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Connecting Connectedness

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Connecting Connectedness

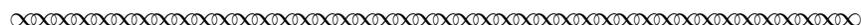
Nicholas A. Scoville*

November 22, 2021

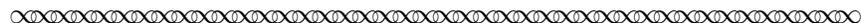
Connectedness has become a fundamental concept in modern topology. The concept seems clear enough – a space is connected if it is a “single piece.” Yet the definition of connectedness we use today is not what was originally written down. As we will see, today’s version of connectedness is a classic example of a definition that took decades to evolve. The first definition of connectedness was given by Georg Cantor. Cantor (1845–1918) is best known for his work in set theory. His work in set theory, however, began with questions concerning Fourier series in an 1872 paper [Cantor, 1872]. In his study of Fourier series, Cantor was interested in finding conditions for when a function has a unique Fourier expansion. This study compelled him to define for the first time some purely topological concepts, including the concepts of a point-set, a neighborhood, and a derived set. Cantor’s early topological investigations were the precursors to a series of six papers published between 1879–1883 that were themselves part of his work on set theory. It is with his fifth paper in this series that we begin this project on connectedness. Given this history, one could trace not only the origins of modern set theory back to Cantor, but also the origins of modern point-set topology.

1 Cantor: A Continuum

We begin our investigation into connectedness with the paper “Ueber unendliche, lineare Punktmanichfaltigkeiten, [Teil] 5” (“On infinite, linear manifolds of sets, Part 5”) [Cantor, 1883]. Cantor’s reason for writing this paper, however, does not appear to have been a consideration of connectedness. Cantor wrote:¹



The point of this examination is to establish a precise criterion — but one that is simultaneously as generally applicable as possible – for designating P as a continuum.



Let’s put ourselves into Cantor’s shoes. What comes to mind when you think of a continuum? A straight line? A closed disk? An open disk? The irrational numbers? Definitions in mathematics tend to be driven by our examples; that is, what do all the examples of objects that you think of as a continuum have in common? Furthermore, what do all the examples that you think of as violating a continuum have in common? The goal is to abstract away the particulars and find the common

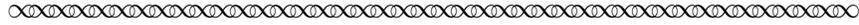
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¹All translations of Cantor, Jordan, and Schoenflies excerpts in this project are due to the project author.

properties that a continuum satisfies. As the quote above indicates, this is what Cantor desired to do in this paper.

Task 1 Consider a straight line, a closed disk, an open disk, and the irrational numbers. Which of these do you consider a continuum? Why?

One concept that Cantor had at his disposal was that of the **derived set**.



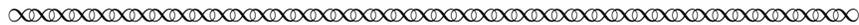
For the sake of brevity I call a given finite or infinite number of ... points on the line a *point-set*. If a point-set is given in a finite interval, a second point-set is generally given along with it, and with the latter a third, etc., which are essential to understanding the first set. In order to define these point-sets, we must begin with the concept of a *limit point of a set*. I define a "limit point of a point-set P " to be a point of the line situated in such a way that each neighborhood of it contains *infinitely* many points of P , and it may happen that the point itself belongs to the set. By a "neighborhood of a point" I mean any interval that has the point *in its interior*. ... Every point of the line is now in a definite relation to a given set P , either being a limit point of P or not, and thereby along with the point-set P the set of limit points of P is *conceptually* given, a set which I wish to denote by P' and call the *first derived point-set of P* . Unless the point-set P' contains only a finite number of points, it also has a derived set P'' , which I call the *second derived point-set of P* . By v such transitions one obtains the concept of the v th derived set $P^{(v)}$ of P [If] P is such that the derivation process produces no change:

$$P = P^{(1)}$$

and therefore

$$P = P^{(\gamma)}$$

[then] such sets P I call **perfect** point-sets.



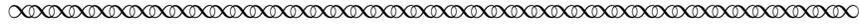
Task 2 Using Cantor's definition of limit point and derived set, determine which of the following sets are perfect in \mathbb{R} .

- (a) $[0, 1]$
- (b) (a, b) , $a < b \in \mathbb{R}$
- (c) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
- (d) $\{\frac{1}{n} : n = 1, 2, 3, \dots\}$
- (e) \mathbb{Q}
- (f) \mathbb{R}
- (g) $[\mathbb{Q} \cap (0, 1)] \cup [-4, 2]$

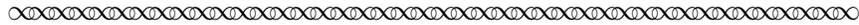
You may have noticed that, for the above examples, each set that you would intuitively think of as constituting a continuum is a perfect set, and each one that you would not think of as a continuum is not perfect. This raises the question: should a perfect set be defined as a continuum?

Task 3 Is the set $[0, 1] \cup [2, 3]$ a perfect set? Should it be a continuum?

As the example above illustrates, being perfect might be a necessary, but not a sufficient condition for a set to be a continuum. The missing piece that Cantor added to the condition of being perfect in order to complete his definition of a continuum was that the set also be **connected**. Cantor wrote:



A [closed and bounded] point-set T is **connected** if for every two of its points t and t' , and arbitrary given positive number ϵ , there always exists a finite number of points t_1, t_2, \dots, t_n of T such that the distances $tt_1, t_1t_2, \dots, t_nt'$ are smaller than ϵ .

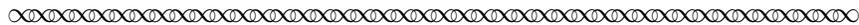


Task 4 Rewrite this definition using modern notation (and possibly terminology), and use that definition to prove that $[0, 1] \cup [2, 3]$ is not connected. Then prove that $[0, 1]$ is connected.

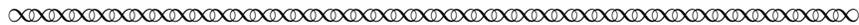
Task 5 According to Cantor's definition, what can be said about the set $[0, 1] - \{\frac{1}{2}\}$ in terms of connectedness?

2 Jordan: Distance between Sets

Cantor's definition of connected appealed to a notion of distance, which leads to a setting that is not as general as that which one encounters in topology today. This approach to the definition was nevertheless taken up by French mathematician Camille Jordan (1838–1922) in his *Cours D'analyse* (*Course on analysis*) [Jordan, 1893].

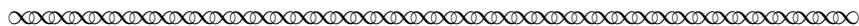


Let E, E' be two [closed] sets with no point in common. The distance between the various points p in E to the various points p' in E' form a set of nonnegative numbers. It is therefore bounded below, and it admits a minimum Δ , positive or 0, we call the **distance** between the sets E, E' . If the distance is greater than 0, we say that the sets E, E' are **separated**.

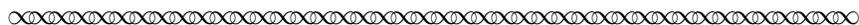


Task 6 Using Jordan's definition, compute the distance between two closed, disjoint disks in \mathbb{R}^2 .

Jordan immediately made the following statement and provided a proof.

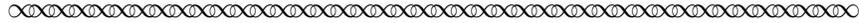


If two bounded and closed sets E, E' which have no points in common have distance Δ , then they will contain at least a couple of points whose mutual distance is precisely $\Delta \dots$

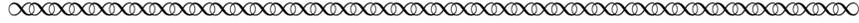


Task 7 Show that this is not necessarily true if the sets in question are not closed.

We are finally ready for Jordan’s definition. Jordan wrote as follows:



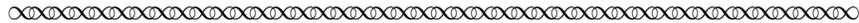
We say that a closed and bounded set E is a **component** if it cannot decompose into several separated sets.



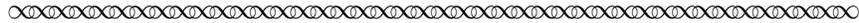
Task 8 Using Jordan’s definition of a component, prove that a closed and bounded set E in \mathbb{R}^n is connected if and only if E itself is a component.

3 Schoenflies: No Distance Required

We have noted that both Cantor and Jordan defined connectedness by appealing to a distance. The German mathematician Schoenflies (1853–1928), who is best known for the Jordan-Schoenflies theorem, instead gave a purely set-theoretic definition of connectedness which did not appeal to the concept of distance. The definition that he gave in his paper “Beiträge zur Theorie der Punktmengen I” (“Contributions to the Theory of Point-sets I”), [Schoenflies, 1904], was simple and elegant.



A perfect set T is called **connected** if it can not be decomposed into subsets, each of which is perfect.



Task 9 Rewrite Schoenflies’ definition of connected with modern terminology and notation.

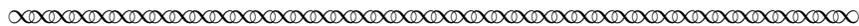
Task 10 Does Schoenflies’ definition give us any better indication of whether or not $[0, 1] - \{\frac{1}{2}\}$ is connected? Why or why not?

Although there are other things that Schoenflies did with this definition, our main purpose here is to note that he abstracted away any need for distance, yet kept the definition very much in the spirit of Jordan and Cantor, as the following task illustrates.

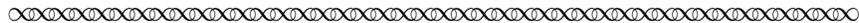
Task 11 Prove that if a set satisfies the Jordan definition, then it must satisfy the Schoenflies definition. Why is the converse not necessarily true?

4 Lennes: The Modern Definition

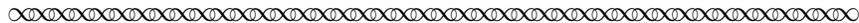
Finally, we turn to Nels Johann Lennes (1874–1951). Lennes was a Norwegian-born mathematician who earned his Ph.D. at the University of Chicago and lived out most of his mathematics career at Montana State University. Lennes was aware of the mathematical thought on connectedness up to this point. In his paper “Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations,” [Lennes, 1911], he wrote:



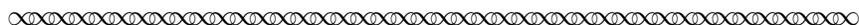
It is apparent that in a geometry possessing linear order and continuity curves and limit curves exist independently of metric properties. . . . Schoenflies uses metric hypotheses in the proof of practically every important theorem dealing with curves and the regions defined by them . . . his treatment makes full use of metric properties.



Lennes then gave his own definition of limit point.



A point ℓ is a **limit-point** of a set of points P if there are points of P other than ℓ within every [neighborhood] of which ℓ is an interior point.



Task 12 Compare Cantor’s definition of limit point with that of Lennes. Are they equivalent? If so, prove it. If not, give a counterexample.

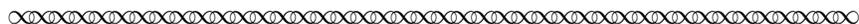
Lennes was interested in the Jordan Curve Theorem, one of the most important and difficult theorems of late 19th and early 20th century mathematics. This theorem states:

Let J be a closed curve in \mathbb{R}^2 which does not self-intersect.
Then $\mathbb{R}^2 - J$ is disconnected with exactly two open, connected components.

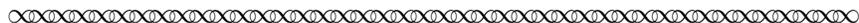
Although easy to state and intuitively obvious, a rigorous and satisfying proof of this fact eluded mathematicians for many years. In order to attempt a rigorous proof, Lennes needed a careful and precise definition of connectedness.



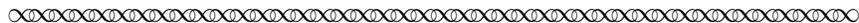
A set of points is a **connected set** if at least one of any two complementary subsets contains a limit point of points in the other set.



Lennes’ definition turns out to be equivalent to the modern definition. To substantiate our claim, here is a definition from a modern classic book on point-set topology [Kelley, 1975, p. 53].



A topological space is **connected** iff X is not the union of two non-void separated [open] subsets.



Task 13 Show that this definition and the one given by Lennes are equivalent. Then determine whether or not $[0, 1] - \{\frac{1}{2}\}$ is connected.

5 Conclusion

We have seen how the definition of connectedness, starting with Cantor, has evolved into the modern definition. For Cantor, connectedness was somewhat of a side note — it needed to be defined in order to properly understand the definition of a continuum. Jordan thought this definition of connected interesting enough to take it up in his own work, and studied it as a concept in its own right. Schoenflies then realized that there was no need to appeal to a notion of “distance” to give a coherent definition of connected. Finally, Lennes tweaked Schoenflies’ definition sufficiently to obtain the one that is used by mathematicians today. So remember this the next time you see a definition in a textbook. The crisp, clean, and pithy definition may have taken some of the world’s greatest mathematicians years to arrive at!

References

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Notes to Instructors

PSP Content: Topics and Goals

While this Primary Source Project (PSP) is designed in part to introduce students to the concept of connectedness, it also serves a broader two-fold purpose. First, it is meant to show where definitions come from. This is seen in Cantor's attempt to study a continuum. This is made most explicit in Task 3 where the student should see that there are perfect point-sets that one ought not consider a continuum. The second purpose is to show how a definition can change over time. Because Cantor was only working with Euclidean space, it made sense for connectedness to be defined in the setting of Euclidean space. As more general spaces started to be explored, more general definitions were called for, some which made sense in these more general settings (like connectedness) while others did not.

Both points (where definitions come from and how definitions change) are just two aspects of the mathematical discovery process that is often hidden from students. As such, it is important to emphasize that this project is showing that there is something much greater to mathematics than applying formulas. It is the hope that this project will help students to see (or even consider for possibly the first time) that mathematics is a human endeavour with both struggles and unclear solutions..

Student Prerequisites

This project does assume some background familiarity with point-set topology. In particular, the students should have been exposed to the concept of derived sets. Otherwise, the definition of a perfect set in Section 1 comes seemingly out of left field.

PSP Design, and Task Commentary

As noted above, the general purpose of this project is two-fold. Its first goal of showing where definitions come from is mostly achieved in this PSP in the section on Cantor, in two different ways. The first way is by illustrating the process through which mathematicians make precise an intuitive idea. The discussion on a continuum is meant to imitate the working out of a definition. We all have in our minds paradigmatic examples of sets that we do and don't consider to be continua. Let us take all those examples, abstract away the particulars, and be left with what they all have in common — the essence of what it means to be a continuum. A good way to accomplish this in class is to ask the students to give examples of what they think are and are not continua, and to write their responses in two columns on the board. This should generate a discussion of some of the properties that students do and don't consider to be essential for a set to be a continuum. There may be students for whom nothing comes to mind when they hear "continuum." That is okay. But by the end of this task, the class as a whole should have a somewhat unified, even if still vague, idea of what 'continuum' means.

The next step in this process is then attempting to precisely formulate the meaning of a continuum in a definition. In order to tease this out, students are asked to recall the definition of a perfect point-set. After a little reflection, it seems that our intuitive idea of what ought and ought not constitute a continuum coincides exactly with that of a perfect set. (In fact, I have had classes where those sets which the class considered a continuum were precisely the perfect point-sets, while sets which were not considered a continuum were precisely those which are not perfect.) Task 3, however, serves as an example to show why equating the two concepts is not appropriate. This leads into the second way in which the project shows where definitions come from, by considering the question: what

property needs to be added to “perfect” in order to exclude examples like the one in Task 3 from being considered a continuum? There is then a need to define this additional property, and the concept that seems to work well is that of being “connected.” Students are then asked to wrestle with Cantor’s original formulation of the definition of connectedness.

The section on Jordan begins to address the second project goal of showing how a definition can evolve, either in its verbal formulation or in its point of view. While Cantor defined what it means for a set to be connected, Jordan added the viewpoint of “separation” as a way to look at connectedness. That is, now we have a positive definition of this concept (connected) as well as a negative concept (separation). The concept of separation can furthermore be quantified in the sense that if a separation exists, we can sometimes assign a number to it (e.g. Task 6). The culmination of Section 2 is in Task 9 where students are asked to show that the two definitions, given by Cantor and Jordan respectively, are equivalent.

The Schoenflies section is brief and meant to be a bit ambiguous. What did he mean by “decomposed into”? A partition? Only two sets? A finite number of sets? Students will wrestle with this question and justify an answer that makes the most sense relative to what Schoenflies was trying to do in Task 10. However, it should be noted that Schoenflies gave a definition that is not equivalent to those given by Cantor and Jordan, since there is no appeal to a metric in Schoenflies’ definition. The final definition due to Lennes is easily motivated with a word about the Jordan Curve Theorem, and it is satisfying for the students to see that this is the first time that we see the current definition being used.

Furthermore, there is a running example of determining whether or not the set $[0, 1] - \{\frac{1}{2}\}$ is connected or not. For both Cantor and Jordan, the concept of connectedness only applied to closed and bounded sets, so that this question would not even make sense to them, a seemingly major drawback of the definition. Depending on how students interpret Schoenflies, students may or may not find the set connected according to his definition. However, it should once again be satisfying to see this simple example, unclear for many years, now easily shown to be disconnected using the Lennes definition.

Suggestions for Classroom Implementation

The entire project should take about two 50-minute class periods in total. As mentioned above, the first day implementation of the project can begin with the professor posing the question “what are some examples and non-examples of a continuum?” to the class. This can then be discussed either by the class as a whole or in small groups with responses shared with the class as a whole. Such an approach does not require the student to do any reading or work before class. After the class shares their ideas, the students can then read through the excerpts and work on the tasks in small groups, sharing their responses with the class or as part of homework to be turned in.

L^AT_EX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or ‘in-class worksheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

Day 1

- **In class discussion (10 minutes):** Before you have handed out the project, write the question “what is a continuum?” on the board. Ask the class for examples of objects (usually mathematical but not necessarily) that are a continuum. List these

in one column on the board and at some point, begin a second column of “non-examples” or things that are not a continuum. You may find some disagreement among students, and you can put these examples in the middle. If it is never brought up, suggest a closed interval, open interval, and disconnected interval a sets for the class to consider and place in a column. You may wish to save this list or return to it after the class has seen Cantor’s definition of a continuum. Then you can see which of the objects fit Cantor’s definition and which do not.

- **Working in groups (15 minutes):** Hand out the project and have students work in small groups on the first two pages up to and including Task 2. Note that Task 1 simply reiterates the in-class discussion so students have a chance to say anything else about a continuum in their groups that they didn’t have a chance to in class.
- **Debrief (5 minutes):** After students work in groups, the next 5 minutes can be spent as a class regrouping and making sure everyone is comfortable with limit points and derived sets.
- **Working in groups (10 minutes):** Students can spend the next 10 minutes working on Tasks 3–4. The goal here is for students to see that more is needed on top of perfect in order to be a continuum.
- **Debrief and Task 5 (10 minutes):** After a brief discussion about what was done in groups, end class by working on Task 5 together. Ask if the answer seems appropriate.
- **Homework:** Write up solutions to all tasks completed in-class and begin read and write up tasks for Section 2 for homework

Day 2

- **In class discussion (10 minutes):** Begin by reviewing what was done the previous day and move into a discussion with the class of the main ideas in Section 2. Ask if there are any questions.
- **Working in groups (20 minutes):** Have students continue working in their groups to both discuss Section 2 tasks as well as begin working through Section 3.
- **Debrief (5 minutes):** Have the class reconvene to discuss what they just worked on. Emphasize that Schoenflies has abstracted away the metric and is now working in a general topological space.
- **Working in groups (15 minutes):** Students can spend the last minutes working on Section 4. Anything they do not finish can be done for homework.

Connections to other Primary Source Projects

The author has a longer version of this project titled *Connectedness: Its Evolution and Applications*, designed to be completed in five 50-minute class periods, that more fully develops the ideas found in this PSP. The main applications in that longer version are the fact that connectedness is a topological invariant and the Invariance of domain Theorem for $n = 1$. Along the way, students learn about limit points, derived sets, perfect sets, homeomorphisms, everywhere dense sets, and other topics typically covered in point-set topology. In addition to the content goals, there is the theme of seeing

how definitions can change over time, and how mathematicians struggle to find the right concepts to express ideas.

In addition to *Connectedness: Its Evolution and Applications*, the following primary source-based projects by the author are also freely available for use in teaching courses in point-set topology. The first three projects listed are full-length PSPs that require 10 and 3 class periods respectively to complete. All others are designed for completion in 2 class periods.

- *Nearness without Distance*
- *From Sets to Metric Spaces to Topological Spaces*
- *Topology from Analysis* (Also suitable for use in Introductory Analysis courses.)
- *The Cantor set before Cantor* (Also suitable for use in Introductory Analysis courses.)
- *The Closure Operation as the Foundation of Topology*
- *A Compact Introduction to a Generalized Extreme Value Theorem*

Classroom-ready versions of these projects can be downloaded from https://digitalcommons.ursinus.edu/triumphs_topology. They can also be obtained (along with their L^AT_EX code) from the author.

Acknowledgments

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