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The Failure of the Euclidean Parallel Postulate and Distance in Hyperbolic Geometry

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Notes to the Instructor

The goal of this series of mini-projects is to introduce the reader to hyperbolic geometry and examine a notion of distance in this new geometry, where the Pythagorean Theorem fails. We first study the hyperbolic parallel postulate (HPP) from Nikolai Lobachevsky's *Geometrical Researches on the Theory of Parallels* [3], where there is more than one line through a point parallel to another line. The first mini-project "The Hyperbolic Parallel Postulate" contains a short excerpt from Lobachevsky's work and examines a few immediate consequences of the HPP such as the existence of "limiting parallels." The first mini-project contains the original source material that is used throughout this sequence of mini-projects and is essential for understanding the development of hyperbolic geometry. The instructor should allow at least two class sessions to cover this first mini-project.

The next mini-project "The Sides of Parallelism" builds on the first and establishes a certain two-fold (left and right) symmetry of limiting parallels. The third mini-project "The Angle of Parallelism" is short, presenting an exercise based on the original source reading in the first project. The next mini-project "The Angle Sum of a Triangle in Hyperbolic Geometry" shows how a triangle with angle sum less than 180° can be constructed in hyperbolic geometry, and, as a result, precludes the existence of squares (or rectangles) in hyperbolic geometry. Recall that the construction of a square is a key step in the proof of the (Euclidean) Pythagorean Theorem. The fifth mini-project examines the distance between limiting parallels. The last mini-project develops a formula for the distance between two points in hyperbolic geometry based on the work of Felix Klein and contains an excerpt from Klein's "On the so-called Non-Euclidean Geometry" [10]. Allow at least two class sessions for this last project.

The complete list of mini-projects is:

1. The Hyperbolic Parallel Postulate

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2. The Sides of Parallelism
3. The Angle of Parallelism
4. The Angle Sum of a Triangle in Hyperbolic Geometry
5. The Distance Between Limiting Parallels
6. Distance in Hyperbolic Geometry

Mini-projects 1 and 6 contain important original source excerpts and require at least two class sessions for coverage. Mini-projects 2, 4 and 5 require at least one class session (with advance reading as homework), while mini-project 3 could be covered in one session. The instructor may wish to assign advance reading of the mini-projects and ask students to prepare specific questions about the reading, especially the original source material. If time permits, the instructor may wish to assign readings from the bibliographic references, particularly if the course has an independent research requirement, such as a term paper. The mini-projects presented here, build on one another, and form the sequel to the series of projects in "The Exigency of the Parallel Postulate and the Pythagorean Theorem," available in this collection of historical projects.

1 The Hyperbolic Parallel Postulate

Purpose: A geometry without the Euclidean parallel postulate (EPP) is difficult to imagine and perhaps unfathomable, since the familiar notion of distance given by the Pythagorean Theorem would no longer hold. Adopting the point of view that the EPP must be true out of philosophical necessity, many scholars attempted to prove the EPP, although all such proofs are flawed. For example, see Adrien-Marie Legendre's (1752–1833) published "proof" of this result [12, pp. 27–29]. In spite of the intellectual bias towards the EPP, several pioneers were bold enough to envision a geometry without the EPP [3, 6, 7, 12] and put in print theorems that would result if through a point A there is more than one line parallel to a given line $\overleftrightarrow{BC}$, (A not on $\overleftrightarrow{BC}$). Among the pioneers were János Bolyai (1802–1860) and Nikolai Lobachevsky (1792–1856), whose work is reprinted (in English translation) in [3]. Many of the results of this new geometry had been anticipated by Carl Friedrich Gauss (1777–1855), although to avoid controversy, Gauss published very little about this. We know from his letters, published posthumously, that Gauss actively pursued this new non-Euclidean geometry and because of his interest, the subject acquired a legitimacy that it otherwise would not have been given. At the time of the initial publications of Bolyai, 1832 [2], and Lobachevsky, 1829 [13], however, the subject was met with disbelief and ridicule. Today the geometry resulting from there being more than one line through a point parallel to another line is called hyperbolic geometry and remains a vibrant area of research [15] as well as a possible model for the universe.

Let's read a short excerpt from Lobachevsky's *Geometrische Untersuchungen zur Theorie der parallelinien* (Geometrical Researches on the Theory of Parallels) [3]. Lobachevsky numbers his statements much like Euclid. The first ten are either axioms or propositions that hold without assuming the EPP. In statement 16, we read what today would be called the hyperbolic parallel postulate (HPP) along with several results that follow quickly from this. Lobachevsky begins this statement with a standard procedure for constructing parallel lines that holds without supposing the EPP or the HPP. Given a point A and a line $\overleftrightarrow{BC}$ (A not on $\overleftrightarrow{BC}$), let D be a point on $\overleftrightarrow{BC}$ with $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$. Then construct line $\overleftrightarrow{AE}$ with $\overleftrightarrow{AE} \perp \overleftrightarrow{AD}$. By the alternate interior angle theorem, line $\overleftrightarrow{AE}$ is parallel to line $\overleftrightarrow{BC} = \overleftrightarrow{DC}$. Lobachevsky writes: "In the uncertainty whether the perpendicular AE is the only line which does not meet DC, we will assume that it may be possible that there are still other lines, for example AG, which do not cut DC, how far soever they may be prolonged." This is a statement of the hyperbolic parallel postulate, that there is more than one line through A parallel to $\overleftrightarrow{BC}$. See Figure (1). Certain questions immediately come to mind. First, if there is more than one line through A parallel to $\overleftrightarrow{BC}$, how many lines through A are parallel to $\overleftrightarrow{BC}$, i.e., do not intersect $\overleftrightarrow{BC}$? A simple exercise shows that there are infinitely many lines through A parallel to $\overleftrightarrow{BC}$. But certainly there are an infinitude of lines through A that are not parallel to $\overleftrightarrow{BC}$. Simply consider a point F on $\overleftrightarrow{BC}$ and connect A to F with a line. Lobachevsky then identifies some angle, $\angle HAD$, where ray $\overrightarrow{AH}$ is between rays $\overrightarrow{AE}$ and $\overrightarrow{AD}$, so that line $\overleftrightarrow{AH}$ serves as a "boundary line" between lines through A parallel to $\overleftrightarrow{BC}$ and lines through

A not parallel to $\overleftrightarrow{BC}$. The goal of this mini-project is to prove that the boundary line $\overleftrightarrow{AH}$ is in fact parallel to $\overleftrightarrow{BC}$, a result that Lobachevsky uses throughout his memoir. Today this boundary line is often called "the limiting parallel" through A to $\overleftrightarrow{BC}$.

Prerequisite Material: The reader may use, without proof, Lobachevsky's first ten statements, which are either axioms or propositions that hold without supposing either the EPP or HPP. Additionally, the reader may use that two (distinct) points determine a unique line.

Lobachevsky writes [3]:

In geometry I find certain imperfections which I hold to be the reason why this science, apart from transition into analytics, can as yet make no advance from that state in which it has come to us from Euclid.

As belonging to these imperfections, I consider the obscurity in the fundamental concepts of the geometrical magnitudes and in the manner and method of representing the measuring of these magnitudes, and finally the momentous gap in the theory of parallels, to fill which all efforts of mathematicians have been so far in vain.

For this theory Legendre's endeavors have done nothing, ...

In order not to fatigue my reader with the multitude of those theorems whose proofs present no difficulties, I prefix here only those of which a knowledge is necessary for what follows.

1. A straight line fits upon itself in all its positions. By this I mean that during the revolution of the surface containing it the straight line does not change its place if it goes through two unmoving points in the surface: (*i.e.,* if we turn the surface containing it about two points of the line, the line does not move.)

2. Two straight lines can not intersect in two points.

3. A straight line sufficiently produced both ways must go out beyond all bounds, and in such way cuts a bounded [plane] into two parts.

4. Two straight lines perpendicular to a third never intersect, how far soever they be produced.

5. A straight line always cuts another in going from one side of it over to the other side: (*i.e.,* one straight line must cut another if it has points on both sides of it.)

6. Vertical angles, where the sides of one are productions of the sides of the other, are equal. This holds of plane rectilinear angles among themselves, as also of plane surface angles: (*i.e.,* dihedral angles.)

7. Two straight lines can not intersect, if a third cuts them at the same angle.

8. In a rectilinear triangle equal sides lie opposite equal angles, and inversely.

9. In a rectilinear triangle, a greater side lies opposite a greater angle. In a right-angled triangle the hypotenuse is greater than either of the other sides, and the two angles adjacent to it are acute.

10. Rectilinear triangles are congruent if they have a side and two angles equal, or two sides and the included angle equal, or two sides and the angle opposite the greater equal, or three sides equal.

...

16. All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes—into *cutting* and *not-cutting*.

The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line*.

From the point A (Fig. 1) let fall upon the line BC the perpendicular AD, to which again draw the perpendicular AE.

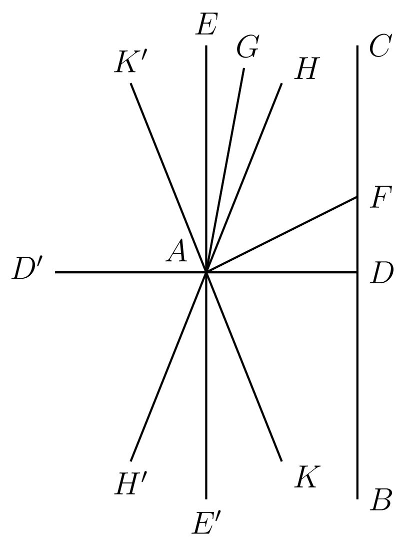

Figure 1: Lobachevsky's Boundary Lines.

In the right angle EAD either will all straight lines which go out from the point A meet the line DC, as for example AF, or some of them, like the perpendicular AE, will not meet the line DC. In the uncertainty whether the perpendicular AE is the only line which does not meet DC, we will assume it may be possible that there are still other lines, for example AG, which do not cut DC, how far soever they may be prolonged. In passing over from the cutting lines, as AF, to the non-cutting lines, as AG, we must come upon a line AH, parallel to DC, a boundary line, upon one side of which all lines AG are such as do not meet the line DC, while upon the other side every straight line AF cuts the line DC.

The angle HAD between the parallel HA and the perpendicular AD is called the parallel angle (angle of parallelism), which we will here designate by $\Pi(p)$ for $AD = p$. ...

If $\Pi(p) < \frac{1}{2}\pi$, then upon the other side of AD, making the same angle $DAK = \Pi(p)$ will lie also a line AK, parallel to the prolongation DB of the line DC, so that under this assumption we must also make a distinction of *sides in parallelism.*

All remaining lines or their prolongations within the two right angles turned toward BC pertain to those that intersect, if they lie within the angle $HAK = 2\Pi(p)$ between the parallels; they pertain on the other hand to the non-intersecting AG, if they lie upon the other sides of the parallels AH and AK, in the opening of the two angles $EAH = \frac{1}{2}\pi - \Pi(p)$, $E'AK = \frac{1}{2}\pi - \Pi(p)$, between the parallels and EE' the perpendicular to AD. Upon the other side of the perpendicular EE' will in like manner the prolongations AH' and AK' of the parallels AH and AK likewise be parallel to BC; the remaining lines pertain, if in the angle $K'AH'$, to the intersecting, but if in the angles $K'AE$, $H'AE'$ to the non-intersecting.

Exercise 1.1. With the set-up as in Lobachevsky's statement 16, prove that there are infinitely many lines through A parallel to $\overleftrightarrow{BC}$. Be sure to organize your work in a step-by-step argument and provide a reason why each step holds.

Strategy of Proof: Consider a ray $\overrightarrow{AL}$ between rays $\overrightarrow{AG}$ and $\overrightarrow{AE}$. Show that $\overleftrightarrow{AL} \parallel \overleftrightarrow{BC}$ by using a proof by contradiction. Assuming that $\overleftrightarrow{AL}$ intersects $\overleftrightarrow{BC}$ at some point X, conclude that $\overleftrightarrow{AL}$ must intersect $\overleftrightarrow{AG}$ at some point Y. (Use Lobachevsky's statement 5). What can be concluded about lines $\overleftrightarrow{AL}$ and $\overleftrightarrow{AG}$?

Exercise 1.2. In a step-by-step argument, prove that the boundary line $\overleftrightarrow{AH}$ is parallel to $\overleftrightarrow{BC}$. Be sure to provide a reason why each step holds.

Strategy of Proof: Using a proof by contradiction, assume that $\overleftrightarrow{AH}$ intersects $\overleftrightarrow{BC}$ at some point X. Let Y be another point on $\overleftrightarrow{BC}$ with X between D and Y. Construct line $\overleftrightarrow{AY}$. Is $\overleftrightarrow{AH}$ still the boundary line between parallels and non-parallels?

2 The Sides of Parallelism

Purpose: Recall that the hyperbolic parallel postulate (HPP) states that given a line $\overleftrightarrow{BC}$ and a point A not on $\overleftrightarrow{BC}$, then there are at least two lines through A parallel to $\overleftrightarrow{BC}$. In result 16 of *Geometrical Researches on the Theory of Parallels* [3], Nikolai Lobachevsky (1792–1856) shows that as a consequence of the HPP, there are in fact an infinite number of lines through A that do not meet $\overleftrightarrow{BC}$. One parallel is constructed from the double perpendicular construction, which does not require a particular version of the parallel postulate (Euclidean or hyperbolic). Let D be a point on $\overleftrightarrow{BC}$ so that $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$. Let $\overleftrightarrow{AE} \perp \overleftrightarrow{AD}$. Then by the corresponding angle theorem (or the alternate interior angle theorem), Propositions 27, 28 of *The Elements*, Book I, [9], $\overleftrightarrow{AE}$ is parallel to $\overleftrightarrow{BC}$. Lobachevsky considers a ray $\overrightarrow{AH}$ between rays $\overrightarrow{AD}$ and $\overrightarrow{AE}$ so that $\overleftrightarrow{AH}$ is a boundary line between lines that meet and do not meet $\overleftrightarrow{BC}$. Line $\overleftrightarrow{AH}$ is then parallel to $\overleftrightarrow{BC}$ and $\angle HAD$ is called the *angle of parallelism*. Note that this construction is performed on one side of $\overleftrightarrow{AD}$, namely the same side of $\overleftrightarrow{AD}$ as point C. Lobachevsky writes: "We must make a distinction of *sides in parallelism*." The goal of this mini-project is to show that the two angles of parallelism on either side of $\overleftrightarrow{AD}$ are actually congruent.

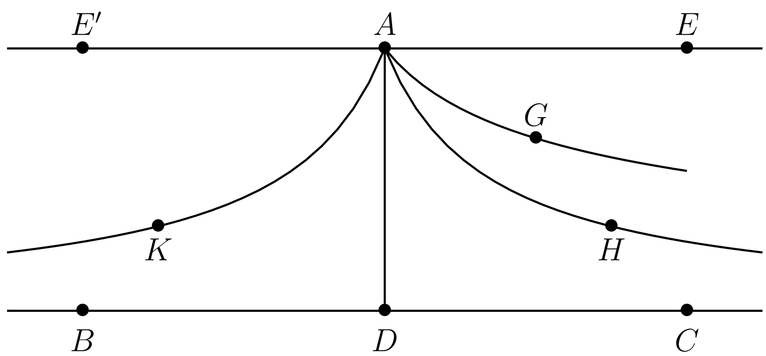

Figure 2: The Sides of Parallelism.

Prerequisite Material: The reader may use without proof that the boundary line $\overleftrightarrow{AH}$ does not intersect $\overleftrightarrow{BC}$. Also, segment duplication, angle duplication and the standard triangle congruency theorems may be used. To state Lobachevsky's result, suppose that line $\overleftrightarrow{AE}$ is extended to line $\overleftrightarrow{EE'}$, where E' is on the opposite side of $\overleftrightarrow{AD}$ as point E. Suppose that $\angle DAK$ is constructed congruent to $\angle DAH$, where ray $\overrightarrow{AK}$ is between rays $\overrightarrow{AE'}$ and $\overrightarrow{AD}$. Lobachevsky writes [3]:

All remaining lines or their prolongations within the two right angles turned toward BC pertain to those that intersect, if they lie within angle HAK. ... [T]hey pertain to the other hand to the non-intersecting [lines] AG if they line upon the other side of the parallels AH and AK.

Exercise 2.1. Let Z' be a point between rays $\overrightarrow{AE'}$ and $\overrightarrow{AK}$. In a step-by-step argument, show that line $\overleftrightarrow{AZ'}$ is parallel to $\overleftrightarrow{BC}$. You may use any of Lobachevsky's points 1–10, Lobachevsky's characterization of a boundary line, Euclid's postulates 1–4, Euclid's propositions 1–28 or Euclid's common notions as a valid reason.

Strategy of Proof: Using an indirect proof, assume that line $\overleftrightarrow{AZ'}$ meets $\overleftrightarrow{BC}$ at some point L' (on the same side of $\overleftrightarrow{AD}$ as point B). Consider a point L on $\overleftrightarrow{BC}$ on the opposite side of $\overleftrightarrow{AD}$ as point B with $DL \simeq DL'$. Use triangle $\triangle ADL$ and the definition of a boundary ray to reach a contradiction.

Exercise 2.2. Let Y' be a point between rays $\overrightarrow{AK}$ and $\overrightarrow{AD}$. In a step-by-step argument, show that $\overleftrightarrow{AY'}$ is not parallel to $\overleftrightarrow{BC}$. Be sure to cite a valid reason for each step.

Strategy of Proof: Construct angle $\angle DAY \simeq \angle DAY'$, where Y is on the same side of $\overleftrightarrow{AD}$ as point C. Must line $\overleftrightarrow{AY}$ meet $\overleftrightarrow{BC}$? If so, consider the resulting triangle and a congruent triangle on the opposite side of $\overleftrightarrow{AD}$.

3 The Angle of Parallelism

Recall Lobachevsky's construction of a limiting parallel. Given line $\overleftrightarrow{BC}$ and a point A not on $\overleftrightarrow{BC}$, let $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$, where D is a point on $\overleftrightarrow{BC}$. Let $\overleftrightarrow{AE} \perp \overleftrightarrow{AD}$, and let $\overleftrightarrow{AH}$ be the limiting parallel to $\overleftrightarrow{BC}$. Lobachevsky uses the notation $\Pi(p) = \angle DAH$ for the angle of parallelism, where $p = \overline{AD}$, suggesting that the angle of parallelism is a function of the length p, which we wish to prove in this mini-project. Consider another line $\overleftrightarrow{RS}$ and a point T not on $\overleftrightarrow{RS}$. Suppose that $\overleftrightarrow{TU} \perp \overleftrightarrow{RS}$, where U is a point on $\overleftrightarrow{RS}$. Let $\overleftrightarrow{TU} \perp \overleftrightarrow{TV}$, and let $\overleftrightarrow{TW}$ be the limiting parallel to $\overleftrightarrow{RS}$. We wish to prove that if $TU \simeq AD$, then $\angle UTW \simeq \angle DAH$. If $\angle UTW \not\cong \angle DAH$, then either (i) $(\angle UTW) < (\angle DAH)$ or (ii) $(\angle UTW) > (\angle DAH)$.

Exercise 3.1. Assuming that $(\angle UTW) < (\angle DAH)$, show how a contradiction can be reached.

Exercise 3.2. Assuming that $(\angle UTW) > (\angle DAH)$, show how a contradiction can be reached.

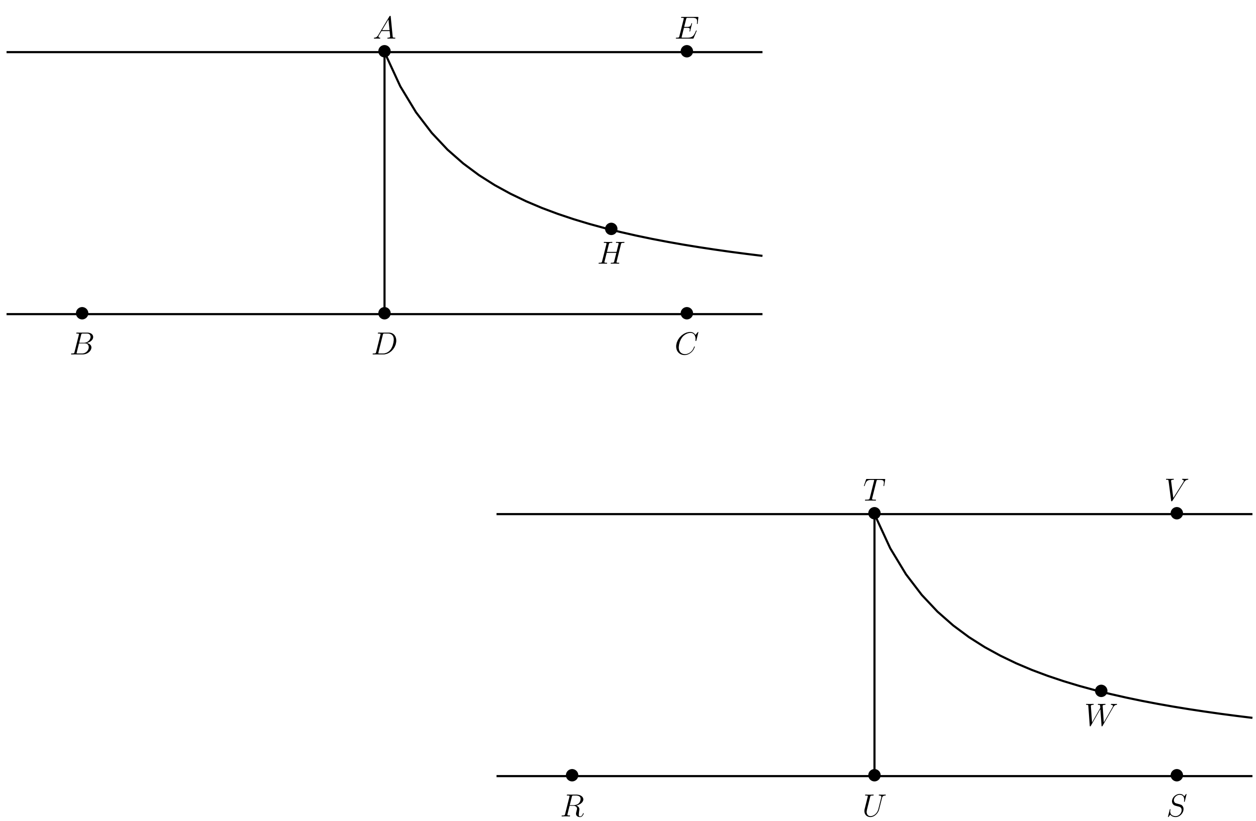

Figure 3: The Angle of Parallelism.

4 The Angle Sum of a Triangle in Hyperbolic Geometry

Purpose: In Euclidean geometry every triangle has angle sum $180°$, a result that relies on the Euclidean parallel postulate. In this mini-project we investigate the angle sum of a triangle in hyperbolic geometry, using the Hyperbolic Parallel Postulate, which states that given a line $\overleftrightarrow{BC}$ and a point A not on $\overleftrightarrow{BC}$, then there is more than one line through A parallel to $\overleftrightarrow{BC}$. We begin with a result from Lobachevsky's *Geometrical Researches on the Theory of Parallels* [3] that demonstrates the construction of an arbitrarily small angle. Using this result, we can construct a triangle between two limiting parallels that has angle sum less than $180°$.

Prerequisite Material: The reader may use the results of neutral geometry, which include Postulates 1–4, Propositions 1–28, and all Common Notions from Book I of Euclid's *Elements* [9]. Additionally, the reader may use the results of §16 from Lobachevsky's *Geometrical Researches on the Theory of Parallels* [3], where it is shown that the angle of parallelism is strictly between $0°$ and $90°$ in hyperbolic geometry. Also, it may be used, without proof, that in neutral geometry the angle sum of a triangle is less than or equal to $180°$.

Lobachevsky writes [3]:

21. *From a given point we can always draw a straight line that shall make with a given straight line an angle as small as we choose.*

Let fall from the given point A upon the given line BC the perpendicular AB; take upon BC at random the point D; draw the line AD; make $DE = AD$, and draw AE.

In the right-angled triangle ABD let the angle $ADB = \alpha$; then must in the isosceles triangle ADE the angle AED be either $\frac{1}{2}\alpha$ or less ... . Continuing thus we finally attain to such an angle AEB, as is less than any given angle. Fig. (4)

Exercise 4.1. In a step-by-step argument, verify the following. Given a line $\mathcal{L}$ and a point A not on $\mathcal{L}$, then there is a point Z on $\mathcal{L}$ with $\angle AZB$ less than any given angle, where B is the point on $\mathcal{L}$ with $\overleftrightarrow{AB}$ perpendicular to $\mathcal{L}$. Be sure to offer a reason why each step holds.

Question 4.1. Does the result in Exercise (4.1) rely on the Euclidean parallel postulate? on the hyperbolic parallel postulate? Why or why not?

Exercise 4.2. In a step-by-step argument prove that in hyperbolic geometry there is a triangle with angle sum less than $180°$. Be sure to offer a reason why each step holds.

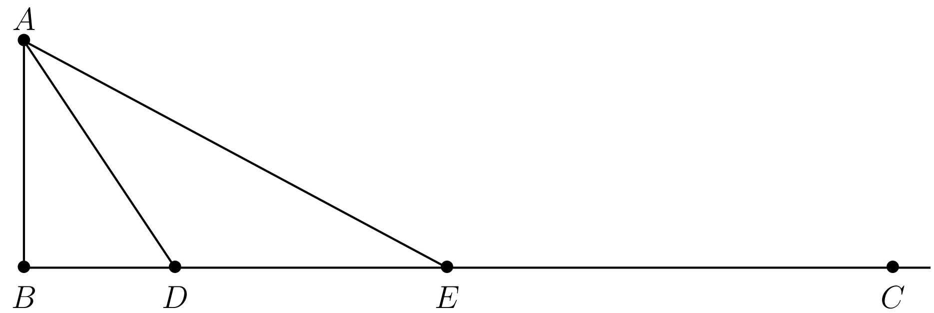

Figure 4: Constructing a Small Angle.

Strategy of Proof: From Lobachevsky's §16, consider a line $\overleftrightarrow{BC}$ and a point A not on $\overleftrightarrow{BC}$. Let $\overleftrightarrow{AH}$ be the limiting parallel (boundary line) through A parallel to $\overleftrightarrow{BC}$. Let $\angle DAH$ be the angle of parallelism, where D is a point on $\overleftrightarrow{BC}$ with $\overleftrightarrow{AD}$ perpendicular to $\overleftrightarrow{BC}$. Let δ be the measure of $\angle DAH$. What can be said about δ in comparison to $90°$? Is there some point Z on line $\overleftrightarrow{BC}$ so that $\angle AZD$ is small enough to guarantee that $\triangle AZD$ has angle sum less than $180°$? How? See the figure below. Be sure to offer a reason why each step holds.

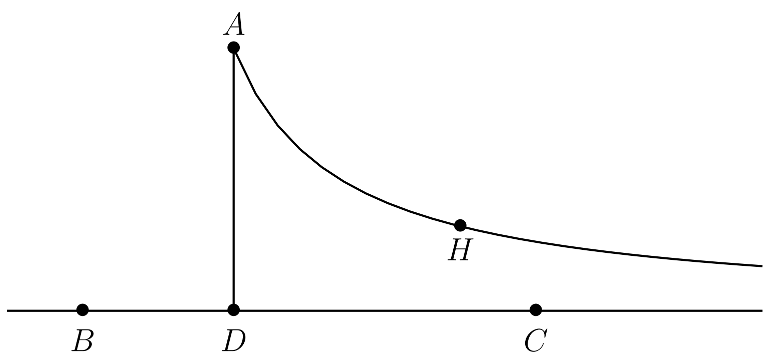

Figure 5: Constructing a Triangle, Exercise (4.2)

It follows from Lobachevsky's work [3] that if one triangle in hyperbolic geometry has angle sum less than $180°$, then every triangle in hyperbolic geometry has angle sum less than $180°$, which the reader may use, without proof, for Exercise (4.3).

Exercise 4.3. In a step-by-step argument, prove that in hyperbolic geometry rectangles do not exist. Be sure to offer a reason why each step holds.

Strategy of Proof: Using an indirect proof, suppose that quadrilateral $ABCD$ is a rectangle. Then sides AB and DC are contained along parallel lines, and sides AD and BC are contained along parallel lines. Furthermore, each of the four angles (at the the four vertices) of quadrilateral $ABCD$ is a right angle. Construct a diagonal such as segment BD. If triangles $\triangle ABD$ and $\triangle BCD$ each have angle sum less than $180°$, can quadrilateral $ABCD$ have four right angles? What is your conclusion about the existence of a rectangle?

Figure 6: A Rectangle, Exercise (4.3).

Epilogue: Since rectangles do not exist in hyperbolic geometry, it follows at once that squares do not exist in this geometry as well, since a square is a special case of a rectangle. Thus, one of the basic constructions in Euclidean geometry and one of the basic steps in the Pythagorean Theorem, the construction of a square, can not be implemented in hyperbolic geometry.

5 The Distance Between Limiting Parallels

Purpose: Given a line $\overleftrightarrow{BC}$ and a point A not on $\overleftrightarrow{BC}$, let ray $\overrightarrow{AH}$ be the limiting parallel through A to $\overleftrightarrow{BC}$. Then $\overrightarrow{AH}$ is Lobachevsky's "boundary line" between lines through A parallel to $\overleftrightarrow{BC}$ and lines through A not parallel to $\overleftrightarrow{BC}$. Let D be a point on $\overleftrightarrow{BC}$ with $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$. Then $\angle DAH$ is Lobachevsky's angle of parallelism and ray $\overrightarrow{AH}$ points in the direct of parallelism. The goal of this mini-project is to show that ray $\overrightarrow{AH}$ approaches line $\overleftrightarrow{BC}$, i.e., the perpendicular distance between $\overrightarrow{AH}$ and $\overleftrightarrow{BC}$ becomes less as points move along $\overrightarrow{AH}$ in the direction of parallelism.

Prerequisite Material: The reader may use, without proof, that in hyperbolic geometry the summit angles of a Saccheri quadrilateral are congruent acute angles. This means that given a quadrilateral $ABCD$ with $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$, $\overleftrightarrow{DC} \perp \overleftrightarrow{BC}$ and $AB \simeq DC$, then $\angle BAD \simeq \angle CDA$. Moreover, both $\angle BAD$ and $\angle CDA$ are acute angles.

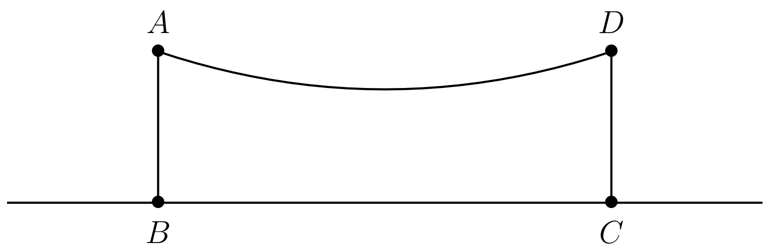

Figure 7: A Saccheri Quadrilateral.

The reader may also use the existence of limiting parallels and that the angle of parallelism is strictly between 0° and 90°. Also, from §17 of Lobachevsky's *Geometrical Researches on the Theory of Parallels* [3] if $\overrightarrow{AH}$ is the limiting parallel to $\overleftrightarrow{BC}$ and $\overrightarrow{HK}$ is the limiting parallel to $\overleftrightarrow{BC}$, then lines $\overleftrightarrow{AH}$ and $\overleftrightarrow{HK}$ coincide. The reader may also use the results of neutral geometry, such as the exterior angle theorem for triangles.

Lobachevsky writes [3]:

24. *The further [limiting] parallel lines are prolonged on the side of their parallelism, the more they approach one another.*

Exercise 5.1. Given quadrilateral $ABCD$ in hyperbolic geometry with $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ and $\overleftrightarrow{DC} \perp \overleftrightarrow{BC}$, in a step-by-step argument show that (i) if $\overline{AB} > \overline{DC}$, then $(\angle BAD) < (\angle CDA)$ and (ii) if $(\angle BAD) < (\angle CDA)$, then $\overline{AB} > \overline{DC}$. Be sure to offer a reason why each step holds.

Strategy of Proof: For (i), construct segment BE along BA so that $BE \simeq CD$. What can be said about quadrilateral $EBCD$? How is $\angle BED$ related to $\triangle AED$? Apply the exterior angle theorem. For (ii), try an indirect proof, and reach a contradiction to result (i).

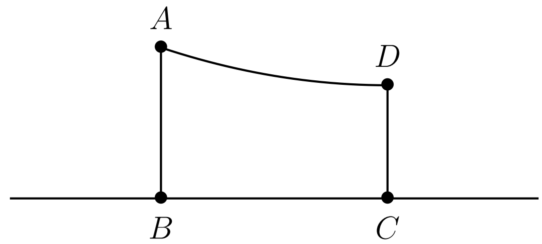

Figure 8: Quadrilateral For Exercise (5.1)

Exercise 5.2. Given a line $\overleftrightarrow{BC}$ and a point A not on $\overleftrightarrow{BC}$, let ray $\overrightarrow{AH}$ be the limiting parallel to $\overleftrightarrow{BC}$. Let $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$, where D is a point on $\overleftrightarrow{BC}$. Let $\overleftrightarrow{HL} \perp \overleftrightarrow{BC}$, where L is a point on $\overleftrightarrow{BC}$. In a step-by-step argument, show that $\overline{AD} > \overline{HL}$. Be sure to offer a reason why each step holds.

Strategy of Proof: Note that if $\overrightarrow{AH}$ is the limiting parallel to $\overleftrightarrow{BC}$ and $\overrightarrow{HK}$ is the limiting parallel to $\overleftrightarrow{BC}$, then line $\overleftrightarrow{AH}$ coincides with line $\overleftrightarrow{HK}$, which does not require proof. What can be said about $\angle DAH$ in comparison to 90°? What can be said about $\angle LHK$ in comparison to 90°? How does $\angle DAH$ compare to $\angle LHA$? Now, apply Exercise 1 above.

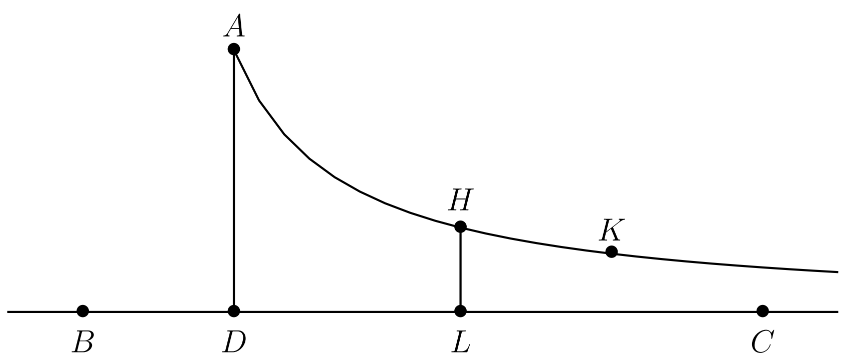

Figure 9: Quadrilateral For Exercise (5.2)

Extra Credit: While the above shows that the distance between two limiting parallels becomes smaller the further the parallels are prolonged, more is true, namely the distance between them becomes arbitrarily small. Prove Lobachevsky's statement: "Not only does the distance between two parallels decrease (Theorem 24), but with the prolongation of the parallels toward the side of parallelism this at last wholly vanishes. [Limiting] parallel lines have, therefore, the character of asymptotes" [3].

6 Distance in Hyperbolic Geometry

Purpose: The goal of this mini-project is to offer a first glimpse into computing distance in hyperbolic geometry based on Felix Klein's (1849–1925) paper "Über die sogenannte Nicht-Euklidische Geometrie," ("On the so-called Non-Euclidean Geometry") published in 1871 [10, 18]. Klein's point of view is to establish a scale on a line in hyperbolic geometry against which distance could be measured, and builds on major contributions of Carl Friedrich Gauss (1777–1855), Bernhard Riemann (1826–1866) and Eugenio Beltrami (1835–1900).

Although Gauss was aware of many results in non-Euclidean geometry, to avoid controversy, he published almost nothing on this subject directly [3, 7, 12]. Instead, Gauss developed the concept of the curvature of a surface, which would serve as the basis for the type of geometry (Euclidean or not) that holds on the surface. We have seen in §3 that the angle sum of a triangle in hyperbolic geometry is less than 180°, while the angle sum of a triangle in Euclidean geometry is 180°. Gauss's idea of curvature is based on comparing the area A_1 of a small triangle on a surface to the area A_2 of a corresponding triangle on a sphere of radius one. The limiting value of the quotient A_2/A_1 as the triangle on the surface approaches a point is the Gaussian curvature at that point. Given a surface in three-space, Gauss develops a formula for curvature involving the first and second derivatives of the surface, which could yield a positive value, a negative value or zero at any given point of the surface [4, 5]. With this construction, the xy-plane has curvature zero (at every point) and is a model for Euclidean geometry, while the unit sphere has curvature $+1$ (at every point) and is, of course, a model for spherical geometry. For more details about Gaussian curvature, see [11]. Gauss proves that if a distance-preserving map between two surfaces exists, then the two surfaces have the same curvature at corresponding points, which is an important relation between distance and curvature. One of Riemann's key ideas [17], however, is that a surface can exist without a prescribed method for determining distance. Distance can be assigned at will (up to a few conditions) and the same surface is amenable to several possible systems of distance measurement, a philosophical point of view adopted by Klein. In an 1868 publication [1] Beltrami realizes that a surface of constant negative curvature is (locally) a model for hyperbolic geometry. Moreover, such a surface can be formed by revolving a tractrix about its axis, forming what today is called a pseudo-sphere (a false sphere), although the boundary of the pseudo-sphere is not a "boundary" in hyperbolic geometry. Once distance on the tractrix or pseudo-sphere is understood, then by mapping (part of) this surface to a second surface establishes a method for determining hyperbolic distance on the second surface. In this mini-project we will determine arc length (distance) on a tractrix and then map the tractrix to a line segment, establishing a hyperbolic scale on the line segment, interpreted in the words of Klein as distance in hyperbolic geometry.

Prerequisite Material: The tractrix, a curve well-known before Gauss, was described in a question posed by Claude Perrault (1613–1688) in 1676 to Willhelm Gottfried Leibniz (1646–1716) [16]. Perrault placed his watch in the middle of a table, pulled the watch chain along the edge of the table, and asked what is the shape of the curve formed by the watch. The resulting curve is called a tractrix. For the purposes of the project, suppose that the watch is initially placed at the point (1, 0) in the xy-plane and that the edge of the table is

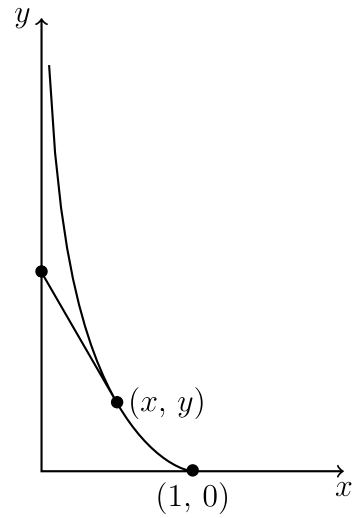

Figure 10: The Tractrix and its Tangent.

given by the y-axis. If the length of the chain is one unit and we begin pulling the free end of chain upward from (0, 0), the slope of the resulting curve is

$$\frac{dy}{dx} = \frac{-\sqrt{1-x^2}}{x}$$

and the equation of the tractrix itself is given by

$$T(x) = -\sqrt{1-x^2} + \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1.$$

To obtain a surface of constant Gaussian curvature -1, revolve the tractrix (in three space) around the y-axis, forming a surface called a pseudo-sphere. The tractrix itself is a ray in the geometry of the pseudo-sphere, and its arc length is a stepping stone to understanding distance in hyperbolic geometry. Again, the arc length of a tractrix had been know for some time with Christiaan Huygens (1629–1695) writing that the rectification (arc length) of the tractrix is given by the quadrature (area) of a hyperbola [8, pp. 408–409].

Let's investigate some of these claims in preliminary exercises before reading the original source by Klein.

Exercise 6.1. Using a right triangle of hypotenuse 1 (the length of the watch chain), show that the slope of the tractrix in Figure (10) is given by

$$\frac{dy}{dx} = \frac{-\sqrt{1-x^2}}{x}, \quad 0 < x \leq 1.$$

Exercise 6.2. Show that a solution to the differential equation $\frac{dy}{dx} = \frac{-\sqrt{1-x^2}}{x}$, $y = 0$ when $x = 1$, is given by

$$y = T(x) = -\sqrt{1-x^2} + \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1.$$

Exercise 6.3. Recall that the arc length of a differentiable curve $y = f(x)$ in the xy-plane between $(a,\, f(a))$ and $(b,\, f(b))$, $a \leq b$, is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Use this to show that the arc length of the tractrix between $(x_0,\, y_0)$ and $(1,\, 0)$, $0 < x_0 \leq 1$, is

$$s(x_0) = -\ln(x_0) = \ln(1/x_0).$$

Exercise 6.4. Find values of x_n for which the arc length of the tractrix satisfies:

(a) $s(x_0) = 0$.

(b) $s(x_1) = 1$.

(c) $s(x_2) = 2$.

(d) $s(x_3) = 3$.

(e) $s(x_n) = n$, where n is a positive integer.

Consider now points $(x_n,\, y_n)$ along the tractrix with arc length n units from $(1,\, 0)$. What would result if the points $(x_n,\, y_n)$ were projected vertically onto the x-axis? What scale of measure along the segment $0 < x \leq 1$ reflects arc length of the tractrix? We now read a brief excerpt from Klein's "On the so-called Non-Euclidean Geometry" [10]. A point in Klein's geometry is denoted by z and he uses multiplication by a constant (a linear transformation) to move z to another point a fixed (unit) distance from z. Klein writes:

[T]he linear transformation which initiates the construction of the scale is given by an equation of the form

$$z' = \lambda z,$$

where λ is a constant determining the transformation. If we now apply this transformation repeatedly to an arbitrary element $z = z_1$, then we obtain the series

$$z_1, \quad \lambda z_1, \quad \lambda^2 z_1, \quad \lambda^3 z_1, \quad \ldots\, ,$$

and this series of elements is our scale. This series is mapped into itself by the generating transformation, as is clear *a priori*.

If we now designate the *scale interval as the unit of displacement*, then the distances of the elements z_1, λz_1, $\lambda^2 z_1$, $\lambda^3 z_1$, ... from the element z_1 are 0, 1, 2, 3, ... respectively.

To be able to measure the distances of other elements from z_1, we now turn to subdividing the scale interval, say into n (equal) parts. This is achieved by subjecting the end element of an interval to a transformation whose nth iterate is the transformation $z' = \lambda z$, i.e., the transformation

$$z' = \lambda^{\frac{1}{n}} z.$$

The nth root here must be chosen so that the element $\lambda^{\frac{1}{n}} z$ lies between the elements z and λz.

When this subdivision is carried out, one can define the distance from z_1 of all points with coordinates of the form

$$z = \lambda^{\alpha + \frac{\beta}{n}} z_1,$$

where α and β are integers. This distance is simply the exponent $\alpha + \frac{\beta}{n}$.

Now by allowing the subdivision of the scale to proceed without limit, it is clear that the distance from z_1 of any element z whatever should be regarded as the exponent α to which λ must be raised so that $\lambda^\alpha z_1 = z$. Here α is any rational or irrational number.

Since obviously $\alpha = \log \frac{z}{z_1} : \log \lambda$, we can also express this as follows:

The distance between an element z and the element z_1 is the logarithm of the quotient $\frac{z}{z_1}$, divided by the constant $\log \lambda$.

The element z_1 here is only chosen by chance as the origin of the scale, and not otherwise specified. One may move it arbitrarily . . . . We therefore have:

The distance between arbitrary elements z and z' is equal to

$$\log \frac{z}{z'} : \log \lambda,$$

as one may verify by taking the difference between the distances of the two elements z and z', namely $\log \frac{z}{z_1} : \log \lambda$ and $\log \frac{z'}{z_1} : \log \lambda$.

Instead of the constant $\frac{1}{\log \lambda}$ we shall now write c for short, ...

Exercise 6.5. Combining Klein's notation above with the notation of Exercise (6.4), we set

$$x_0 = z_1, \quad x_1 = \lambda z_1, \quad x_2 = \lambda^2 z_1, \quad x_3 = \lambda^3 z_1, \quad \ldots \ .$$

What is the numerical value of z_1 in this interpretation (first review Exercise (6.4))? What is the value of λ in this interpretation? With the values of z_1 and λ above and the value of x_2 from Exercise (6.4), is $x_2 = \lambda^2 z_1$? Explain your work.

Exercise 6.6. Given that $\lambda^\alpha z_1 = z$, show that

$$\alpha = \frac{\log(z/z_1)}{\log \lambda}.$$

Exercise 6.7. Let $D(z, z_1)$ denote the hyperbolic distance between z and z_1, so that

$$D(z, z_1) = c \log(z/z_1)$$

for some constant c. If points z_1, z' and z are collinear, with z' between z_1 and z, show that

$$D(z, z') = D(z, z_1) - D(z', z_1).$$

Exercise 6.8. Show that $D(z, z) = 0$.

Exercise 6.9. Verify what Klein calls "the additivity of measure." If z, z' and z'' are collinear with z' between z and z'', then

$$D(z, z'') = D(z, z') + D(z', z'').$$

Epilogue: The above exercises show how Klein's writing can be interpreted as a scale for hyperbolic distance when a tractrix is projected vertically unto the line segment $0 < x \leq 1$. To understand how Klein constructs a distance formula between two arbitrary points in hyperbolic geometry requires a more detailed study of his paper [10]. First note that when we set $x_0 = z_1$ in Exercise (6.4), x_0 is given the Euclidean distance of the point x_0 from the origin, O, although in hyperbolic geometry O is infinitely far from x_0. Also note that the tractrix is only a ray in hyperbolic geometry, since it has an endpoint, (1, 0) in our example. Given two points A and B in hyperbolic geometry, we must first consider the line $\mathcal{L}$ determined by A and B. Given a model for hyperbolic geometry in a bounded (Euclidean) region, the line $\mathcal{L}$ meets the boundary of the region in two points, which Klein calls 0 and ∞. The (finite) Euclidean distances of A and B from 0 to ∞ are used to determine numerical values for z and z' that appear in the formula $c \log(z/z')$ to determine the hyperbolic distance from A to B.

The LaTeX code of this entire Primary Source Project (PSP) is available from the author by request (`jlodder@nmsu.edu`). The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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