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Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean

Janet Heine Barnett

Colorado State University-Pueblo, janet.barnett@csupueblo.edu

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Just prior to his 19^{th} birthday, the mathematical genius Carl Friedrich Gauss (1777–1855) began a "mathematical diary" in which he recorded his mathematical discoveries for nearly 20 years. Among these discoveries was the existence of a beautiful relationship between three particular numbers:

- the ratio of the circumference of a circle to its diameter, or π;
- a specific value of a certain (elliptic) integral, which Gauss denoted by $\varpi = 2\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$;
- a number called "the arithmetic-geometric mean" of $\sqrt{2}$ and 1, which he denoted by $M(\sqrt{2}, 1)$.

Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice that historian Adrian Rice calls "Gaussian Guesswork" in his *Math Horizons* article subtitled "Why 1.19814023473559220744... is such a beautiful number" [Rice, November 2009].

This short project is one of a set of four projects that looks at the power of Gaussian guesswork via the story of his discovery of this beautiful relationship. Based on excerpts from Gauss' mathematical diary [Gauss, 2005] and related manuscripts, this particular project focuses on how infinite sequences are used to define the arithmetic-geometric mean. We begin in Section 1 with the definition, some examples and basic properties of the arithmetic-geometric mean. In Section 2, we then look briefly at how the arithmetic-geometric mean is related to the Gaussian Guesswork story.

1 The Arithmetic-Geometric Mean

Although Gauss appears to have discovered the arithmetic-geometric mean when he was only 14 years old,[1] he published very little about it during his lifetime. Much of what we know about his work in this area instead comes from a single paper [Gauss, 1799] that was published as part of his mathematical legacy (or, as the Germans would say, as part of his *Nachlass*) only after his death. In the first excerpt[2] from this paper that we will read in this project, Gauss began by defining two related infinite sequences.

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*Department of Mathematics and Physics, Colorado State University - Pueblo, Pueblo, CO 81001 - 4901; `janet.barnett@csupueblo.edu`.

[1]Gauss himself reminisced about his 1791 discovery of this idea in a letter [Gauss, 1917] that he wrote to his friend Schumacher much later, in 1816. Although his memory of the exact date may not be accurate, Gauss was certainly familiar with the arithmetic-geometric mean by the time he began his mathematical diary in 1796.

[2]All Gauss translations in this project were prepared by George W. Heine III (Math and Maps), 2017. To set them apart from the project narrative, original source excerpts are set in sans serif font and bracketed by the following symbol at their beginning and end: ∞∞∞

Let $\left\{\begin{matrix} a, & a_1, & a_2, & a_3, \dots \\ b, & b_1, & b_2, & b_3, \dots \end{matrix}\right\}$ be two sequences[3] of magnitudes formed by this condition: that the terms of either correspond to the mean between the preceding terms, and indeed, the terms of the upper sequence have the value of the arithmetic mean, and those of the lower sequence, the geometric mean, for example,

$$a_1 = \frac{1}{2}(a+b),\ b_1 = \sqrt{ab},\ a_2 = \frac{1}{2}(a_1+b_1),\ b_2 = \sqrt{a_1b_1},\ a_3 = \frac{1}{2}(a_2+b_2),\ b_3 = \sqrt{a_2b_2}.$$

But we suppose a and b to be positive reals and [that] the quadratic [square] roots are everywhere taken to be the positive values; by this agreement, the sequences can be produced so long as desired, and all of their terms will be fully determined and positive reals [will be] obtained.

Since this may be the first time that you are hearing about the 'geometric mean' of two positive numbers a, b, notice that one way to think about it is simply as the length s of a side of a square that has the same area as a rectangle with sides of length a and b respectively; this gives us the formula $s^2 = ab$, or $s = \sqrt{ab}$. Along with the more familiar 'arithmetic mean,' computed via the formula $\frac{1}{2}(a+b)$, the geometric mean is one of several 'averages' that were studied by mathematicians as far back as the ancient Greeks. The terminology 'arithmetic' and 'geometric' to describe these two different types of means also dates back to ancient Greek mathematics.

In his 1799 *Nachlass* paper, Gauss gave four specific examples of sequences (a_n) and (b_n) defined by way of the arithmetic and geometric means as described above; let's pause to look at the first two of these now.

Example 1: $a = 1, \quad b = 0.2$

$a = 1.00000\,00000\,00000\,00000\,0$	$b = 0.20000\,00000\,00000\,00000\,0$
$a_1 = 0.60000\,00000\,00000\,00000\,0$	$b_1 = 0.44721\,35954\,99957\,93928\,2$
$a_2 = 0.52360\,67977\,49978\,99964\,1$	$b_2 = 0.51800\,40128\,22268\,36005\,0$
$a_3 = 0.52080\,54052\,86123\,66484\,5$	$b_3 = 0.52079\,78709\,39876\,24344\,0$
$a_4 = 0.52080\,16831\,12999\,95414\,3$	$b_4 = 0.52080\,16380\,99375$
$a_5 = 0.52080\,16381\,06187$	$b_5 = 0.52080\,16381\,06187$

Here a_5, b_5 differ in the 23rd decimal place.

[3] Gauss himself used prime notation (i.e., a', a'', a''') to denote the terms of the sequence. In this project, we instead use indexed notation (i.e., a_1, a_2, a_3) in keeping with current notational conventions. To fully adapt Gauss' notation to that used today, we could also write $a_0 = a$ and $b_0 = b$.

Task 1 This task examines Example 1 from Gauss' paper.

(a) Verify that the values given by Gauss in the previous excerpt are correct. Are you able to use your calculator to obtain the same degree of accuracy (21 decimal places!) that Gauss obtained by hand calculations?

(b) Write three observations about the two sequences in this example. Use a full sentence to state each of your observations.

Task 2 In Example 2 from his paper, Gauss set $a = 1, \quad b = 0.6$.

(a) Use a calculator or computer to compute the next four terms of each sequence.

$a = 1.00000\,00000\,00000\,00000\,0$	$b = 0.60000\,00000\,00000\,00000\,0$
$a_1 =$	$b_1 =$
$a_2 =$	$b_2 =$
$a_3 =$	$b_3 =$
$a_4 =$	$b_4 =$

(b) In Task 1, part (b), you made three observations about Gauss' Example 1. Do the same general patterns hold for Example 2? If so, why do you think this is? If not, in what way(s) are the two examples different?

With these two examples in hand, let's now go back to look at some of the general properties that Gauss claimed will always hold for such sequences. The following excerpt includes three such properties, which we will examine in further detail in the tasks below. For now, read through each of these carefully and compare them to the observations that you made in Task 1, part (b).

Moreover, we first offer here the following observations:

I. If $a = b$, all of the terms of either sequence will be $= a = b$.

II. If however a, b are unequal, then $(a_1 - b_1)(a_1 + b_1) = \frac{1}{4}(a-b)^2$, whence it is concluded that $b_1 < a_1$, and also that $b_2 < a_2$, $\quad b_3 < a_3$ etc., i.e. any term of the lower sequence will be smaller than the corresponding [term] of the upper. Wherefore, in this case, we suppose also that $b < a$.

III. By the same supposition it will be that $a_1 < a$, $b_1 > b$, $a_2 < a_1$, $b_2 > b_1$ etc.; therefore the upper sequence constantly decreases, and the lower constantly increases; thus it is evident that each [sequence] has a limit; these limits are conveniently expressed[4] a_∞, b_∞.

The first of Gauss' observations needs little proof — if $a = b$, then clearly the arithmetic mean is $\frac{1}{2}(a+b) = \frac{1}{2}(2a) = a$ and the geometric mean is $\sqrt{ab} = \sqrt{a^2} = a$; thus, $(a_n) = (a) = (b_n)$ gives just one constant sequence. In the following tasks, you will verify Gauss' next two observations.

[4] In keeping with the notation currently in use today, we have replaced Gauss' use of superscripts (a^∞, b^∞) to denote the limiting values with subscripts (a_∞, b_∞) throughout this project.

Task 3 This task examines Gauss' Observation II:

> II. If however a, b are unequal, then $(a_1 - b_1)(a_1 + b_1) = \frac{1}{4}(a - b)^2$, whence it is concluded that $b_1 < a_1$, and also that $b_2 < a_2$, $b_3 < a_3$ etc., i.e. any term of the lower sequence will be smaller than the corresponding [term] of the upper. Wherefore, in this case, we suppose also that $b < a$.

(a) Use the definitions of a_1, b_1 to verify that $(a_1 - b_1)(a_1 + b_1) = \frac{1}{4}(a - b)^2$.

(b) Now explain why the equality in part (a) allows us to conclude that $a_1 > b_1$.

(c) Finally, explain why we are now able to conclude that $b_n < a_n$ for all values of n.

Task 4 This task examines Gauss' Observation III:

> III. By the same supposition it will be that $a_1 < a$, $b_1 > b$, $a_2 < a_1$, $b_2 > b_1$ etc.; therefore the upper sequence constantly decreases, and the lower constantly increases; thus it is evident that each [sequence] has a limit; these limits are conveniently expressed by a_∞, b_∞.

To see what Gauss meant by the phrase "the same supposition" here, look back at his statement of Observation II, and notice that he ended that earlier observation with the assertion that "... we suppose also that $b < a$."

(a) Use the assumption that $b < a$ and the definitions of the two sequences to verify Gauss' claim that (a_n) is a strictly decreasing sequence and that (b_n) is a strictly increasing sequence.

Consider Gauss' next assertion: "... thus it is evident that each [sequence] has a limit."

Write a convincing explanation why this conclusion must hold, given what we know about these two sequences thus far. Do you agree with Gauss that this conclusion is 'evident'?

(b) In your answer to part (a), you may have mentioned that the two sequences in question are *bounded* — that is, bounded both above and below. Whether or not you did so, state the values of the upper and lower bounds for the sequence (a_n), and for the sequence (b_n). Why it is important that (a_n) is bounded, as well as increasing? Similarly, why it is important that (b_n) is bounded, as well as decreasing?

(c) Now find a theorem in your Calculus textbook (in the chapter that considers infinite sequences) that could also be used to conclude that the sequences (a_n) and (b_n) converge. Give both the name of this theorem and its full statement.

(d) Do you think that the notation a_∞, b_∞ that Gauss used to denote the limits of these sequences is appropriate? Why or why not? Based on what you've seen in Examples 1 and 2, how do you expect the values of a_∞ and b_∞ to be related?

Gauss made one final observation about the sequences (a_n), (b_n), which you have perhaps already predicted yourself:

IV. Finally, from $\frac{a_1-b_1}{a-b} = \frac{(a-b)}{4(a_1+b_1)} = \frac{a-b}{2(a+b)+4b_1}$, it follows that $a_1 - b_1 < \frac{1}{2}(a-b)$, and in the same way, $a_2 - b_2 < \frac{1}{2}(a_1 - b_1)$ etc. Hence, it is concluded that $a-b$, $a_1 - b_1$, $a_2 - b_2$, $a_3 - b_3$ etc. forms a strictly decreasing sequence and the limit itself is $= 0$. Thus $a_\infty = b_\infty$, i.e., the upper and lower sequences have the same limit, which always remains below the one and above the other.

We call this limit the *arithmetic-geometric mean* between a and b, and denote it by $M(a, b)$.

Task 5 This task examines Gauss' Observation IV from the previous excerpt.

(a) Verify that $\frac{a_1-b_1}{a-b} = \frac{(a-b)}{4(a_1+b_1)}$.
Hint: Start with the equality $(a_1 - b_1)(a_1 + b_1) = \frac{1}{4}(a-b)^2$ from Gauss' Observation II.

(b) Now verify that $\frac{(a-b)}{4(a_1+b_1)} = \frac{a-b}{2(a+b)+4b_1}$.

Combining the results of parts (a) and (b), we now have the equality $\frac{a_1-b_1}{a-b} = \frac{a-b}{2(a+b)+4b_1}$. Use this equality to explain why the following hold:

(i) $a_1 - b_1 < \frac{1}{2}(a-b)$

(ii) $a_2 - b_2 < \frac{1}{2}(a_1 - b_1) < \frac{1}{4}(a-b)$

(iii) Taking n to be any arbitrary value, $a_n - b_n < \left(\frac{1}{2}\right)^n (a-b)$

(c) Based on part (b), why can we now conclude (with Gauss) that '$a - b$, $a_1 - b_1$, $a_2 - b_2$, $a_3 - b_3$ etc forms a strictly decreasing sequence and the limit itself is $= 0$'? That is, why does $\lim_{n\to\infty}(a_n - b_n) = 0$? Also explain why this allowed Gauss to conclude that $a_\infty = b_\infty$.

Now that we have verified Gauss' claim that $a_\infty = b_\infty$, his definition of the *arithmetic-geometric mean* $M(a, b)$ of two positive numbers a and b as the common value of those two limits makes sense! Stating this in modern notation, we have:

$$M(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n, \quad \text{where } a_0 = a,\ b_0 = b,\ a_{n+1} = \tfrac{1}{2}(a_n + b_n) \text{ and } b_{n+1} = \sqrt{a_n b_n}.$$

This is precisely how $M(a, b)$ is still defined today, although it is often denoted instead by $\mathrm{agm}(a, b)$ or $\mathrm{agM}(a, b)$. Today, the arithmetic-geometric mean is used to construct fast algorithms for calculating values of elementary transcendental functions and some classical constants, like π — examples 1 and 2 above show just how rapidly this process generally converges. In the next section of this project, we will consider why Gauss himself became interested in the arithmetic-geometric mean.

2 Why *is* 1.19814023473559220744... such a beautiful number?

The complete answer to this question requires a bit more mathematics than we will study in this short project. In this closing section, we summarize just the highlights. We begin by looking at Gauss' fourth example of the arithmetic-geometric mean in his 1799 *Nachlass* paper.

Example 4: $a = \sqrt{2}, \quad b = 1$

$a = 1.41421\,35623\,73095\,04880\,2$	$b = 1.00000\,00000\,00000\,00000\,0$
$a_1 = 1.20710\,67811\,86547\,52440\,1$	$b_1 = 1.18920\,71150\,02721\,06671\,7$
$a_2 = 1.19815\,69480\,94634\,29555\,9$	$b_2 = 1.19812\,35214\,93120\,12260\,7$
$a_3 = 1.19814\,02347\,93877\,20908\,3$	$b_3 = 1.19814\,02346\,77307\,20579\,8$
$a_4 = 1.19814\,02347\,35592\,20744\,1$	$b_4 = 1.19814\,02347\,35592\,20743\,9$

Notice that the first 20 decimal places of the limit value for this example is precisely the number that appears in title of this section. But why would Gauss or others have considered $M(\sqrt{2}, 1)$ to be particularly beautiful?

In fact, Gauss' initial acquaintance with the number 1.19814023473559220744... had nothing to do with the arithmetic-geometric mean, but was related instead to his efforts to evaluate certain types of integrals. One such integral and its value will be quite familiar to you; namely,

$$\int_0^1 \frac{1}{\sqrt{1-t^2}}dt = \frac{\pi}{2}$$

Make sure you believe that this is a correct value ... and then consider what you might do to evaluate the following quite-similar-looking integral:

$$\int_0^1 \frac{1}{\sqrt{1-t^4}}dt$$

Stuck? If so, then you're in good company! Evaluating integrals of the form $\int_0^x \frac{1}{\sqrt{1-t^n}}dt$ for $n > 2$ is notoriously difficult[5] ... but also necessary for problems such as determining the arc length of ellipses and other curves that naturally arise in astronomy and physics. For instance, the integral $\int_0^1 \frac{1}{\sqrt{1-t^4}}dt$ gives the arc length of a curve known as a *lemniscate*.

Based on entries in his mathematical diary, we know that Gauss himself began to study elliptic integrals in September 1796, at the ripe old age of 19. Following an analogy suggested by the fact that $\pi = 2\int_0^1 \frac{1}{\sqrt{1-t^2}}dt$, he defined a new constant by setting $\varpi = 2\int_0^1 \frac{1}{\sqrt{1-t^4}}dt$ and used power series techniques to calculate ϖ to twenty decimal places, finding that[6]

$$\varpi = 2.62205755429211981046\ldots.$$

[5]For $n = 3$ and $n = 4$, $\int_0^1 \frac{1}{\sqrt{1-t^n}}dt$ is called an *elliptic* integral; for $n > 5$, it is called *hyperelliptic*.

[6]The symbol 'ϖ' that Gauss used to denote this specific value is called "varpi;" it is a variant of the Greek letter π.

Task 6 This task looks at the relationship between ϖ, π, and $M(\sqrt{2}, 1)$.

Recalling that $\pi = 3.14159265258979323846\ldots$ and $M(\sqrt{2}, 1) = 1.19814023473559220744\ldots$, try to find a relationship between these two numbers and the value of ϖ given above.

Three years after taking up the study of elliptic functions, Gauss wrote the following entry in his mathematical diary on May 30, 1799 [Gauss, 2005]:

We have established that the arithmetic-geometric mean between $\sqrt{2}$ and 1 is π/ϖ to 11 places; the proof of this fact will certainly open up a new field of analysis.

It took Gauss another year to fully prove that his guesswork about the numerical relationship $M(\sqrt{2}, 1) = \frac{\pi}{\varpi}$ was correct. The "new field of analysis" that opened up in connection with this proof led him well beyond the study of elliptic functions of a single real-valued variable, and into the realm of functions of several complex-valued variables. Today, a special class of such functions known as the 'theta functions' provide a powerful tool that is used in a wide range of applications throughout mathematics — providing yet one more piece of evidence of Gauss' extraordinary ability as a mathematician and a guesswork genius!

References

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Carl Friedrich Gauss. *Mathematisches Tagebuch [Mathematical Diary], 1796 - 1814*. Ostwalds Klassiker der Exakten Wissenschaften [Ostwald's Classics of the Exact Sciences]. Verlag Harri Deutsch, Frankfurt am Main, 2005. Edited by K. R. Biermann.

Adrian Rice. Gaussian Guesswork, or why 1.19814023473559220744... is such a beautiful number. *Math Horizons*, pages 12–15, November 2009.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is one of a set of three 1–2 day projects designed to consolidate student proficiency of the following traditional topics from a first-year Calculus course:

- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution
- Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve
- Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean

Each of these PSPs can be used either alone or in conjunction with any of the other two. All three are based on excerpts from Gauss' mathematical diary [Gauss, 2005] and related primary texts that will introduce students to the power of numerical experimentation via the story of his discovery of a relationship between three particular numbers: the ratio of the circumference of a circle to its diameter (π), a specific value (ϖ) of the elliptic integral $u = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$; and the Arithmetic-Geometric Mean of $\sqrt{2}$ and 1. Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice referred to as "Gaussian Guesswork" by historian Adrian Rice in his *Math Horizons* article subtitled "Why 1.19814023473559220744... is such a beautiful number" [Rice, November 2009].

The primary content goal of this particular PSP is to consolidate students' understanding of sequence convergence, and especially the Monotone Convergence Theorem.

Student Prerequisites

In light of the project's content goals, it is assumed that students have had some introduction to the study of sequence convergence, including the statement of the Monotone Convergence Theorem, and that they are familiar with the associated notation and basic vocabulary. Familiarity with integration techniques sufficient to confirm that $\int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{\pi}{2}$ will also be useful for reading the closing section of the project; however, evaluation of this or any other integral is not required for completion of the student tasks in this particular PSP.

PSP Design, and Task Commentary

This project begins with a brief introduction to the historical context of Gauss' discovery. In Section 1, students then read Gauss' definition of the two sequences needed for a definition of the arithmetic-geometric mean, and work through two of his numerical examples for such sequences. They then read Gauss' observations about these examples, and complete proofs of the general properties that he derived from those observations. In Task 4 (based on Gauss' Observation III), the Monotone Convergence Theorem is invoked in order to conclude that each of the two sequences in question does converge. In Task 5 (based on Gauss' Obervation IV), students then prove that these limits are the same, a fact that naturally leads to the definition of the arithmetic-geometric mean. By way of conclusion, Section 2 then ties the mathematical details of this particular project to the larger story of Gauss' amazing discovery.

Suggestions for Classroom Implementation

Classroom implementation of this and other PSPs in the collection may be accomplished through individually assigned work, small-group work and/or whole-class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project. To reap the full mathematical benefits offered by the PSP approach, students should be required to read assigned sections in advance of in-class work, and to work through primary source excerpts together in small groups in class.

The author's method of ensuring that advance reading takes place is to require student completion of "Reading Guides" (or "Entrance Tickets"), for which students receive credit for completion, but with no penalty for errors in solutions. A typical Reading Guide will include "Classroom Preparation" exercises (drawn from the PSP Tasks) for students to complete prior to arriving in class. They may also include "Discussion Questions" that ask students only to read a given task and jot down some notes in preparation for class work. On occasion, other tasks are assigned as follow-up to a prior class discussion.

In completing assigned Reading Guides, students are asked to strive to answer each question correctly, but to think of the guide as preparatory work for class, not as a final product (e.g., formal polished write-ups are not expected). Students who arrive unprepared to discuss assignments on days when group work is conducted based on advance reading are not allowed to participate in those groups, but are allowed to complete the in-class work independently. Reading Guides are collected at the end of each class period for instructor review and scoring (again, for completeness only), thereby providing helpful feedback to the instructor about individual and whole-class understanding of the material, in addition to supporting students' advance preparation efforts.

A sample Reading Guide (based on the Day 1 Advanced Preparation Work suggested in the Sample Implementation Schedule given below) appears in the Appendix at the end of these Notes.

LaTeX code of the entire PSP is available from the author by request to facilitate preparation of reading guides or 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 75-minute class period)

To complete this project in its entirety, the following implementation schedule is recommended. Instructors teaching 50-minute class periods can easily adapt this to a 2-day schedule as well, with small-group work on Section 1 tasks continuing into Day 2.

- **Advance Preparation Work** (to be completed before class):
  Read pages 1–3, completing Tasks 1 and 2 for class discussion along the way, per the sample Reading Guide in the Appendix to these Notes.
- **Day 1: In-Class Work**
  - Brief whole-class or small-group comparison of answers to Tasks 1 and 2.
  - Small-group work on Tasks 3–5 (supplemented by whole-class discussion as deemed appropriate by the instructor). Note that these three tasks form the mathematical core of this PSP. They are based on the excerpt at the bottom of page 3, which is recommended for advance preparation work. If the advance preparation assignment does not include reading the latter half of page 3, then students will need additional class time to read that half page prior to starting on Task 3.
  - Time permitting, individual or small-group reading of last half of page 5 (below Task 5).
- **Follow-up Assignment** (to be completed prior to the next class period):

  As needed, read the last half of page 5; also read Section 2 (pages 6–7), completing Task 6 along the way. This assignment could also be made part of an advance preparation "Reading Guide" if a second day of in-class small-group work is desired. Note, however, that the answer to Task 6 is revealed at the top of page 7.

- **Day 2 (optional): Closing Discussion (5–30 minutes)**
  - In addition to a brief (5–10 minute) whole-class discussion of Section 2 and/or student questions on Section 1, students could be asked to share how they approached Task 6. (Again, the answer to that Task is revealed at the top of page 7, but how students approached it could make for interesting discussion.)
  - Instructors who wish to share additional details about the story told in this project during their closing comments can find those in the author's other two "Gaussian Guesswork" PSPs; adding this further commentary will, of course, take up more class time.

- **Homework (optional)**: A complete formal write-up of student work on Tasks 3–5 could be assigned, to be due at a later date (e.g., one week after completion of the in-class work).

Connections to other Primary Source Projects

Links to all three "Gaussian Guesswork'' PSPs (described earlier in these Notes) are as follows:

- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution
  `https://digitalcommons.ursinus.edu/triumphs_calculus/8/`
- Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve
  `https://digitalcommons.ursinus.edu/triumphs_calculus/3/`
- Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean
  `https://digitalcommons.ursinus.edu/triumphs_calculus/2/`

The following additional projects based on primary sources are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name of each is given (together with the general content focus, if this is not explicitly given in the project title). With the exception of the final two projects in the list (which require 4 and 6 days respectively for full implementation), each of these can be completed in 1–2 class days. Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_calculus`.

- Investigations Into d'Alembert's Definition of Limit (calculus version), Dave Ruch
- L'Hôpital's Rule, Danny Otero
- The Derivatives of the Sine and Cosine Functions, Dominic Klyve
- Fermat's Method for Finding Maxima and Minima, Kenneth M Monks
- Beyond Riemann Sums: Fermat's Method of Integration, Dominic Klyve
- How to Calculate π: Buffon's Needle (calculus version), Dominic Klyve (integration by parts)
- How to Calculate π: Machin's Inverse Tangents, Dominic Klyve (infinite series)
- Euler's Calculation of the Sum of the Reciprocals of Squares, Kenneth M Monks (infinite series)
- Fourier's Proof of the Irrationality of e, Kenneth M Monks (infinite series)
- Bhāskara's Approximation to and Mādhava's Series for Sine, , Kenneth M Monks (approximation, power series)
- Braess' Paradox in City Planning: An Application of Multivariable Optimization, Kenneth M Monks
- Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem, Abe Edwards
- The Radius of Curvature According to Christiaan Huygens, Jerry Lodder

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For more information about TRIUMPHS, visit https://blogs.ursinus.edu/triumphs/.

APPENDIX

This appendix provides a 'Sample Reading Guide' that illustrates the author's method for assigning advance preparation work in connection with classroom implementation of primary source projects. More detail concerning these guides is included in the subsection "Suggestions for Classroom Implementation" of the Notes to Instructors for this project.

Note that the full text of each assigned Task is reproduced on the guide, with blank space for students' responses deliberately left below each question. This not only makes it easier for students to jot down their thoughts as they read, but also makes their notes more readily available to them during in-class discussions. This practice also makes it easier for the instructor to efficiently review each guide for completeness, and for students to review their own notes and instructor feedback once it is returned to them.

SAMPLE READING GUIDE: Advance Preparation Work for Day 1

Background Information: The goal of the advance reading and tasks assigned in this 2-page reading guide is to familiarize students with the definition and examples of the two sequences that define the arithmetic-geometric mean in order to prepare them for in-class small group work on Tasks 3–5 .

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Reading Assignment: *Gaussian Guesswork: Sequences and the Arithmetic-Geometric Mean*, pp. 1–3

1. Read the introduction on page 1.
   Any questions or comments?

2. In Section 1, read pages 1–2.
   Any questions or comments?

3. **Class Prep** Complete **Task 1** from page 3 here:

   Task 1 This task examines Example 1 from Gauss' paper.

   (a) Verify that the values given by Gauss in the previous excerpt are correct. **Are you able to use your calculator to obtain the same degree of accuracy (21 decimal places!) that Gauss obtained by hand calculations?**

   (b) Write three observations about the two sequences in this example.
   Use a full sentence to state each of your observations.

4. **Class Prep** Complete **Task 2** from page 3 here:

Task 2 In Example 2 from his paper, Gauss set $a = 1, \quad b = 0.6$.

(a) Use a calculator or computer to compute the next four terms of each sequence.

$a = 1.00000\,00000\,00000\,00000\,0$ $\qquad$ $b = 0.60000\,00000\,00000\,00000\,0$

$a_1 =$ $\qquad$ $b_1 =$

$a_2 =$ $\qquad$ $b_2 =$

$a_3 =$ $\qquad$ $b_3 =$

$a_4 =$ $\qquad$ $b_4 =$

(b) In Task 1, part (b), you made three observations about Gauss' Example 1.
Do the same general patterns hold for Example 2?
If so, why do you think this is? If not, in what way(s) are the two examples different?

5. READ the rest of page 3, starting below Task 2.

Write at least one question or comment about this part of the reading.