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Solving a System of Linear Equations Using Ancient Chinese Methods

Mary Flagg
University of St Thomas, flaggm@stthom.edu

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Solving a system of linear equations is a skill essential to the modern mathematician, and one that is generally the first topic in an introductory linear algebra course. The standard modern algorithm is called Gaussian elimination, leading students to believe that it was invented by Carl Friedrich Gauss (1777–1855). However, the algorithm has a much longer history.

The ancient Chinese text *The Nine Chapters on the Mathematical Art* [Shen et al., 1999] [1] compiled by the first century BCE, has a chapter entitled *Fangcheng* which is translated as 'Rectangular Arrays'. This chapter introduces the Fangcheng Rule, which is a general method for solving a system of linear equations using rectangular arrays. Although the Chinese viewed the problems in a completely different context than the modern notions of equations with variables, their technique is equivalent to Gaussian elimination. The purpose of this lesson is to use the Fangcheng Rule of the ancient Chinese to introduce Gaussian elimination.

1 Historical Background

Systems of linear equations were not commonly part of ancient mathematics. There are a few isolated problems preserved, with no general solution method, Grcar [2011a]. The one exception to this rule is the mathematics of ancient China in the text *The Nine Chapters on the Mathematical Art*, originally compiled by the first century BCE. In Western mathematics, symbolic algebra began to develop in the late Renaissance, and systems of linear equations appeared as exercises in algebra textbooks in the sixteenth and seventeenth centuries. This 'schoolbook elimination' used the power of symbolic algebra to solve a system of equations (linear or possibly nonlinear) using a series of substitutions to eliminate one variable at a time. Contributors to these methods included texts by Michel Rolle [1690], Isaac Newton [1720], the banker Nathaniel Hammond [1742] and Thomas Simpson [1755].

A very compelling need for an efficient method of solving simultaneous linear equations arose in the early nineteenth century with the development of the method of least squares. The two men recognized for independently developing the least squares method are Adrien–Marie Legendre and Carl Friedrich Gauss [Grcar, 2011a, p. 178]. The method of least squares was developed to address the problem of making accurate predictions from collected data, and involved solving a system of linear equations. Gauss used the method of least squares when calculating the orbit of the asteroid Ceres in 1794 or 1795, but the method was not published until 1809 Gauss [1809]. Gauss' method

*Department of Mathematics, Computer Science and Cooperative Engineering, University of St. Thomas, Houston, Texas 77096; flaggm@stthom.edu.

[1] For the Chinese text, see Shu-Chun Guo [1990]

appeared after Legendre had published his version Legendre [1805]. Only years later was it called Gaussian elimination in honor of Gauss [Grcar, 2011a, p. 209].

Matrix algebra was developed in the middle of the nineteenth century by Arthur Cayley [1858]. It is remarkable that the ancient Chinese found the power of using a rectangular array to solve systems of linear equations nearly 2000 years earlier!

The *Nine Chapters on the Mathematical Art*, hereafter referred to as the *Nine Chapters* for brevity, dominated the early history of Chinese mathematics [Shen et al., 1999, p. 1]. It played a central role in Chinese mathematics equivalent to that of Euclid's *Elements* in Western mathematics. It remains the fundamental source of traditional Chinese mathematics. The *Nine Chapters* is an anonymous text, compiled across generations of mathematicians. It is believed that the original text was compiled by the first century BCE, but it is difficult to date precisely. Western mathematical ideas were not introduced into China until the first Chinese translation of Euclid's *Elements* by Xu Guangqi (1562–1633) and Matteo Ricci (1552–1610) appeared in 1606 [Shen et al., 1999, p. 21].

The *Nine Chapters* is a series of 246 problems and their solutions organized into nine chapters by topic. The topics indicate that the text was meant for addressing the practical needs of government, commerce and engineering. The problems and solutions do not generally include an explanation of why a particular solution method worked. Unlike the Greek emphasis on proofs, the Chinese emphasized algorithms for solving problems. This does not mean that they did not know why an algorithm worked, it only shows that the most important goal was to show students how to perform the calculations correctly.

The chapters of the book demonstrate that an extensive body of mathematical knowledge was known to the ancient Chinese:

1. Rectangular Fields: This chapter is concerned with land measurement and gives the formulas for finding the areas of fields of several shapes.

2. Millet and Rice: Chapters 2 and 3 contain a variety of problems from agriculture, manufacturing and commerce.

3. Distribution by Proportion

4. Short Width: The problems in this chapter involve changing the dimensions of a field while maintaining the same area and includes algorithms for finding square roots and working with circles.

5. Construction Consultations: This chapter includes formulas for volumes of various solids.

6. Fair Levies: The problems in this chapter come from taxes and distribution of labor.

7. Excess and Deficit: The rule of double false position for solving linear equations is used to solve a variety of problems in this chapter. [2]

8. Rectangular Arrays: The Fangcheng Rule is introduced to solve systems of linear equations.

[2] Double false position refers to a method of solving a linear equation using trial and error by using a series of prescribed steps to obtain the correct solution from information reported on incorrect guesses, and is still a viable method today.

9. Right-angled Triangles: This chapter includes the *Gougu Rule*, known to Western mathematicians as the Pythagorean Theorem.

The noted Chinese mathematician Liu Hui, who flourished in the third century CE, published an annotated version in the year 263 [Shen et al., 1999, p. 3] with detailed explanations of many of the solution methods. We know very little of his life beyond his notes accompanying the text of the *Nine Chapters*. Fortunately, in his introduction he gives his motivation for annotating the text. Read the following portion of Liu Hui's introduction and discover his reasons for publishing his comments.

> I read the *Nine Chapters* as a boy, and studied it in full detail when I was older. [I] observed the division between the dual natures of Yin and Yang [the positive and negative aspects] which sum up the fundamentals of mathematics. Thorough investigation shows the truth therein, which allows me to collect my ideas and take the liberty of commenting on it. Things are known to belong to various classifications. Just as the branches of a tree are to its trunk, so are a multitude of things to an archetype. Therefore I have tried to explain the whole theory as concisely as possible, with spatial forms shown in diagrams, so that the reader should have a reasonably good all-around understanding of it.[3]

The *Nine Chapters* does not provide a diagram with the Fangcheng Rule. Since the text does not give any visual clues as to how the procedure was physically performed, it is difficult for modern readers to translate the instructions. Liu Hui's commentary will prove invaluable as we attempt to decipher the ancient Chinese instructions for solving rectangular arrays. This lesson will examine both the original text and the corresponding commentary together to discover how they applied the Fangcheng Rule to systems of linear equations.

2 Counting Rod Arithmetic

Modern mathematicians have calculators and computers available to perform tedious arithmetic. Before the advent of calculating machines, all computation had to be carried out by hand. Complex computations required methods of keeping track of the numbers. Ancient China developed a very efficient system of computation by physically manipulating counting rods. Counting rods and rod arithmetic were used in China from 500 BCE until approximately 1500 CE when counting rods were gradually replaced with the abacus [Shen et al., 1999, pp. 11–17].

2.1 The Counting Rods

China developed a base-ten place value system for numerals. Counting rods were used to represent the digits 1–9 and the arrangement of the rods on a counting board indicated the place value. Counting rods were small bamboo sticks, approximately 2.5 mm in diameter and 15 cm long. The rods were laid out either upright or horizontally, as in Figure 1. The numbers 1–5 were represented by laying

[3] [Shen et al., 1999, pp. 52-58]

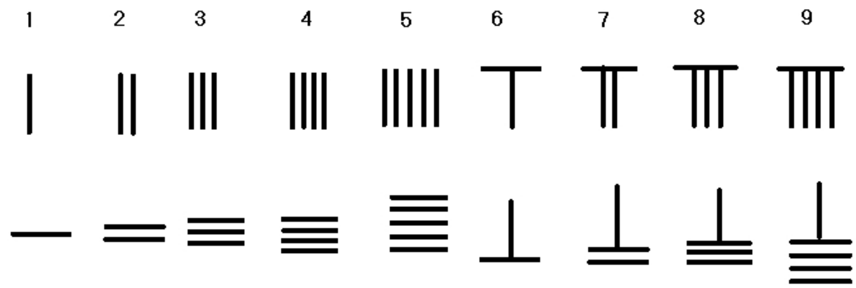

Figure 1: Vertical and Horizontal Counting Rod Numerals

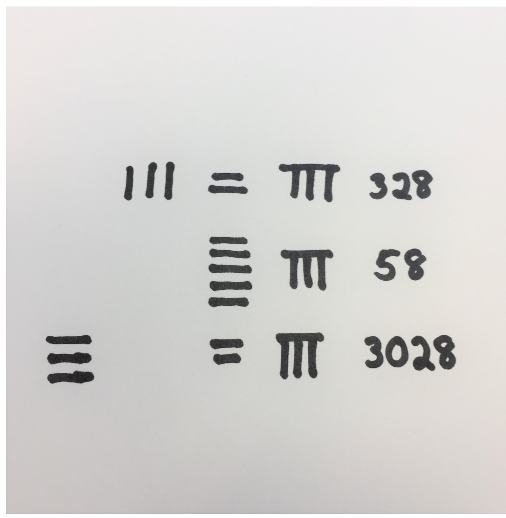

Figure 2: Examples of Rod Numbers

the corresponding number of rods side by side, either horizontally or vertically. One horizontal rod set atop a number of vertical rods, or a vertical rod on top of some horizontal rods each represent five units in the digits 6, 7, 8 and 9. Numbers were formed by alternating upright numerals for units, hundreds, etc., with horizontal numerals for tens, thousands, etc. Places with zeros were left blank since there was no symbol for zero in the counting rod system. The alternating horizontal and vertical numerals helped distinguish the places in the base ten numeral. The alternating orientation of the counting rods also served as a point of demarcation when one of the digits in the number was zero and the place in the written numeral was left blank.

Figure 2 illustrates the usefulness of the alternating orientation of the counting rods in the representations of the numbers 328, 58, and 3028. Notice that the alternating directions of the rods for numerals of successive powers of ten separates the 3 in the hundreds place and the 2 in the tens place, easily distinguishing 328 from 58. The counting rod representation of 3028 differs from that of 328 by the fact that the 3 is also horizontal, which indicates that there is zero in the hundreds place in 3028. Numbers with more than one consecutive zero, like 2003 or 400005, would need some context to help the reader interpret the space between the nonzero digits since the alternating horizontal and vertical rods would not obviously mark the missing digit. Do you see why zeros are so useful in our modern numerals?

Counting rod arithmetic was performed by manipulating the counting rods on a counting board.

Unfortunately, we have no visual record of counting boards or how counting rod arithmetic was performed. Addition was carried out by the usual rules of combining the rods, replacing ten rods with one in the place to the left, and replacing five rods with one when recording digits greater than five. Subtraction was performed by taking away rods and borrowing one rod from one place and replacing it with ten rods in the next place to the right or replacing one rod with five to have enough rods to take away the required amount.

2.2 The Multiplication Algorithm

A multiplication problem was set up by placing rods representing the first number to be multiplied in the top row of the counting board, and the rods representing the second number in the bottom row. As the procedure is performed, the answer is worked out in the middle row. The process of manipulating the rods kept no record of the intermediate steps, unlike our paper and pencil algorithm for multiplication. An example of how multiplication was performed will help us appreciate the value of counting rod arithmetic; 48×67 will be illustrated.

Arrange the numbers on the counting board with each digit in a column representing a power of ten. All multiplication diagrams that follow will label the place values in each column in the header row, with the actual multiplication problem below. The first number to be multiplied is laid in the top row and the second number is placed in the bottom row, with its ones digit in the same column as the tens digit in the first number in the top row.

1000	100	10	1
		4	**8**
	6	**7**	

(1)

Why shift the 67 one place from where we expect it to be? The number is not 670. The multiplication begins by performing 67×4, but the 4 is really 40 and we are going to shift the 67 to the left one column so that the answer to 67×40 is placed in the columns with the correct place value. Begin by multiplying the 4 by the 6 and placing the answer in the middle row with the last digit in the same column as the 6.

1000	100	10	1
		4	**8**
2	4		
	6	**7**	

(2)

Multiply the 4 by the 7 and place its last digit in the column with the 7, then add and carry where necessary.

1000	100	10	1
		4	**8**
2	$4 + 2 = 6$	8	
	6	**7**	

(3)

Next, remove the 4 in the top row and slide the 67 in the bottom row over one place to the right. The second number is slid to the right because the next step is to multiply 67×8, and shifting corrects the place values of the 67.

1000	100	10	1
			8
2	6	8	
		6	**7**

(4)

Then multiply the bottom 6 by the 8 in the top row. Place the ones digit in the column with the 6, the tens place in the second number.

1000	100	10	1
			8
2	$6+4$	$8+8$	
		6	**7**

(5)

Add the numbers in the middle row, carrying when needed.

1000	100	10	1
			8
3	1	6	
		6	**7**

(6)

Finally multiply the 8 by the 7 in the bottom row and add that answer to the numbers in the middle row.

1000	100	10	1
			8
3	1	$6+5$	6
		6	**7**

(7)

The last step is to add the numbers in the middle row and delete the numbers in the top and bottom row, showing only the answer.

1000	100	10	1
3	2	1	6

(8)

Task 1 The procedure for multiplication is actually easier to perform with the rods than it is to explain in words. Try multiplying 38×82 using counting rods. Use flat toothpicks or other small rods as counting rods and lay out a grid on your work space. Follow the steps for multiplication outline in the example above. How does this procedure compare to the algorithm you learned for multiplication? Where do you see a modern equivalent to shifting the multiplier to the left in the middle of our modern algorithm?

2.3 The Sign Rule

Negative numbers sometimes appear in the solution process of systems of linear equations, even when the final answers are positive. Negative numbers first appear in Problem 3 of Chapter 8 of the *Nine Chapters*. The Sign Rule for arithmetic with negative numbers is included in the *Nine Chapters* following Problem 3 to explain how to handle addition and subtraction with negative numbers. Negative and positive numbers were distinguished by differently-colored counting rods, generally red for positive and black for negative.

The Sign *Zhengfu* Rule

Like signs subtract. Opposite signs add. Positive without extra, make negative; negative without extra make positive.

Opposite signs subtract; same signs add; positive without extra, make positive; negative without extra, make negative.[4]

Task 2 The Sign Rule is actually two rules. The first rule (first paragraph) explains how to perform $a - b$ for two numbers a and b depending on their signs. The phrase 'Positive without extra make negative' in the first paragraph refers to $0 - b$ when b is positive. The second paragraph explains how to calculate $a + b$ depending on the signs of each. Explain the rules in your own words.

The following is a portion of Liu Hui's commentary on The Sign Rule. Note that the counting rods in different colors allowed the Chinese to represent excess as positive and deficit as negative in the normal course of solving practical problems.

Now there are two opposite kinds of counting rods for gains and losses, let them be called positive and negative [respectively]. Red counting rods are positive, black counting rods are negative. Alternately distinguish [positive as] upright and [negative as] slanting. The rule for rectangular arrays [comprises] operations on the red and black entries from left to right. However, whether to add or subtract varies, so red and black [counting rods] are used to cancel one another. The operations of subtract or addition depend on the two types of entries in each column.[5]

Task 3 The Sign Rule explains how to add or subtract positive and negative numbers, yet it makes no explicit mention of the sign of the answer. Why do you think this is? (Hint: Liu's Commentary gives us the clue that the rule was meant for rod arithmetic.)

[4] [Shen et al., 1999, p. 404]
[5] [Shen et al., 1999, pp. 404-405]

2.4 The Array on the Counting Board

Chapter 8 of the *Nine Chapters* starts by posing a problem. The English translation is presented below.

> Problem 1: Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 dou. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 dou. Tell: how much paddy does one bundle of each grain yield? Answer: Top grade paddy yields $9\frac{1}{4}$ dou [per bundle]; medium grade paddy $4\frac{1}{4}$ dou; [and] low grade paddy $2\frac{3}{4}$ dou.[6]

Paddy is rice. The term dou is an ancient unit of volume which translates to approximately 2 liters in modern units [Shen et al., 1999, p. 10]. Problem 1 contains measures that are not familiar to modern readers, yet the basic structure of the problem looks exactly like many of the word problems in a modern linear algebra textbook.

Task 4 Write Problem 1 as a system of linear equations. For this system of equations, create a modern word problem unrelated to rice harvesting. For example, one could substitute different items on a lunch menu for the different grades of paddy, and prices of these items instead of yield of grain. Be creative!

The Fangcheng Rule for solving these types of problems appears immediately after Problem 1. The general rule is given by walking through the specific solution to Problem 1. The Fangcheng Rule starts by explaining how to organize the numbers on a counting board.

> The Fangcheng Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. Similarly for the middle and left column.[7]

Traditional Chinese was read from top to bottom and then from right to left. Therefore, each combination of paddy and its yield is arranged in a column, with the numbers in order from top to bottom, starting from the right side of the counting board. The number of bundles of top grade paddy was in the first row in each column, the medium grade in the second row, the low grade paddy in the third row and the yield in the bottom row.

[6] [Shen et al., 1999, p. 399]

[7] The text for the Fangcheng Rule and the commentary by Liu Hui comes from [Shen et al., 1999, pp. 399-400].

We will construct the array in the same way that the ancient Chinese did, as a grid with rows and columns. We will also use modern numerals throughout the rest of the lesson, but keep in mind that the Chinese used counting rod numerals and physically performed the arithmetic operations on the rods. Students are encouraged to do each of Tasks 5–9 on a separate sheet of paper, leaving room for solving the arrays in tasks later in the lesson.

Task 5 Set up Problem 1 from Chapter 8 of the *Nine Chapters* using the instructions from the first two sentences of the Fangcheng Rule.

The pattern for Problem 1 is supposed to be followed when solving the other problems in Chapter 8. The problems become both more difficult to set up on the counting board and more difficult to solve.

Problem 3: Now there are 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 4 bundles of low grade paddy. Yield is each less than 1 dou. The top grade plus medium, the medium grade plus low [and] the low grade plus top, in each case adding one bundle, then the yield is one dou. Tell: What is the yield of 1 bundle of top, medium [and] low grade paddy?[8]

Liu Hui's commentary helps us understand the question, as the wording was probably difficult for readers in his time, and definitely is not clear to modern readers.

Lay down 2 bundles of top grade paddy at the top of the right column, 3 bundles of medium grade paddy at the middle of the middle column, [and] 4 bundles of low grade paddy at the bottom of the left column; the 1 bundle to be added and the one dou yields in the appropriate places. Items that are added or borrowed in each column can follow this example.

Liu's commentary is still not completely clear. He is telling the reader that neither 2 bundles of top grade paddy, 3 bundles of medium grade paddy, nor 4 bundles of low grade paddy will yield a single dou of grain. However, adding the yield of just one bundle of the appropriate grade paddy will bring the yield to exactly one dou.

Task 6 Liu tells us that 2 bundles of top grade paddy and one bundle of medium grade paddy yields one dou of grain. Therefore, the right column of the array should have a 2 in the top row for 2 bundles of top grade paddy and a 1 in the medium grade paddy second row with the 1 dou of grain in the bottom row. Use the Fangcheng Rule and Liu's commentary to finish setting up the array for Problem 3. Remember that the Chinese counting rod arithmetic had no symbol for zero, so leave those entries blank!

[8] [Shen et al., 1999, p. 404]

The next problem should sound familiar from similar problems in high school algebra classes.

Problem 7: Now there are 5 cattle [and] 2 sheep costing 10 liang of silver. 2 cattle [and] 5 sheep costs 8 liang of silver. Tell: what is the cost of a cow and a sheep, respectively?[9]

The liang is an ancient Chinese unit of weight which corresponds to approximately 16 grams. or a little more than a half of an ounce, in modern units [Shen et al., 1999, pp. 9-11].

Task 7 Set up Problem 7 from the *Nine Chapters*.

Problem 8 in the *Nine Chapters* requires the use of negative numbers in the formation of the array.

Problem 8: Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 cash. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8 pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?[10]

Task 8 Set up Problem 8 from the *Nine Chapters*. The Chinese considered selling as positive and buying as negative. Use negative numbers to represent a deficit. Read carefully; the numbers are not in the same order in each sentence!

Task 9 Set up the following modern problem using the instructions from the Fangcheng Rule: Three apples, 1 loaf of bread and 1 quart of milk cost $7. One apple, 3 loaves of bread and 1 quart of milk cost $10. Two apples, 4 loaves of bread and 3 quarts of milk cost $18. Find the cost of each individual item.

The statements of the above problems from the *Nine Chapters* and the modern problem are reproduced in Appendix A for easy reference. These problems will be used throughout this lesson to practice the Fangcheng Rule and modern solution techniques.

[9] [Shen et al., 1999, p. 408]
[10] [Shen et al., 1999, p. 409]

3 Solving by the Fangcheng Rule

The Fangcheng Rule gives step-by-step instructions for solving problems on a counting board that are equivalent to systems of linear equations. The Rule can be broken into two separate steps. The first step is sometimes called **forward elimination** in modern mathematics, since the procedure uses an equation, starting with the first, to remove variables from the equations that follow. As we will see, the ancient Chinese version of forward elimination will result in the array resembling the shape of a triangle, and this form of the array will be referred to as **triangular form** or **lower triangular form**.

The algorithm for solving the triangular array will be referred to as **substitution** since the simplest method of solution is to start by solving the equation with only one variable, and then substitute the known values of variables into the remaining equation to solve for one variable at a time.

The ancient Chinese method of forward elimination is equivalent to the modern method, yet the procedure for substitution presented in the Fangcheng Rule is different from modern methods. To highlight the similarities and differences in the procedure and the resulting arithmetic, we will separate the elimination and substitution steps.

3.1 The Rule

Before we begin a careful explanation of each step of the Chinese method, read the English translation of the Fangcheng Rule. The term shi in the Fangcheng Rule is the yield of grain or fruit. It refers to the seeds of rice as it comes off the plant and before it is husked. It also means the constant in the equation [Shen et al., 1999, p. 400].

The Fangcheng Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. Similarly for the middle and left column. Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge. Again multiply the next [and] follow the pivoting [11]. Then use the remainder of the medium grade paddy in the middle column to multiply the left column and pivot. The remainder of the low grade paddy in the left column is the divisor, the entry below is the dividend. The quotient is the yield of low grade paddy. To solve for the medium grade paddy, use the divisor [of the left column] to multiply the shi in the middle column then subtract the value of the low grade paddy. To solve for the top grade paddy also take the divisor to multiply the shi of the right column then subtract the values of the low grade and the medium grade paddy. Divide by the number of bundles of top grade paddy. This is the number of bundles of top grade paddy. The constants are divided by the divisors. Each gives the dou of yield [of one bundle].

[11] The word 'pivot' used here as a modern term the translator chose to use to describe the process, it is not the original Chinese word.

Task 10 Try solving your array for Problem 1 of the *Nine Chapters* using the Fangcheng Rule before reading further. At what step were the instructions unclear?

Did you follow the whole rule? Probably not! Don't be discouraged. The fact that Liu Hui gave detailed commentary to the Fangcheng Rule in his edition from 263 CE indicates that other ancient Chinese mathematicians needed help as well! The original students would have likely read the rule in conjunction with a visual demonstration of the procedure on a counting board. Since we no longer have any record of the visual part of the lesson, it will take a little more effort to translate the verbal description.

In this section we will use Liu Hui's commentary to help us understand how to perform the Fangcheng Rule. We will use modern numerals to aid in understanding, but the layout will correspond to the ancient Chinese format in columns. Read Liu's introduction to the Fangcheng Rule.

> The character *cheng* means comparing quantities. Given several different kinds of item, display [the number for] each as a number in an array with the sums (shi) [at the bottom]. Consider [the entries in] each column as rates, 2 items corresponds to a quantity twice, 3 items corresponds to a quantity 3 times, so the number of items is equal to the corresponding [number]. They are laid out in columns [from right to left], [and] therefore called a rectangular array (*fangcheng*). [Entries in each] column are distinct from one another and [these entries] are based on practical examples.

Task 11 What is the significance of Liu's statement that 'Entries in each column are distinct from one another'? Why is it important that the numbers are based on practical examples?

3.2 Forward Elimination in the *Nine Chapters*

We will now work through the Fangcheng Rule one step at a time, using Liu's commentary to help us understand the procedure. Our goal in this section is to reduce the array to triangular form.[12]

> Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 dou of grain. Similarly for the middle and left column.
>
> Liu's Commentary: This is the general rule [for arrays]. It is difficult to comprehend in mere words, so we simply use paddy to clarify. Lay down the middle and left column like the right column.

[12] In this section the source text will be labeled as part of the original (Fangcheng) Rule or from Liu's commentary to make the distinction clear.

Task 12 Why is the Fangcheng Rule given using the numbers in a specific example instead of as a general procedure?

The array for Problem 1 is the following:

1	2	3
2	3	2
3	1	1
26	34	39

(9)

The first step of the solution procedure is the following:

Rule: Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge.

Liu's Commentary: The meaning of this rule is: subtract the column with smallest [top entry] repeatedly from the columns with larger [top entries], then the top entry must vanish. With the top entry gone, the column has one item absent. However, if the rates in one column are subtracted [from another column], this does not affect the proportions of the remainders. Eliminating the top entry means omitting one item from the sum (shi). In this way, subtract adjacent columns from one another. Determine whether [the sum is] positive or negative. Then one can obtain the answer. First take top grade paddy in the right column to multiply the middle column. This means homogenizing and uniformizing. To homogenize and uniformize means top grade paddy in the middle column also multiplies the right column. For the sake of simplicity, one omits saying homogenize and uniformize. From the point of view of homogenizing and uniformizing this reasoning is natural.

Lui first uses the terms 'homogenize and uniformize' in Chapter 1 of the *Nine Chapters* after Problem 9 when he is explaining the rules for adding fractions. The term refers to the process of multiplying the numerator and denominator of each fraction by a specific factor in order to create equivalent fractions over a common denominator [Shen et al., 1999, pp. 70-72]. In the case of the Fangcheng Rule, 'homogenizing and uniformizing' refers to multiplying a column by the specified number in order for an entry to cancel when one column is subtracted by another.

We are instructed to first multiply the middle column by 3, the number for the top grade paddy in the right column. To multiply a column by a number means to multiply each entry in that column by the given number. After multiplying the middle column by 3, the middle column now has a larger top number than the right column. Liu tells us to subtract the right column repeatedly from the middle column until the top number vanishes (is zero). Subtracting the right column from the middle column means to replace each entry in the middle column by the difference between that number and the number on the same row of the right column. Modern mathematicians would record a zero if that was the result of subtraction. However, the ancient Chinese did not use a symbol for zero in counting board arithmetic, so we will follow the traditional procedure and leave the space blank.

Task 13 Use the instructions in this piece of the Fangcheng Rule to eliminate the number for the top grade paddy in the middle column of the array for Problem 1.

Liu next explains how to continue the process of elimination.

Liu's Commentary: Again eliminate the first entry in the left column. Again, use the two adjacent columns to eliminate the medium grade paddy.

Task 14 Follow the same procedure Liu outlines.

(a.) Eliminate the 1 in the top row of the left column.

(b.) Follow the elimination procedure for the middle and left column with the medium grade paddy.

At this point your array should resemble a triangle. Today this is called *lower triangular form* or simply *triangular form*.

The array for Problem 1 should now be in lower triangular form. Practice this procedure with the other Problems from the *Nine Chapters* and the modern problem by completing the following tasks.

Task 15 Reduce the array you created in Task 6 for Problem 3 from the *Nine Chapters* to lower triangular form using the elimination procedure the Fangcheng Rule. Note that this problem involves using negative numbers in the elimination process.

Task 16 Reduce the array you created in Task 7 for Problem 7 from the *Nine Chapters* to lower triangular form using the elimination procedure the Fangcheng Rule.

Task 17 Reduce the array for the modern problem created in Task 9 to lower triangular form using the Fangcheng Rule.

The Fangcheng Rule, together with the Sign Rule, were used to solve problems that involved excess and deficit in the form of both positive and negative numbers. However, problems arose from practical examples, so the final solutions were always positive. In triangular form, the column containing only one unknown and a yield should be positive. Consider again Problem 8 in the *Nine Chapters*.

Problem 8: Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 coins. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8 pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?

Answer: Cattle price 1200; sheep price 500; pig price 300.

Method: Use the Array Rule rule: lay down 2 cattle, 5 sheep, positive; 13 pigs, negative; surplus coins positive; next 3 cattle, positive; 9 sheep, negative; 3 pigs, positive; next 5 cattle, negative; 6 sheep, positive; 8 pigs, positive. Deficit coins, negative. Calculate using the Sign Rule.[13]

Task 18 Perform elimination using the Fangcheng Rule on the array created for Problem 8 (the array was created in Task 8) to reduce it to lower triangular form. Follow the steps prescribed below. This is the order suggested by Liu in his commentary to Problem 8.

(a.) Multiply the middle column by 2.

(b.) Replace the middle column with the middle column minus three times the right column.

(c.) Multiply the left column by 2 and then add 5 times the right column to eliminate the cattle number in the left column.

(d.) Use the sheep number in the left column to eliminate the sheep number in the middle column.

Is the resulting array in triangular form? How can it be transformed to triangular form? Why did Liu eliminate the sheep number in the middle column instead of eliminating the sheep number in the left column as the last step?

3.3 Substitution: Solving the Triangular Array

Using the Fangcheng Rule to transform an array into lower triangular form is only the first half of the process of solving the system of linear equations. Substitution, the second half of the solution procedure, may be performed in several different ways. We will first examine the instructions in the Fangcheng Rule. Next, we will use Liu Hui's commentary on the Fangcheng Rule to explain how the resulting formulas arise by following his algorithm. Finally, we will introduce the modern algorithm, called **back substitution**, and compare the arithmetic in the Fangcheng Rule and in back substitution to discover each method's advantages and disadvantages.

The array for Problem 1 is in lower triangular form after the elimination steps finished in Task 14 is:

		3
	5	2
36	1	1
99	24	39

(10)

[13] [Shen et al., 1999, p. 409]

The original Fangcheng Rule instructions explain how to find the values for the yields of one bundle of each grade of paddy. Recall that the shi is the amount of grain in the yield (the number in the last row of each column).

Rule: The remainder of the low grade paddy in the left column is the divisor, the entry below is the dividend. The quotient is the yield of the low grade paddy. To solve for the medium grade paddy, usee the divisor [of the left column] to multiply the shi in the middle column then subtract the value of the low grade paddy. To solve for the top grade paddy, also take the divisor to multiply the shi of the right column then subtract the values of the low grade and the medium grade paddy. Divide by the number of bundles of top grade paddy. This is the number of bundles of top grade paddy. The constants are divided by the divisors. Each gives the dou of yield [in one bundle].[14]

The instructions in the Fangcheng Rule begin easily enough. It is fairly straightforward to solve for the yield of the low grade paddy. However, the instructions for the medium grade and the top grade paddy may not be clear to modern readers unfamiliar with the manipulations on a counting board.

Task 19 Use the instructions to solve the lower triangular array for Problem 1. Although finding the low grade paddy yield is a simple division problem, the numbers in the left column will be needed for the rest of the procedure, so that division problem is delayed until the last step. Explicitly write down the arithmetic problem as well as the answers.

Step 1 Multiply the number of bundles of low grade paddy in the left column by the yield in the middle column and subtract the number of bundles of low grade paddy in the middle column times the yield in the left column, then divide that answer by the number of bundles of medium grade paddy in the middle column. The result is the number of dou in the shi of the medium grade paddy.

Step 2 For the top grade paddy, multiply the number bundles of low grade paddy times the yield in the shi of the right column and then subtract the number of bundles of medium grade paddy in the right column times the dou in the shi of the medium grade paddy from Step 1 and subtract the number of dou in the shi of the low grade paddy. Divide that result by the number of bundles of top grade paddy in the right column. The result is the number of dou in the shi of the top grade paddy.

Step 3 The final step is to divide the number of dou in the shi for each grade of paddy by the number of bundles of low grade paddy in the left column. The results are the final answers.

Did you obtain 153 as your answer for Step 1 and 333 for your answer to Step 2? The answer for Step 3 should be the solution given by the *Nine Chapters*. Where do the number answers for Step 1 and Step 2 come from?

[14] [Shen et al., 1999, p. 400]

In [Hart, 2011, p. 95] the instructions are translated into solution steps for a general system of three equations and three unknowns in triangular form. Consider the general array below with modern variable notation. Note that the subscript notation reflects the fact that traditional Chinese was read from top to bottom and then right to left, so the first number counts the columns, starting from the right, and the second number counts the rows, starting from the top.

		a_{11}
	a_{22}	a_{12}
a_{33}	a_{23}	a_{13}
b_3	b_2	b_1

(11)

The steps for the general array with three equations and three unknowns are as follows.

Step 1 Calculate

$$B_2 = (a_{33}b_2 - a_{23}b_3) \div a_{22}.$$

Step 2 Using the result from the first step, calculate

$$B_1 = (a_{33}b_1 - a_{12}B_2 - a_{13}b_3) \div a_{11}.$$

Step 3 Divide by the number in the third row of the left column, a_{33}, to solve for the unknowns.

$$x_1 = B_1 \div a_{33}.$$
$$x_2 = B_2 \div a_{33}.$$
$$x_3 = b_3 \div a_{33}.$$

Task 20 Is there a difference between using the instructions in Task 19 or the general formula given above when solving Problem 1? Would you prefer the instructions spelled out in words as in Task 19 or the general formula with variables? Why?

Task 21 Solve the lower triangular array for Problem 3 in the *Nine Chapters* created in Task 15 using the Fangcheng Rule procedure.

Task 22 Solve the lower triangular array for Problem 7 in the *Nine Chapters* created in Task 16 using the Fangcheng Rule procedure.

Task 23 Solve the lower triangular array for Problem 8 in the *Nine Chapters* created in Task 18 using the Fangcheng Rule procedure.

Task 24 Solve the lower triangular array for the Modern Problem created in Task 17 using the Fangcheng Rule procedure.

Task 25 (Optional) Translate the Chinese lower triangular array into modern equations and solve the system by substituting known values of the variables into each successive equation. Compare your formulas for the answer with the ones given above.

This part of the Fangcheng Rule is not obvious. Liu Hui admits that the procedure given is complicated, and offers a variation on the rule. He begins by explaining how to find the yield of the medium grade paddy.

Liu's Commentary: After eliminating the top grade and the medium grade paddy, the remaining [bottom entry] is the yield of not just one bundle of low grade paddy. To reduce the yield on all the bundles one should take the number of bundles of paddy as divisor. Display this. Take the number of bundles of low grade paddy and multiply [the entries in] the second column. Merge, eliminate the entries of the low grade paddy. The *shi* is then divided by the number of bundles to give the number of *dou* of medium grade paddy. The calculations involved are complicated and inefficient, hence an alternative rule is introduced for simplification. However, if the old rule has to be used, this is a variation.

Liu begins by eliminating the low grade paddy entry in the middle column.

Task 26 Follow the instructions by solving the triangular array for Problem 1 by multiplying the middle column by the low grade paddy number in the left column. Then subtract the left column from the middle column and divide the middle column by 5. Did you notice that this Task performs Step 1 of the Fangcheng Rule procedure for solving a general triangular array? You should obtain the following array:

		3
	36	2
36		1
99	153	39

(12)

Now read the rest of the instructions for solving the array.

Liu's Commentary:These are the values for the medium [and] low grade paddy. Calculate the yield for one bundle of low grade paddy first. Substitute in the middle column to find [the yield of] the medium paddy. First the displayed value. Subtract the shi [in the left column]. Although the yield of one bundle of low grade paddy is given in the left column in taking the divisor as denominator. From the point of view of rates this is not possible. Hence first take the divisor to multiply the constant [in the middle column] to uniformize and use the divisor as [common] denominator, then subtract the low grade paddy constant. Take the yield of one bundle of low grade paddy. Let it multiply the number of bundles of low grade paddy [in the middle column]. This is the lieshi displayed for the low grade paddy [in the middle column]. Subtract it from the shi which is the value for the medium grade paddy.

This is the value for the 3 types of paddy in the right column. Now the values for the low grade and medium grade paddy have been found. Let them multiply the number of bundles of paddy in the right column [respectively]. Then, as before, subtract the lieshi [in the right column] from the displayed values.

Treat the 3 values similarly. If the remainder is smaller than the divisor, take the latter as denominator. Simplify the denominators and numerators when possible.[15]

The instructions present a variation on solving for the yield of one bundle of medium grade paddy, then continue by using these values to eliminate unknowns in the right column. The next tasks accomplish the needed arithmetic using further elimination steps.

Task 27 Finish the solution process on the above array by performing the following steps:

1. Multiply the right column by 36.
2. Subtract twice the middle column from the right column.
3. Subtract the left column from the right column. Then divide the resulting right column by 3.
4. Divide each column by 36, giving the final answer.

Task 28 Follow the pattern outlined in Tasks 26 and 27 to solve the system represented by the triangular array for Problem 3 in the *Nine Chapters* obtained in Task 15.

4 Modern Algorithms for Elimination and Substitution

4.1 Elimination in Modern Mathematics

Modern solution methods for systems of linear equations take full advantage of the power of symbolic algebra. The unknown quantities in the statement of the problem are assigned variable representations and the stated conditions are translated into equations. Let x be the yield of one bundle of top grade paddy, y the yield of one bundle of medium grade paddy and z be the yield of one bundle of low grade paddy. Then the equations for Problem 1 of the *Nine Chapters* have the following form:

$$\begin{array}{ccccccc} 3x & + & 2y & + & z & = & 39 \\ 2x & + & 3y & + & z & = & 34 \\ x & + & 2y & + & 3z & = & 26 \end{array}$$

When manipulating the equations, the essential arithmetic in the problem comes down to operations with the coefficients of the variables and the yield numbers. Matrix notation is introduced as a compact way to record this information. The system of equations is transformed into a rectangular array of numbers called an **augmented matrix**. In an augmented matrix, the numbers in each row represent the constants from a single equation (both the coefficients multiplying each variable and

[15] [Shen et al., 1999, p. 400]

the constant on the other side of the equation) and each column represents the coefficients for a single variable, except the last column which represents the constants on the right side of the equations. The augmented matrix for the system of equations for Problem 1 becomes:

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{pmatrix}$$

Compare the augmented matrix for Problem 1 with the layout on the Chinese counting board. Recall that traditional Chinese was read from top to bottom and then right to left, and modern English is read from left to right and then top to bottom. So the top right of the Chinese array would be the first number read, and the top left of the modern array is the first number read. Rotate the Chinese array counterclockwise to move the top grade paddy number from the Chinese first position to the modern first position and the result is the augmented matrix!

Task 29 Use modern notation to express Problem 3 in the *Nine Chapters* first as a system of linear equations, then as the equivalent augmented matrix. Be sure to label your variables.

The ancient Chinese elimination steps involved either multiplying a column by a constant or subtracting one column from another column. Modern elimination does essentially the same procedures using rows instead of columns. However, modern mathematics demands a careful description of the operations that may be performed on an augmented matrix that do not change the solution set to the system of equations. The allowable operations for modern elimination are known as **elementary row operations**. There are three typse of elementary row operations that may be performed on an augmented matrix.

- Type 1: Replace a row by a nonzero multiple of that row. A Type 1 row operation on Row i multiplies every entry in Row i by the same nonzero constant c and will be denoted $R(i) \leftarrow c \times R(i)$, where the arrow $\leftarrow$ denotes replacement.
- Type 2: Replace a row by the result of adding that row to a multiple of anther row. A Type 2 row operation, replacing Row i with Row i plus a constant k times Row j, will be denoted $R(i) \leftarrow R(i) + k \times R(j)$.
- Type 3: Interchange two rows. Interchanging two rows i and j will be denoted $R(i) \leftrightarrow R(j)$.

Elementary row operations are performed one at a time.

Task 30 Elementary row operations do not change the solution set of a system of linear equations. Justify this statement for each type of row operation.

1. Type 1: If a row is multiplied by a nonzero constant, prove that the solution set does not change.
2. Type 2: Show that (x, y, z) is simultaneously a solution to the equations represented by $R(i)$ and $R(j)$ if and only if it is simultaneously a solution to the equations represented by $R(i)$ and $R(j) + k \times R(i)$ for any nonzero constant k and any natural numbers i, j.

3. Type 3: What does switching two rows in the augmented matrix correspond to relative to its associated system of linear equations? Does switching two rows alter the solution set? Explain.

Task 31 Elementary row operations are reversible. Reversing a Type 3 row operation is obvious; one simply switches the two equations back to the original configuration. How are Type 1 and Type 2 row operations reversed?

The purpose of elementary row operations is to eliminate nonzero entries in some equations in order to make the system easier to solve. The first goal is to transform the augmented matrix to one that is in upper triangular form.

The modern linear algebra term that encompasses upper triangular form of a matrix is **echelon form**. An *echelon* is a military formation of troops or equipment arranged in parallel rows with the end of each row projecting farther than the one before. A matrix is in echelon form if the first nonzero entry in each row is to the right of the first nonzero entry in the row above it, and rows of all zeros are at the bottom of the matrix. Echelon form is more general than upper triangular form because matrices with any number of rows or columns can be brought to echelon form via elementary row operations, whereas not all matrices can be brought to a triangular form this way. The following matrix is in echelon form (Check this!).

$$\begin{pmatrix} 3 & 2 & 1 & 3 & 12 \\ 0 & 4 & 6 & 24 & 3 \\ 0 & 0 & 0 & 2 & 13 \end{pmatrix}$$

Task 32 Which of the following matrices is in echelon form? Justify your answers.

$$A = \begin{pmatrix} 3 & 5 & -2 & 4 \\ 0 & 0 & 53 & 5 \\ 0 & 1 & 12 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 2 & 4 & 5 \\ 0 & 0 & 1 & 5 & 7 \\ 0 & 0 & 0 & 2 & 13 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 0 & 5 & 7 \\ 0 & 5 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row operations may be performed in any order, yet typically one starts in the top left corner of the matrix and works on eliminating nonzero entries below the diagonal in each column, in order from left to right and top to bottom. The systematic approach avoids the chance that one row operation will undo the progress made by the previous row operations.

The procedure will be demonstrated with the augmented matrix for Problem 1. A **pivot position** is defined as the position of the first nonzero entry in a row of the echelon form of the matrix. The number in that position is called the **pivot** in that row (or column) of the matrix, and will be used as a multiplier to perform row operations of Type 1 or Type 2 that leave that position unchanged and eliminate nonzero entries in the pivot column and rows below the pivot. The procedure starts with a nonzero number in the first row and left column. If this position contains a zero, the first step is to switch two rows to put a nonzero number in that position. For Problem 1, the first pivot is the 3, which is highlighted in the matrix:

$$\begin{pmatrix} \boxed{3} & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{pmatrix}$$

At this point, the algorithm will follow the inspiration of the ancient Chinese because their method does not result in fractions in the echelon form of the matrix. Since the other numbers in the first column are not divisible by the pivot number 3, the first steps is to replace Row 2 by 3 times Row 2, $R(2) \leftarrow 3 \times R(2)$. Next, multiply Row 3 by 3 as well, $R(3) \leftarrow 3 \times R(3)$.

$$\longrightarrow \begin{pmatrix} \boxed{3} & 2 & 1 & 39 \\ 6 & 9 & 3 & 102 \\ 3 & 6 & 9 & 78 \end{pmatrix}$$

The algorithm next proceeds by performing row operations of Type 2 using the first row as the row to be added or subtracted from the other rows. First, we replace Row 2 with the operation $R(2) \leftarrow R(2) - 2 \times R(1)$. Then, we replace Row 3, $R(3) \leftarrow R(3) - R(1)$.

$$\longrightarrow \begin{pmatrix} \boxed{3} & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 3 & 6 & 9 & 78 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{3} & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 4 & 8 & 39 \end{pmatrix}$$

The next pivot is the 5 in the second row and second column. To eliminate the 4 below it, multiply the third row by 5, $R(3) \leftarrow 5 \times R(3)$, and replace Row 3 with $R(3) \leftarrow R(3) - 4 \times R(2)$.

$$\longrightarrow \begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & \boxed{5} & 1 & 24 \\ 0 & 4 & 8 & 39 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & \boxed{5} & 1 & 24 \\ 0 & 20 & 40 & 195 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{pmatrix}$$

The matrix is now in echelon form. We will use the modern term *upper triangular form* instead of echelon form to emphasize its correspondence with lower triangular form in the Chinese array.

Task 33 Compare the steps just presented with the Fangcheng Rule steps on columns instead of rows that you performed in Tasks 13 and 14.

Task 34 Other orders of row operations are often suggested in modern linear algebra texts. Often the first step suggested is to switch rows if necessary to place a 1 in the first row and first column. Start with the matrix for Problem 1 and switch Row 3 and Row 1 in order to put a 1 in the upper left corner. Then pivot about the 1 by subtracting multiples of Row 1 from Row 2 and Row 3 to eliminate the nonzero entries below the 1 in the first column. What happens? Would you suggest this technique over the one outlined above? Does the order of the row operations change the final solution? How do you know?

The pattern for reducing the matrix from Problem 1 suggests the following steps for transforming a matrix into upper triangular form with row operations.

1. The first pivot position is the upper left corner. If this entry is a zero, perform a Type 3 row operation to switch Row 1 with another row which has a nonzero first entry. This entry is the first pivot p_1.

2. For each row with a nonzero entry in the first column below the first pivot, multiply that row by p_1.

3. For $j \geq 2$, if the first entry $a_{j,1}$ in Row j is nonzero, replace Row j with the operation $R(j) \leftarrow R(j) - a_{j,1} \times R(1)$. If the first entry $a_{j,1}$ is zero, do not change that row. Note that the notation $a_{j,k}$ refers to the number in row j and column k of the augmented matrix.

4. The second pivot position is in the second row and second column. If this position contains a zero, switch Row 2 with a row below it to put a nonzero number in the pivot position.

5. The second pivot p_2 is now the entry in the second row and second column. Multiply each row below Row 2 by p_2 if its entry in the second column is not zero.

6. For $j \geq 3$, if the entry $a_{j,2}$ in the second column of Row j is nonzero, replace Row j with the operation $R(j) \leftarrow R(j) - a_{j,2} \times R(2)$.

7. If there are more than 3 rows, continue by finding the next pivot in the next column from the left and use the pivot to eliminate nonzero entries below it in the same pattern.

This procedure assumes that the matrix has a pivot in every column corresponding to a variable, which is the case when the system has a unique solution. Further practice with elementary row operations will illustrate how variations in the order of the steps are often done to make the arithmetic easier. The reader is encouraged to practice with this procedure.

Task 35 Create an augmented matrix for Problem 3 in the *Nine Chapters* and use the general procedure to reduce the matrix to upper triangular.

Task 36 Create an augmented matrix for Problem 7 in the *Nine Chapters* and use the general procedure to reduce the matrix to upper triangular.

Task 37 Create an augmented matrix for Problem 8 in the *Nine Chapters* and use the general procedure to reduce the matrix to upper triangular.

Task 38 Create an augmented matrix for the modern problem of Task 6 and use the general procedure to reduce the matrix to upper triangular.

4.2 Back Substitution

Liu Hui alludes to the most intuitive way to solve a triangular array in his explanation of the Fangcheng Rule. The following is the beginning of his instructions.

Liu: These are values for the medium [and] low grade paddy. Calculate the yield for one bundle of low grade paddy first. Substitute in the middle column to find [the yield of] the medium paddy.[16]

[16] [Shen et al., 1999, p. 400]

If the yield of one bundle of low grade paddy is known, then the middle column involving only the low grade and the medium grade paddy has only one unknown, which can be found. Then, the right column has only the top grade paddy as unknown, and its value may be found. The modern method that corresponds to this idea called is **back substitution**. This section demonstrates the process of back substitution on an upper triangular augmented matrix.

Consider the augmented matrix which corresponds to Problem 1 in the *Nine Chapters*.

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{pmatrix}$$

The last row, when translated back into equation form, reads $36x_3 = 99$. This gives $x_3 = \frac{99}{36} = 2\frac{3}{4}$.

The second row of the matrix is equivalent to the equation $5x_2 + x_3 = 24$. The value of x_3 can be substituted into this equation. Solving the equation for x_2 results in the following:

$$x_2 = \frac{24 - 2\frac{3}{4}}{5} = 4\frac{1}{4}.$$

To finish the process, substitute the values of x_3 and x_2 into the equation corresponding to the first row, $3x_1 + 2x_2 + x_3 = 39$, to solve for x_1:

$$x_1 = \frac{39 - 2 \times 4\frac{1}{4} - 2\frac{3}{4}}{3} = 9\frac{1}{4}.$$

The forward elimination algorithm transforms the array from the top line down, and then the back substitution algorithm works backwards from the bottom up – hence 'back' substitution. The procedure is straightforward to generalize: translate the upper triangular array into equations and solve them one at a time, starting from the bottom. Substitute the known values into subsequent equations as the process proceeds.

Task 39 Solve the upper triangular matrix for Problem 3 from the *Nine Chapters* created in Task 35 using back substitution.

Task 40 Solve the upper triangular matrix for Problem 8 from the *Nine Chapters* created in Task 37 using back substitution.

Task 41 Solve the upper triangular matrix from the Modern Problem created in Task 38 using back substitution.

5 Arithmetic Comparisons and Conclusions

The systems of linear equations we have examined in this lesson all have a single unique solution. The method used to obtain this solution does not change the final answer. However, the amount and difficulty of the arithmetic that is involved in finding the solution changes with the choice of method for substitution. Furthermore, the precise order for performing the elimination or substitution operations may change the difficulty of the arithmetic. For example, an arbitrary elementary row operation could be performed on an augmented matrix, followed immediately by the reverse operation. This involves extra arithmetic with no benefit. Roger Hart points out that the Fangcheng Rule's counter-intuitive method of solving the triangular array usually had the advantage of introducing fractions only in the last step of the substitution process [Hart, 2011, pp. 99-110].

Task 42 Compare the arithmetic for the Fangcheng Rule solution to Problem 3 of the *Nine Chapters* with the modern technique of forward elimination followed by back substitution. (Tasks 15, 21 and 28 perform the Fangcheng Rule solution algorithm and Tasks 29, 35 and 39 solve Problem 3 by the modern method.) Which method would you prefer to practice if you had to manually solve similar problems on a regular basis? Why?

The Fangcheng Rule in Chapter 8 of the *Nine Chapters* was conceived about 2000 years ago, yet it feels very modern!

The Fangcheng Rule demonstrated the important consideration of designing a technique that keeps the arithmetic as simple as possible. We saw that the substitution step prescribed by the original rule was counterintuitive, yet had the advantage of delaying the introduction of fractions until the last step.

Liu Hui puts the importance of wisely choosing how to apply the Fangcheng Rule to make the arithmetic as simple as possible in his comments on Problem 18. Problem 18 in the *Nine Chapters* has five variables and five equations. Clearly, the Fangcheng Rule requires many steps. As he introduces his new variation on the rule, Liu explains why it is important to think before blindly starting to calculate. We will not delve into the *New Array Rule* that Liu proposed because its complexity will not be enlightening. However, the following excerpt is instructive for students of mathematics at any level:

Liu: ...Some clumsy students of exact science use this Rule mechanically, or arrange a lot of counting rods on a carpet. They are meticulous but vulnerable, without thinking it irrational to behave so, but rather [saying] "the more the better". All the algorithms, even though each applies to special subjects individually, are mutually connected by basic principles. For certain problems, this [rule] succeeds; however, they are really quite fallible, and cannot be regarded as simple. Then there is a special solution. The skillful butcher always leaves plenty of room for the play of his knife [when slicing] through the muscles in dissecting an ox, so that the blade remains sharp even after a lengthy period. Mathematical rules are just like the blade, and simplicity conforms to the butcher's way. So if one takes care with the blade, one can solve problems both promptly and with few mistakes. ... [17]

Task 43 Does Liu Hui give us good advice? Have you experienced problems for which this advice would have helped make solving easier?

Although systems of linear equations may be solved using a computer in modern times, the question of streamlining the arithmetic remains an important consideration. The Chinese were amazingly ahead of their time!

[17] [Shen et al., 1999, p. 426]

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Appendix A: The Problems

Text from the *Nine Chapters* [Shen et al., 1999] for the problems in the Tasks and the Modern Problem in the Tasks are copied here for easy reference.

Problem 1: Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much paddy does one bundle of each grain yield? Answer: Top grade paddy yields $9\frac{1}{4}$ *dou* [per bundle]; medium grade paddy $4\frac{1}{4}$ *dou*; [and] low grade paddy $2\frac{3}{4}$ *dou*.

Problem 3: Now there are 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 4 bundles of low grade paddy. Yield is each less than 1 *dou*. The top grade plus medium, the medium grade plus low [and] the low grade plus top, in each case adding one bundle, then the yield is one *dou*. Tell: What is the yield of 1 bundle of top, medium [and] low grade paddy?

Problem 7: Now there are 5 cattle [and] 2 sheep costing 10 *liang* of silver. 2 cattle [and] 5 sheep costs 8 *liang* of silver. Tell: what is the cost of a cow and a sheep, respectively?

Problem 8: Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 cash. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8 pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?

Modern Problem: Three apples, 1 loaf of bread and 1 quart of milk cost \$7. One apple, 3 loaves of bread and 1 quart of milk cost \$10. Two apples, 4 loaves of bread and 3 quarts of milk cost \$18. Find the cost of each individual item.

References for the Problems in the Tasks:

- Problem 1 is the subject of Tasks 4, 5, 13, 14, 19, 20, 26, 27, 33, 34 and 42.
- Problem 3 is the subject of Tasks 6, 15, 21, 28, 29, 35 and 39.
- Problem 7 is the subject of Tasks 7, 16, 22 and 36.
- Problem 8 is the subject of Tasks 8, 18, 23, 37 and 40.
- The Modern Problem is worked in Tasks 9, 17, 24, 38 and 41.

Appendix B: Results of Forward Elimination

This lesson used Problems 1, 3, 7 and 8 in the *Nine Chapters* as well as a Modern Problem to practice elimination and substitution. Later Tasks ask students to solve the Chinese array or matrix starting from the result of an earlier exercises. The answers to the Tasks concerned with setting up the array and performing forward elimination on the array are printed here for each of these Problems. Note that the upper or lower triangular form printed here is the result of following the algorithms in this lesson in the order given. Upper (lower) triangular form is not unique, other forms are possible.

Problem 1 in the *Nine Chapters*

The Chinese array for Problem 1 is as follows:

1	2	3
2	3	2
3	1	1
26	34	39

The Chinese array in lower triangular form:

		3
	5	2
36	1	1
99	24	39

The augmented matrix for Problem 1 is as follows:

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{pmatrix}$$

The augmented matrix for Problem 1 in upper triangular form:

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{pmatrix}$$

Problem 3 in the *Nine Chapters*

The Chinese array for Problem 3 is as follows:

1		2
	3	1
4	1	
1	1	1

The Chinese array in lower triangular form:

		2
	3	1
25	1	
4	1	1

The augmented matrix for Problem 3 is as follows:

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 4 & 1 \end{pmatrix}$$

The augmented matrix for Problem 3 in upper triangular form:

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 25 & 4 \end{pmatrix}$$

Problem 7 in the *Nine Chapters*

The Chinese array for Problem 7 is as follows:

2	5
5	2
8	10

The Chinese array in lower triangular form:

	5
21	2
20	10

The augmented matrix for Problem 7 is as follows:

$$\begin{pmatrix} 5 & 2 & 10 \\ 2 & 5 & 8 \end{pmatrix}$$

The augmented matrix for Problem 7 in upper triangular form:

$$\begin{pmatrix} 5 & 2 & 10 \\ 0 & 21 & 20 \end{pmatrix}$$

Problem 8 in the *Nine Chapters*

The Chinese array for Problem 8 is as follows:

−5	3	2
6	−9	5
8	3	−13
−600		1000

The Chinese array in (not quite) lower triangular form:

		2
37		5
−49	48	−13
3800	14400	1000

The augmented matrix for Problem 8 is as follows:

$$\begin{pmatrix} 2 & 5 & -13 & 1000 \\ 3 & -9 & 3 & 0 \\ -5 & 6 & 8 & -600 \end{pmatrix}$$

The augmented matrix for Problem 8 in upper triangular form:

$$\begin{pmatrix} 2 & 5 & -13 & 1000 \\ 0 & 37 & -49 & 3800 \\ 0 & 0 & 48 & 14400 \end{pmatrix}$$

The Modern Problem

The Chinese array for the Modern Problem is as follows:

2	1	3
4	3	1
3	1	1
18	10	7

The Chinese array in lower triangular form:

		3
	8	1
36	2	1
90	23	7

The augmented matrix for the Modern Problem is as follows:

$$\begin{pmatrix} 3 & 1 & 1 & 7 \\ 1 & 3 & 1 & 10 \\ 2 & 4 & 3 & 18 \end{pmatrix}$$

The augmented matrix for the Modern Problem in upper triangular form:

$$\begin{pmatrix} 3 & 1 & 1 & 7 \\ 0 & 8 & 2 & 23 \\ 0 & 0 & 36 & 90 \end{pmatrix}$$

The solution to the Modern Problem One apple costs \$0.75, one loaf of bread costs \$2.25 and on quart of milk costs \$2.50.

Notes to Instructors

This lesson introduces Gaussian elimination on the augmented matrix for small systems of linear equations. There are no prerequisites for the students beyond high school algebra. The lesson was intended for a linear algebra class, however it has been used in an algebra class for pre-service teachers and in a history of math class.

The instructor in a typical linear algebra class would use this lesson in place of the introductory textbook section on reducing an augmented matrix to echelon form and solving by back substitution. The *Nine Chapters on the Mathematical Art* does not address the issue of systems without a unique solution, therefore this lesson does not go into any detail on the techniques for handling more complicated systems. The instructor should follow this lesson with the more general instructions for elimination from a standard textbook. A linear algebra instructor may choose to implement the full PSP, or focus on the comparison of the ancient Chinese methods with modern methods of reducing an array to triangular form (echelon form) using elementary row operations.

Linear Algebra Full Implementation

It is estimated that it will take 2–2.5 weeks to implement the full PSP with a linear algebra class. A mixture of student individual preparation and in-class activities is optimal. The following plan may be implemented in a introductory linear algebra class with a 75 minute class period. Instructors teaching a 50 minute class should allow 3 class periods for each two days listed.

- Preparation: Introduce the PSP and ask students to read Section 1, Section 2, and answer some or all of Tasks 1–5.
- Class 1: The Array on the Counting Board and The Fangcheng Rule: Sections 2.4 and 3.1
  - Class discussion on preparatory reading and rod arithmetic, including the algorithm for multiplication.
  - Group work on Tasks 5–9
  - Class discussion of results on Tasks 5–9 on setting up a Chinese array.
  - Section 3.1 Tasks 10 and 11 if time permits, or as assigned homework for the next class.
- Class 2: Reducing the Chinese Array to Triangular Form: Section 3.2.
  - Questions and comments from trying the Fangcheng Rule by doing Tasks 10 and 11
  - Group work on reducing the array for Problem 1 in the *Nine Chapters* using Tasks 12–14
  - Group work practicing the technique on another problem chosen from Tasks 15–18.
  - Assign Tasks 15–18 problems not done in class as homework.
- Class 3: Chinese Substitution: Section 3.3.
  - Class review of reducing an array to triangular form
  - Group work on Task 19
  - Group work on another problem chosen from Tasks 20–24
  - Optional group work or whole class discussion of an alternate approach to the Chinese formula following the procedure in Tasks 26–28

  - Assign Section 4.1 up until Task 32 as preparation for the next class.

- Class 4: Modern Elimination and Substitution: Section 4.

  - Discussion of modern notation and elementary row operations
  - Group work chosen from Tasks 33–38 on reducing a matrix to upper triangular form
  - Group work chosen on Back Substitution, Section 4.2
  - Follow-up homework assignment from Section 5 on comparing modern and ancient Chinese arithmetic

Central Focus Implementation in a Linear Algebra Class

The essential ideas of elimination can be presented in a linear algebra class by spending approximately 4 class instruction hours on Sections 2.4, 3.1, 3.2 and 4.1.

- Preparation: Assign Section 1 Historical Background, Section 2.1 on counting rod numerals and Section 2.4 The Array on the Counting Board as preparatory reading, along with answering Tasks 4 and 5.

- Hour 1: Section 2.4 The Array on the Counting Board group work (20 minutes) on Tasks 6–9 followed by class discussion on the answers (10 minutes). Read Section 3.1 and try Task 10 in their groups. Assign pages 12–14 up through Task 14 as homework.

- Hour 2: Section 3.2 Forward Elimination in the *Nine Chapters*. Work in groups on Tasks 14–18.

- Hour 3: Finish Section 3.2 with sharing student work on the full procedure of translating a word problem into an array in lower triangular form, then start on Section 4.1 on modern elimination.

- Hour 4: Section 4.1 Modern Elimination and comparison with the Fangcheng Rule.

Ancient Chinese Arithmetic for Pre-Service Teachers or History Students

In an algebra class for pre-service teachers or a history of math class, the section on Counting Rod Arithmetic (Section 2) presents the ancient Chinese system of numeration used for computation and explains the Chinese algorithm for multiplying multi-digit whole numbers, the techniques for working with negative numbers and the construction of the rectangular array for solving a system of linear equations. An instructor could implement Sections 1 and 2 to cover counting rod arithmetic or Sections 1–3.2 to cover counting rod arithmetic and reducing an array to triangular form using the Chinese algorithm. The following discussion lists several 1-2 class day lesson options for other courses besides linear algebra.

- Lesson on Chinese Arithmetic (One Class Session)

  - Background reading Section 1 and Section 2.1
  - Open with discussion of background material and Chinese counting rods

  - Student group work with physical manipulatives on Chinese rod numerals. Flat toothpicks, coffee stirrers and matchsticks have been used successfully as counting rods.
  - Student group work on Section 2.2 on multiplication and Task 1.

- Lesson on Arithmetic with Signed Numbers (One Class Session)
  - Background reading Section 1. and Section 2.1
  - Discussion and questions from the background reading.
  - Student group work with physical manipulatives on Chinese rod numerals using flat toothpicks, coffee stirrers, or matchsticks.
  - Student group work on Section 2.3 on the Sign Rule including Tasks 2 and 3.

- Lesson on Solving a System of Linear Equations on a Counting Board (One Week of Class)
  - Background reading Section 1 and Section 2.1 if neither arithmetic lesson was done previously.
  - Open with a discussion of background reading and Chinese arithmetic using counting rods on a counting board.
  - Student group work on the beginning of Section 2.4 including Tasks 4 and 5 on setting up the array for Problem 1 in the *nine Chapters*.
  - Group work on practice applying the technique with problems chosen from Tasks 6–9.
  - The Fangcheng Rule Section 3.1, Tasks 10 and 11 as homework or in class work.
  - Reducing a system of equations to triangular form using the Fangcheng Rule with Problem 1 including Tasks 12–14.
  - Practice with the Fangcheng Rule in class or as homework utilizing Tasks 13–18.

Content and Intentions of the Tasks

The Tasks in this lesson are intended to help the reader practice the techniques as they read. Instructors are encouraged to require students to keep a notebook with all their work (showing each step in each algorithm) so that they can compare the complexity of the arithmetic using multiple approaches. The same four problems from the *Nine Chapters* and the same modern problem appear throughout the sections on elimination and substitution using both the ancient Chinese and the modern techniques. The purpose is to analyze the differences in the algorithms with essentially the same arithmetic operations in different orders. Appendix B provides the arrays and the reduced arrays after forward elimination for each of the Problems in the *Nine Chapters*. The purpose of including these hints are to let students check their work at the intermediate stage and to start later problems in the substitution section with the correct array.

The section on Counting Rod Arithmetic is important to understanding the use of rectangular arrays in Chinese algebra. An example of multiplication is shown to illustrate the very modern techniques used to multiply multi-digit numbers. Practicing the Chinese algorithm with Task 1 is useful even if done with modern numerals, as it forces students out of familiar patterns of arithmetic. Instructors who teach pre-service teachers might want to spend time on this section and actually create counting rods and learn to add, subtract and multiply Chinese style. Flat toothpicks, coffee stirrers and matchsticks have been used successfully as counting rods. A comparison with the standard algorithms with base ten blocks would be enlightening.

Task 3 on the Sign Rule is designed to have students understand that the rule was written for working with counting rods. Simply following the rule with modern numbers does not fully explain how the rule works. For example, if $a = 3$ and $b = 7$ then $a - b = -4$ as we know. The Sign Rule says that like signs subtract, so we would have three red (positive) rods and try to take away 7 red rods. Of course, we take away three and then we cannot take any more away. However, the rule says 'Positive without extra, make negative', meaning the four remaining positive rods that we need to subtract are made negative (black) and thus the answer is -4. The physical operations with the rods dictates the Sign Rule. This is a particularly meaningful exercise for pre-service teachers, since it presents an unfamiliar twist on rules for arithmetic with signed numbers.

Task 11 asks students to think about why Liu states that 'Entries in each column are distinct from one another'. The point is that the columns are linearly independent. The fact that the problems come from practical examples implies that they have solutions.

The lesson emphasizes the Fangcheng Rule's order of operations that delays the introduction of fractions until the last step. Notice that the problems did not in general have simple integer answers.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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