $(a + \epsilon b)$  have length  $A$  and deviate  $v$  degrees from the absolute unit, and let the line  $(c + \epsilon d)$  have length =  $C$  and deviation =  $u$ ; then according to §9:

$$a + \epsilon b = A \cos v + A\epsilon \sin v$$

and

$$c + \epsilon d = C \cos u + C\epsilon \sin u,$$

so that

$$a = A \cos v,$$

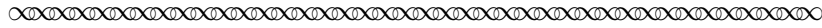
$$b = A \sin v,$$

$$c = C \cos u,$$

$$d = C \sin u$$

(§3), but according to §4  $(a + \epsilon b)(c + \epsilon d) = AC[\cos(v + u) + \epsilon \sin(v + u)] = AC[\cos v \cos u - \sin v \sin u + \epsilon(\cos v \sin u + \cos u \sin v)]$ , (§8). Consequently, by replacing  $AC \cos v \cos u$  by  $ac$  and  $AC \sin v \sin u$  by  $bd$ , etc., we get what was to be proved.

From this it follows that even if the added lines of the sum are not all direct there is no need for an exception to the known rule, on which the theory of equations and theory of integral functions and their *Divisores simplices* are based, namely, when two sums are to be multiplied, then each of the added quantities in the one sum must be multiplied by every term in the second sum. Therefore one may be assured that when an equation is about straight lines and its root is of the form  $a + \epsilon b$ , then one is dealing with an indirect line. but if one wanted [sic] to multiply two lines that are not both in the same plane as the absolute unit, then the above mentioned rule would have to be abandoned. This is the reason why I omit multiplication of such lines.



**Task 22**

(a) Use

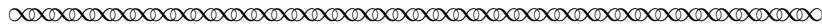
$$(a + \epsilon b)(c + \epsilon d) = ac - bd + \epsilon(ad + bc)$$

to compute the product  $(3 + \epsilon 4)(6 + \epsilon 5)$ .

- (b) What happens to a complex number when it is multiplied by a real number (a number with only imaginary part)? How does this arise algebraically? Geometrically? Discuss in terms of rotations.
- (c) What happens to a complex number when it is multiplied by a number with no real part (a number with no imaginary part)? How does this arise algebraically? Geometrically? Discuss in terms of rotations.

### 4.1 Division

We now turn to division of lines. Notice that Wessel defines division in terms of multiplication.

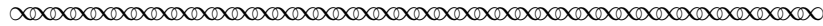


#### §11

The quotient multiplied by the divisor must be equal to the dividend. Thus it need not be proved that these lines must lie in the same plane with the absolute unit, because it follows immediately from the definition in §4. Similarly, it is easily seen that the quotient must deviate from the absolute unit by the angle  $v - u$ , if the dividend deviates by the angle  $v$ , and the divisor by the angle  $u$ , both from the unit.

Consider for instance the case when  $A(\cos v + \epsilon \sin v)$  is to be divided by  $B(\cos u + \epsilon \sin u)$ ; then the quotient is  $\frac{A}{B}[\cos(v - u) + \epsilon \sin(v - u)]$ , because  $\frac{A}{B}[\cos(v - u) + \epsilon \sin(v - u)] \cdot B(\cos u + \epsilon \sin u) = A(\cos v + \epsilon \sin v)$ .

$\epsilon \sin u) = A(\cos v + \epsilon \sin v)$  according to §7. This is the case since  $\frac{A}{B}[\cos(v-u) + \epsilon \sin(v-u)]$  multiplied by the divisor  $B(\cos u + \epsilon \sin u)$  is equal to the dividend  $A(\cos v + \epsilon \sin v)$ , and hence the quotient we are looking for is  $\frac{A}{B}[\cos(v-u) + \epsilon \sin(v-u)]$ .



We'll develop a more modern statement and proof of Wessel's result in §11 in the next Task. It will be your job to fill in some of the details by consulting Wessel's argument.

**Task 23** Fill in the blanks (1)–(7) in the statement and proof below to give a formal justification of Wessel's claim in §11.

**Proposition 1.** *Let  $A, B$  be real numbers,  $u, v$  angles. Then*

$$\frac{A \quad (1)}{B \quad (2)} = \frac{A \quad (3)}{B \quad (4)}.$$

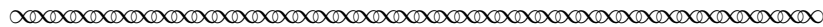
*Proof.* Observe that

$$\begin{aligned} \frac{A}{B}[\cos(v-u) + \epsilon \sin(v-u)] \cdot B(\cos u + \epsilon \sin u) &= \underline{\hspace{10em}} \quad (5) && \text{by } \underline{\hspace{1em}} \quad (6) \\ &= A[\cos v + \epsilon \sin v]. \end{aligned}$$

Dividing the first and last equation by  $\underline{\hspace{1em}} \quad (7)$  yields the desired equality. □

**Task 24** Use the above formula to compute  $\frac{1}{\cos u + \epsilon \sin u}$  without the denominator.

## 5 Inverses, division, modulus, and conjugate



### §12

If  $a, b, c, d$  are direct lines, and the indirect  $a + \epsilon b$  and  $c + \epsilon d$  are coplanar with the absolute unit, then

$$\frac{1}{c + \epsilon d} = \frac{c - \epsilon d}{c^2 + d^2},$$

and the quotient

$$\frac{a + \epsilon b}{c + \epsilon d} = (a + \epsilon b) \cdot \frac{1}{c + \epsilon d} = (a + \epsilon b) \cdot \frac{c - \epsilon d}{c^2 + d^2} = \frac{ac + bd + \epsilon(bc - ad)}{c^2 + d^2}$$

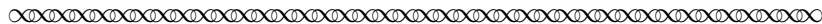
because according to §9 one may substitute  $a + \epsilon b = A(\cos v + \epsilon \sin v)$  and  $c + \epsilon d = C(\cos u + \epsilon \sin u)$ , and hence  $c - \epsilon d = C(\cos u - \epsilon \sin u)$  according to §3, and because  $(c + \epsilon d)(c - \epsilon d) = c^2 + d^2 = C^2$  (§10), it follows  $\frac{c - \epsilon d}{c^2 + d^2} = \frac{1}{C}(\cos u - \epsilon \sin u)$ , (§10),



or  $\frac{c-\epsilon d}{c^2+d^2} = \frac{1}{C}(\cos(-u) - \epsilon \sin(-u)) = \frac{1}{c+\epsilon d}$ , (§11); when this is multiplied by  $a + \epsilon b = A(\cos v + \epsilon \sin v)$  one gets

$$(a + \epsilon b) \cdot \frac{c - \epsilon d}{c^2 + d^2} = \frac{A}{C}(\cos(v - u) + \epsilon \sin(v - u)) = \frac{a + \epsilon b}{c + \epsilon d},$$

§11.



**Task 25**

We will rewrite the above in proposition-proof format below. It is your task to fill in the blanks (1)-(4).

**Proposition 2.** Let  $a + \epsilon b$  and  $c + \epsilon d$  be lines. Then  $\frac{1}{c+\epsilon d} = \frac{c-\epsilon d}{c^2+d^2}$  and  $\frac{a+\epsilon b}{c+\epsilon d} = \frac{(a+\epsilon b)(c-\epsilon d)}{c^2+d^2}$

*Proof.* Write  $c + \epsilon d = C(\cos u + \epsilon \sin v)$ . By §3,  $c = C \cos u$  and  $d = \underline{\hspace{1cm} (1) \hspace{1cm}}$ , so  $-d = \underline{\hspace{1cm} (2) \hspace{1cm}}$ . Thus  $c - \epsilon d = \underline{\hspace{1cm} (3) \hspace{1cm}}$ . By §10,  $c^2 + d^2 = C^2$ . Hence

$$\begin{aligned} c - \epsilon d &= C(\cos u - \epsilon \sin u) \\ \frac{c - \epsilon d}{c^2 + d^2} &= \frac{\cos u - \epsilon \sin u}{C} \end{aligned}$$

We thus have

$$\begin{aligned} \frac{c - \epsilon d}{c^2 + d^2} &= \frac{\cos(-u) + \sin(-u)}{C} \\ &= \frac{1}{C(\cos u + \epsilon \sin u)} && \text{By Task } \underline{\hspace{1cm} (4) \hspace{1cm}} \\ &= \frac{1}{c + \epsilon d} \end{aligned}$$

whence the first claim. Multiplying this by  $a + \epsilon b$ , we obtain

$$\frac{a + \epsilon b}{c + \epsilon d} = \frac{(a + \epsilon b)(c - \epsilon d)}{c^2 + d^2}.$$

□

A couple of very important concepts arose in the above computation. In addition to the complex number,  $c + \epsilon d$ , we saw that the related complex number  $c - \epsilon d$  and their product, the real number  $c^2 + d^2$  appeared more than once. In your work in Task 20, a similar expression showed up. In general, if  $a + \epsilon b$  is a complex number, its **(complex) conjugate** or **conjugate** is given by  $\overline{a + \epsilon b} = a - \epsilon b$ . Furthermore, the **absolute value, length, magnitude** or **modulus** of  $a + \epsilon b$  is given by  $|a + \epsilon b| = \sqrt{a^2 + b^2}$ .

**Task 26**

In the following, let  $z = a + \epsilon b$  and  $w = c + \epsilon d$ . Prove that

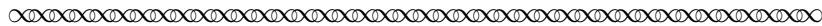
(a)  $\overline{\overline{z}} = z$ .

- (b)  $\overline{z \pm w} = \bar{z} \pm \bar{w}$ .
- (c)  $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ .
- (d)  $\overline{\frac{z}{w}} = \frac{\bar{z}}{\bar{w}}$
- (e)  $|z| = |\bar{z}|$
- (f)  $|z \cdot w| = |z| \cdot |w|$
- (g)  $|-z| = |z|$
- (h)  $|z + w|^2 = (z + w)\overline{(z + w)}$

**Task 27** Let  $z = a + \epsilon b$ . Prove that  $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ .

## 6 De Moivre's theorem

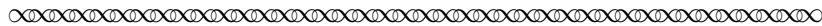
In this section, we see how Wessel somewhat matter-of-factly proves an important result called de Moivre's theorem.



### §13

If  $m$  is an integer,

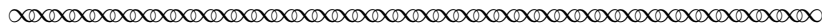
... we always have  $\cos \frac{v}{m} + \epsilon \sin \frac{v}{m} = (\cos v + \epsilon \sin v)^{\frac{1}{m}}$  and therefore, when  $m$  and  $n$  are both integers  $(\cos v + \epsilon \sin v)^{\frac{n}{m}} = \cos \frac{n}{m}v + \epsilon \sin \frac{n}{m}v$ .



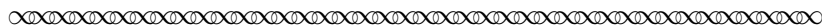
Your task in the next exercise is to prove de Moivre's theorem.

**Task 28** Use results found in §7 and §11 to prove de Moivre's theorem. That is, prove that  $(\cos v + \epsilon \sin v)^{\frac{n}{m}} = \cos \frac{vn}{m} + \epsilon \sin \frac{vn}{m}$ .

Wessel continues with an application of this powerful theorem. We will give others below.



From this one may find the value of expressions like  $\sqrt[n]{(b + c\sqrt{-1})}$  or  $\sqrt[m]{[a + \sqrt[n]{(b + c\sqrt{-1})}]}$ ; for instance  $\sqrt[3]{(4\sqrt{3} + 4\sqrt{-1})}$  designates a straight line, whose length is = 2, and whose angle with the absolute unit is  $10^\circ$ .



To verify Wessel's claim, we use the formula established in Task 20 and compute

$$\begin{aligned}\sqrt[3]{4\sqrt{3} + 4\epsilon} &= (4\sqrt{3} + 4\epsilon)^{\frac{1}{3}} \\ &= (8 \cos(30^\circ) + \epsilon \sin(30^\circ))^{\frac{1}{3}} \\ &= 2(\cos(10^\circ) + \sin(10^\circ)).\end{aligned}$$

We then easily read off that this line has length 2 and angle  $10^\circ$ .

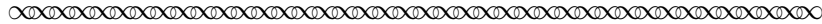
**Task 29** Find the length and angle of the line  $(\sqrt{2 + \epsilon\sqrt{12}})^3$ .

These are other interesting uses of de Moivre's theorem. For example, we can use it to compute positive powers of a complex number. Let  $z = 3 - \epsilon\sqrt{3}$ . We then compute

$$\begin{aligned}(3 - \epsilon\sqrt{3})^5 &= (\sqrt{12}(\cos 150^\circ + \epsilon \sin 150^\circ))^5 \\ &= (2\sqrt{3})^5(\cos 30^\circ + \epsilon \sin 30^\circ) \\ &= 32 \cdot 3^{\frac{5}{2}} \left( \frac{\sqrt{3}}{2} + \frac{\epsilon}{2} \right) \\ &= 432 + 16 \cdot 3^{\frac{5}{2}}\epsilon.\end{aligned}$$

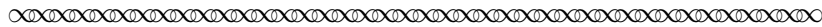
**Task 30** Let  $z = 1 - \epsilon$  and compute  $z^{10}$ .

## 7 Periodicity



### §14

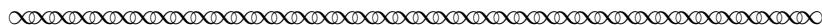
When two angles have equal sine and equal cosines, then their difference is either 0 or  $\pm 4$  right angles, or a multiple of  $\pm 4$  right angles; and conversely, when the difference between two angles is 0 or  $\pm 4$  right angles, taken once or several times, then their sines as well as their cosines are equal.



**Task 31** (a) Rewrite Wessel's claim as an if and only if statement by filling in the blanks (1)–(3) below:

“ $\cos(v) = \cos(u)$  and \_\_\_\_\_ (1) \_\_\_\_\_ if and only if  $u - v =$  \_\_\_\_\_ (2) \_\_\_\_\_  $\cdot k$ , where  $k$  is \_\_\_\_\_ (3) \_\_\_\_\_”

(b) Prove the claim you just made.



§15<sup>1</sup>

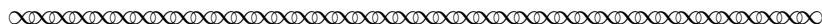
If  $m$  is an integer and  $2\pi = 360^\circ$ , then  $(\cos v + \epsilon \sin v)^{\frac{1}{m}}$  attains only the following  $m$  different values  $\cos \frac{v}{m} + \epsilon \sin \frac{v}{m}, \cos \frac{2\pi+v}{m} + \epsilon \sin \frac{2\pi+v}{m}, \dots, \cos \frac{(m-1)2\pi+v}{m} + \epsilon \sin \frac{(m-1)2\pi+v}{m}$ , because the numbers by which  $2\pi$  is multiplied in the preceding sequence are in arithmetic progression  $1, 2, 3, 4, \dots, m - 1$ .



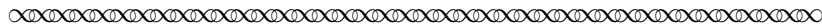
The rest of this section is devoted to following Wessel’s proof of this last claim. He breaks it up into the following parts:

1. Replacing  $2\pi$  with  $-2\pi$  in the sequence above produces exactly the same complex numbers.
2. All the values in the sequence are distinct from one another.
3. Extending the sequence does not produce any different values.
4. No angle that corresponds to one of the  $m$ th roots of  $\cos v + \epsilon \sin v$  can be other than those already in the above sequence.

We begin by showing step 1.



Therefore the sum of any two [terms  $n, u$  of the sequence  $1, 2, \dots, m - 1] = m$ , when one is as far from 1 as the other is from  $m - 1$ , and if their number is odd, then twice the middle one  $= m$ ; hence, when one adds  $\frac{(m-n)2\pi+v}{m}$  to  $\frac{(m-u)2\pi+v}{m}$ , and the former in the sequence is as far from  $\frac{2\pi+v}{m}$  as  $\frac{(m-n)2\pi+v}{m}$  is from  $\frac{(m-1)2\pi+v}{m}$ , then the sum  $= \frac{2m-u-n}{m}2\pi + \frac{2v}{m} = 2\pi + \frac{2v}{m}$ .



Admittedly, it is fairly difficult to see what Wessel is trying to communicate here. However, his claims are not too difficult to verify when one realizes he is making claims about two integers between 1 and  $m - 1$ . In his paragraph above, he uses  $n, u$  to denote the two integers. Thus, the claim

Therefore the sum of any two  $= m$ , when one is as far from 1 as the other is from  $m - 1$

means

$$\text{“If } u - 1 = (m - 1) - n, \text{ then } u + n = m.\text{”}$$

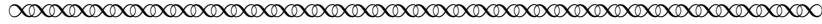
In this light, the claim is obvious.

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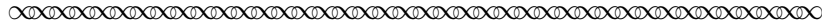
<sup>1</sup>In this section and the following, Wessel defines  $\pi = 360^\circ$ . We have chosen to use the familiar definition  $2\pi = 360^\circ$  here to avoid the unnecessary confusion of Wessel’s unorthodox use of the symbol  $\pi$ .

**Task 32**

Assuming that  $u - 1 = m - 1 - n$ , prove that  $\frac{(m-n)2\pi+v}{m} + \frac{(m-u)2\pi+v}{m} = 2\pi + \frac{2v}{m}$ .

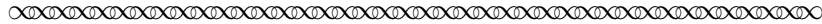


But to add  $\frac{(m-n)2\pi}{m}$  is the same as subtracting  $\frac{(m-n)(-2\pi)}{m}$ ; and since the difference is  $2\pi$ , then  $\frac{(m-n)(-2\pi)+v}{m}$  has the same cosine and sine as  $\frac{(m-u)2\pi+v}{m}$ , according to §14; similarly,  $\frac{(m-u)(-2\pi)+v}{m}$  and  $\frac{(m-n)2\pi+v}{m}$  have the same cosine and sine; thus  $-2\pi$  does not give any other values than  $+2\pi$ .

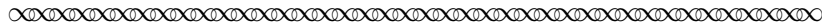
**Task 33**

Following Wessel, give a formal proof of the above. Conclude that “ $-2\pi$  does not give any other values than  $2\pi$ .” That is, replacing  $2\pi$  with  $-2\pi$  in the expressions  $\cos \frac{v}{m} + \epsilon \sin \frac{v}{m}$ ,  $\cos \frac{2\pi+v}{m} + \epsilon \sin \frac{2\pi+v}{m}$ ,  $\dots$ ,  $\cos \frac{(m-1)2\pi+v}{m} + \epsilon \sin \frac{(m-1)2\pi+v}{m}$  produces exactly the same complex numbers.

The second part of the claim follows quite easily.



But that none of these are equal follows from the fact that the difference between two of the angles in the sequence is always less than  $2\pi$ , and never = 0. Neither does one get more values by continuing the sequence, because then one gets the angles  $2\pi + \frac{v}{m}$ ,  $2\pi + \frac{2\pi+v}{m}$ ,  $4\pi + \frac{4\pi+v}{m}$ , etc., so according to §14 the values of their cosine and sine are the same as before. If the angles were to fall outside the sequence, then  $2\pi$  was not multiplied by an integer in the numerator, and the angles taken  $m$  times could not produce an angle which subtracted from  $v$  gave 0, or  $\pm 2\pi$ , or a multiple of  $\pm 2\pi$ ; hence, neither could the  $m$ th power of such an angle have cosine and sine =  $\cos v + \epsilon \sin v$ .



## 8 Roots of unity

As an interesting and important application of this work, we continue investigating some of the ideas begun in Section 6 concerning roots of complex numbers and equations. One of the most celebrated applications of complex analysis is that it provides a proof for the Fundamental Theorem of Algebra.

**Theorem 3.** *Every non-constant polynomial of a single complex variable with complex coefficients has at least one complex root. In particular, every polynomial with real coefficients has at least one complex root.*

Although a proof of this theorem is beyond the scope of this project (but see [Conway, 1978] for a proof), we may begin to lay some of the groundwork towards this result. Let  $z = r(\cos u + \epsilon \sin u)$  and  $w = s(\cos v + \epsilon \sin v)$  be two complex numbers and consider the equation

$$w^n = z.$$

Using de Moivre's theorem, we know that

$$s^n(\cos nv + \epsilon \sin nv) = w^n = r(\cos u + \epsilon \sin u),$$

and furthermore we can assert that

$$\begin{aligned} s^n &= r \\ \cos nv &= \cos u \\ +\epsilon \sin nv &= +\epsilon \sin u. \end{aligned}$$

By our work in Section 7, we conclude that

$$v = \frac{u + 2k\pi}{n}.$$

The key insight here is to note that as  $k$  takes on values from 0 to  $n - 1$ , the modulus stays fixed at  $\sqrt[n]{r}$  but the angle changes. Hence for each integer value  $0 \leq k \leq n - 1$ , we obtain the  $n^{\text{th}}$  **roots of the complex number**  $z$  given by

$$w_k = \sqrt[n]{r} \left( \cos \left( \frac{u + 2k\pi}{n} \right) + \epsilon \sin \left( \frac{u + 2k\pi}{n} \right) \right).$$

**Task 34** Find all fourth roots of  $i$ .

We may now define the  $n^{\text{th}}$  **roots of unity** to be solutions to the equation  $z^n = 1$ .

**Task 35** (a) Prove that the  $n^{\text{th}}$  roots of unity are given by

$$\cos \frac{2k\pi}{n} + \epsilon \sin \frac{2k\pi}{n}.$$

(b) Show that there is an  $n^{\text{th}}$  root of unity  $\zeta_n$  such that if  $w$  is any  $n^{\text{th}}$  root of unity, then  $\zeta_n^k = w$  for some  $0 \leq k \leq n - 1$ .

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## Notes to Instructors

### PSP Content: Topics and Goals

This Primary Source Project (PSP) draws on the work of Casper Wessel (1745–1818), a Norwegian map surveyor, following closely his 1797 paper *Om Directionens analytiske Betegning, et Forsog, anvendt fornemmelig til plane og sphaeriske Polygoners Oplosning* (On the Analytical Representation of Direction. An Attempt Applied Chiefly to Solving Plane and Spherical Polygons).

The goal of this project is to introduce the basics of complex numbers from a geometric perspective, including important and fundamental algebraic connections. There are many ways that instructors could use this project in their classroom. It could be used in the first 1-2 weeks of an undergraduate or possibly graduate introduction to complex analysis course, as part of a capstone course, or even as some type of special project in a trigonometry or pre-calculus course that introduces complex numbers. The latter, of course, would require more support from the instructor.

### Student Prerequisites

Little mathematical background is needed to understand this project. It does not formally assume anything but some rudimentary trigonometry and even that is in theory built from the ground up by Wessel. However, instructors using this PSP in a trigonometry or pre-calculus course will likely need to scaffold considerably. Students who have prior courses that involve writing proofs can be asked to formally prove many of the results. The project builds off of certain first principles, most especially in the section on trigonometric functions. In this sense, the project may be said to be axiomatic in nature, which can lead to discussions of rigor and careful attention to logical details.

### PSP Design, and Task Commentary

This PSP is designed as a stand-alone introduction to the arithmetic and geometric properties of the complex numbers.

Section 2 provides the geometric background that is crucial to the project.

Task 2 asks students to try to interpret a long passage of Wessel, and they may need some support in doing so. The overarching concept that Wessel seems to be going for is given at the beginning of his third full paragraph; that is, the relationship that a line through a given point can have with a given line. If the students are told this after struggling to come to their own answers, then they can be asked what all the relationships between a given line and a point are. After they think about this, students can then reread Wessel and see if they can find their answers in his writing.

Tasks 3–12 guide the students through basic terminology and geometric/physical operations on lines, including how to add and multiply lines, commutativity of each operation, introduction of the unit line (+1), and decomposition into orthogonal components. Students in an upper-level class can likely work through many of these on their own.

These tasks set the stage for Section 3, where Wessel introduces, with no prior mention, the definition of cosine to a circular arc. Things get subtle here, and students will be tempted to fall back on their knowledge of cosine from trigonometry. As mentioned above, this provides the instructor an opportunity to emphasize the axiomatic development of this PSP, and the importance of using only what is provided. It can also lead to a discussion on how it is hard for us to set aside our modern notions of mathematics when reading from an original source about a topic that has evolved over time.



The remainder of Section 3 introduces basic terminology of complex numbers and deduces that the value of  $\epsilon$  is in fact  $\sqrt{-1}$ .

Section 4 and 5 focus on key arithmetic properties of complex numbers, beginning with the distributive property of multiplication over addition, and heading into division, inverses, the modulus, and the conjugate. Modern notation is used and Task 26 asks students to prove a set of 8 common properties of complex numbers and their associated arithmetic.

Section 6 uses Wessel's proof to establish De Moivre's theorem and then ask students to use the result to simplify complicated expressions and raise complex numbers to powers.

Section 7 addresses the periodicity of complex numbers, and Section 8 concludes the PSP with the roots of unity.

L<sup>A</sup>T<sub>E</sub>X code of this entire PSP is available from the authors and can be used to facilitate preparation of advanced preparation or reading guides, 'in-class worksheets' based on tasks included in the project, homework assignments, and slide decks. The PSP itself can also be modified by instructors as desired to better suit their goals for the course. The authors would be thrilled to receive a copy of any modifications or supplementary material produced that you're willing to share with them.

### **Sample Implementation Schedule (based on a 50-minute class period)**

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students.

- Day 1: Although Wessel's writing may be difficult to interpret, the mathematics is not too difficult, and it is not unreasonable to assign Section 1–2 for the first day that this project is implemented. The first 15 minutes of class can begin with a discussion of what the students read and their answers to the questions. It is important that students understand this section well, as it is critical for the rest of the project. Hence, a delicate balance must be struck between allowing the students to share and discuss their ideas about Sections 1–2 versus making sure the students understand the content. The next 20 minutes can be spent working in small groups on Section 3.1. Again, because understanding this section is so crucial, it is worth spending the last 15 minutes of class debriefing this section, ensuring the students understand the content. Students can share their ideas about the complex plane on the board or any other IBL technique may be used. A nice write up of the tasks in the first two sections should be assigned as homework (either group homework or individual homework). Students should also be assigned reading the rest of Section 3 for homework.
- Day 2: This class can begin with the instructor proposing two problems on the board— convert a particular expression from rectangular to polar, and another expression from polar to rectangular. Students can spend the first five minutes of class, either in groups or as individuals, to figure out the solutions, and a group or individual may be called on to present the solution(s) to the class on the board. This will help remind the students what they read about in Section 3.3. At this point, a similar schedule to day 1 may be followed; that is, students can spend 20 minutes in groups working through Section 4 on multiplication and division. After the class reconvenes as a whole, the class can work through Tasks 23 and 24 as a group. The last 10 minutes may then be used for students to individually read and work on Section 5. A write up of Section 3.3, 3.4, and Section 4 may be assigned as homework.

- Day 3: Tasks 26 and 27, while in a certain sense basic exercises, are good ones to ensure everyone is on the same page. To that end, part of this day can be devoted to making sure the entire class sees the solution to these tasks. (Task 26 is actually comprised of 8 different exercises). One possibility is to assign a different problem to pairs of students and give the pairs 10 minutes to work on their problem. Each group can then present their solution to the class, which could take up to 30 minutes depending on how much a group might struggle with a problem. The remainder of the class period can be spent either individually reading the discussion of de Moivre's theorem or by having the professor lead a discussion about the importance and interest of de Moivre's theorem.
- Day 4: One can spend an entire day on de Moivre's theorem, depending on how familiar the instructor is with it. There are many interesting applications of it, and one can go far beyond what we have here. Otherwise, students can spend 20 minutes working through this section in groups with 15 minutes given to debrief this section. The last 15 minutes can be spent again working in groups reading about periodicity with the exercises assigned as homework.
- Day 5: This final day introduces root of unity which is a very important concept not only in complex analysis, but abstract algebra, number theory, and others. Like de Moivre's theorem, one can pull in many interesting and varied applications of roots of unity. For the material in this section, students can prepare Tasks 34 and 35 as homework for this day, or work in groups on one of those tasks, present it to the class, and complete the other one as homework.

### **Connections to other Primary Source Projects**

The PSP entitled “The Logarithm of  $-1$ ”, written by Dominic Klyve, treats the logarithm function on negative and complex numbers. While historically that work came before that of Wessel's, in the context of a modern complex analysis course, implementation of that PSP would generally come later in the term than the current one.

The current project connects loosely with Danny Otero's PSP entitled “A Genetic Context for Understanding the Trigonometric Functions”.

Nick Scoville's project “Sets to Metric Spaces to Topological Spaces” is also axiomatic in nature in the same sense as the current PSP.

The most recent versions of these PSPs can be found on the TRIUMPHS website.

### **Recommendations for Further Reading**

For an instructor looking to dive more deeply into the work of Wessel, reading his original paper would be strongly encouraged.

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