Analysis

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Henri Lebesgue and the Development of the Integral Concept

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In an important text published in 1854, the celebrated German mathematician Bernhard Riemann (1826–1866) presented the approach to integration that is still known by his name today. In fact, Riemann devoted only a small portion (5–6 pages) of his text to the question of how to define the integral. Over two decades later, the French mathematician Gaston Darboux (1842–1917), an admirer of Riemann's ideas, provided the rigorous reformulation of the Riemann integral which is learned in most undergraduate-level real analysis courses in his publication *Mémoire sur les fonctions discontinues* (*Memoir on discontinuous functions*) [Darboux, 1875]. Using the precise definitions in the reformulation, Darboux also provided rigorous proofs of the fundamental properties of Riemann integrable functions, including the following:

- Every continuous function is integrable.
- If f is integrable, then the function $F(x) = \int_a^x f(y)dy$ is continuous in x.
- If f is continuous at x_0, then the function $F(x) = \int_a^x f(y)dy$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

Despite these useful properties, Riemann's version of integration was not perfect. Just over twenty-five years later, the French mathematician Henri Lebesgue (1875–1941) formulated a new integral concept with the goal of addressing certain weaknesses of Riemann's version. Lebesgue began his work on integration immediately after he finished his undergraduate work at the age of 22 and completed his doctoral dissertation, *Intégrale, Longueur, Aire* (*Integral, Length, Area*) [Lebesgue, 1902], just five years later. In this project, we will examine excerpts from a later paper, "Sur le développement de la notion d'intégrale" ("On the development of the integral concept") [Lebesgue, 1927], in which Lebesgue used somewhat less technical terms to describe the essential idea of what is now called the *Lebesgue integral.* Our primary goals in studying this particular paper will be to gain insight into the Riemann integral and its relative strengths and weaknesses, and to examine how the underlying idea of the Lebesgue integral differs from that of the Riemann integral.

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1 Shortcomings of the Riemann Integral: A First Glimpse

We begin with an excerpt from the introduction to Lebesgue's doctoral thesis.[1]

> It is known that there are non-integrable derivative functions, when one adopts . . . the definition of integral that was given by Riemann; so that integration, as defined by Riemann, does not in all cases solve the fundamental problem of integral calculus:
>
> Find a function with a given derivative.
>
> It may therefore seem natural to look for another definition of the integral, so that, for a wider range of cases, integration is the inverse operation of derivation.

Notice that the problem of finding a function with a given derivative can be rephrased as follows: given a function f, can we find an antiderivative (also called a primitive function) F such that $F' = f$? Task 1 gives a reminder about why the Riemann integral *does* solve this problem for a certain special class of functions.

Task 1 Recall that the following theorem holds for the Riemann integral (as was first rigorously proven by Darboux):

If f is continuous at x_0, then $F(x) = \int_a^x f(y)dy$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

Explain how this solves the problem of finding a function with a given derivative when the given derivative is a continuous function.

Taking Task 1 into account, we see that every continuous function is indeed antidifferentiable. Thus, a function that is Riemann integrable but not antidifferentiable (i.e., not itself a derivative) must necessarily be discontinuous. Although the construction of a discontinuous function that is Riemann-integrable but not antidifferentiable is beyond the scope of this project, Task 2 gives us a glimpse into a related difficulty with the Riemann integral.

[1] All translations of Lebesgue excerpts in this project were prepared by the project author.

$\boxed{\textbf{Task 2}}$ Consider the sequence of functions (f_n) where $f_n : [0,1] \to \mathbb{R}$ is defined for each $n \in \mathbb{Z}^+$ by[2]

$$f_n(x) = \begin{cases} 1 & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \end{cases},$$

where the sets A_n are defined by

$$A_n = \{p/q : p, q \in \mathbb{Z}^+ \wedge \gcd(p,q) = 1 \wedge q \leq n\} \cup \{0\}.^3$$

(a) Use theorems about Riemann integrals to explain why each of the individual functions f_n is Riemann integrable on $[0,1]$. (Feel free to use a modern textbook as needed to remind yourself about these theorems.)

(b) What is the value of the individual Riemann integrals $\int_0^1 f_n(x)dx$? Explain.

(c) Given $x \in [0,1]$, explain why $\lim_{n\to\infty} f_n(x) = f(x)$, where f is the Dirichlet function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

(In other words, show that (f_n) converges pointwise to f.)

(d) Use the definition of the Riemann integral to explain why f is *not* Riemann integrable on $[0,1]$.

(e) Finally, explain why the following equation *fails* to hold when Riemann integration is used:

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

2 The History of the Integral Concept According to Lebesgue

We now turn to our reading of Lebesgue's 1927 paper on the development of the integral concept, which was based on a talk that he delivered at a meeting of the Danish Mathematical Society in Copenhagen. Lebesgue began his talk with a discussion of the prehistory of his notion of integration.

[2] Alternatively, we could accomplish this same result by using the fact that the set of rational numbers $\mathbb{Q}$ is countable to enumerate the elements of $\mathbb{Q} \cap [0,1]$ as $\{x_k : k \in \mathbb{Z}^+\}$, and then defining a different sequence of functions $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \begin{cases} 1 & \text{if } x \in \{x_1, x_2, \ldots, x_n\} \\ 0 & \text{otherwise} \end{cases}$.

[3] For example, $A_6 = \{0, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, 5/6\}$.

Gentlemen,

Leaving aside all technicalities, we will examine together the successive modifications, the enrichments of the notion of integral, and how other notions used in recent research on the functions of real variables have arisen. Before Cauchy, there was no definition of the integral, in the current meaning of the word "definition." One simply said which areas had to be added or subtracted to obtain the integral $\int_a^b f(x)dx$.

For Cauchy, a definition was necessary; for, it was with him that the concern for rigor which is the characteristic of modern mathematics appeared. Cauchy defined continuous functions and the integrals of such functions pretty much as we do now. To arrive at the integral of $f(x)$, it was for him sufficient to form the sums

$$S = \sum f(\xi_i)(x_{i+1} - x_i), \tag{1}$$

which surveyors and mathematicians of all time periods have used for the approximate calculation of areas, and to then deduce $\int_a^b f(x)dx$ from these sums by a passage to the limit.

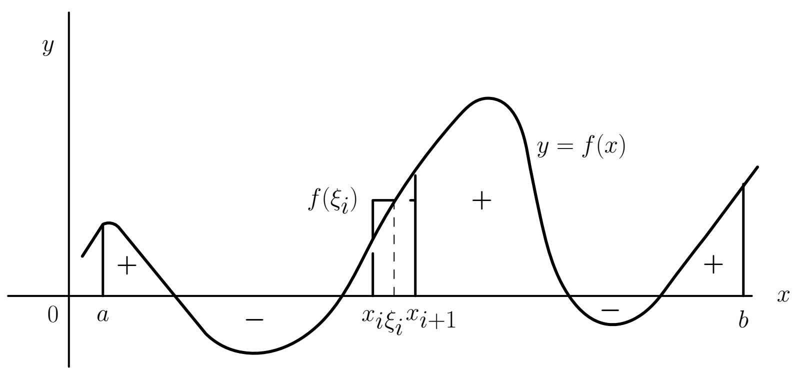

Figure 1

Except, while the legitimacy of such a passage was obvious for those who started from the notion of area, Cauchy had to demonstrate that the sum S did indeed approach a limit under the conditions that he was considering. An analogous necessity is imposed any time that the use of an experiential notion is replaced by a purely logical definition. It should be added that the inherent interest of the defined object is then no longer obvious, it can only be recovered through the study of the properties of this object [that are implied by the definition].

This is the price of logical progress. That which Cauchy achieved is so considerable that it has a sort of philosophical magnitude. It is often said that Descartes reduced geometry to algebra; I would say more willingly that, by the use of coordinates, he reduced all geometries to that of the line and that the latter, by giving us the notions of continuity and irrational number, has allowed algebra to reach its current scope.

For the reduction of geometries to that of the line to be completed, however, it remained to eliminate a certain number of notions relating to multi-dimensional geometries, such as length of a curve, area of a surface, volume of a body. This is precisely the progress that Cauchy achieved. After him, in order to carry out the full arithmetization of analysis, mathematicians had only to construct the linear continuum from the integers.

And, now, should we confine ourselves to doing pure analysis? No. Certainly, everything we do can be translated into the arithmetic language, but if we gave up having direct, geometric, intuitive views, if we were reduced to pure logic that does not allow us to choose between all that are correct, we would hardly be able to think of many questions, and certain notions — most of the ones we are going to examine today, for example — would escape us completely.

Task 3 According to Lebesgue's description of the early history of the integral in the previous excerpt:

(a) How was the integral defined before Cauchy?

(b) What was Cauchy's motivation for providing a definition of the integral? Do you agree with Cauchy that this was an important reason to give a definition?

(c) What new difficulties arose because of Cauchy's new approach to defining the integral? Identify at least two such difficulties. Of these, which do you think is the greater obstacle for someone who might try to learn about integration starting with Cauchy's definition of the integral, and why?

(d) What progress did Cauchy's approach make possible? Be specific! Do you agree with Lebesgue that this was progress? Why or why not?

Task 4 In the last paragraph of the preceding excerpt, Lebesgue discussed the question: "And, now, should we confine ourselves to doing pure analysis?" What did Lebesgue seem to mean by this question, and how did he answer it? To answer these questions, it will also be useful to look back at the two paragraphs immediately preceding the last paragraph of this excerpt (starting with "This is the price of ..." and "For the reduction of geometries to be complete ...," respectively).

Let's return now to our reading of Lebesgue's discussion of the history of integration, which he continued by looking at Riemann's approach.

For a long time, certain discontinuous functions had been integrated. Cauchy's definition still applied to these integrals, so it was natural to seek, as Riemann did, the full scope of that definition.[4]

If $\underline{f_i}$ and $\overline{f_i}$ denote the lower and upper bounds of $f(x)$ in (x_i, x_{i+1}), then S lies between

$$\underline{S} = \sum \underline{f_i}(x_{i+1} - x_i) \quad \text{and} \quad \overline{S} = \sum \overline{f_i}(x_{i+1} - x_i).$$

Riemann showed that it suffices that

$$\overline{S} - \underline{S} = \sum (\overline{f_i} - \underline{f_i})(x_{i+1} - x_i)$$

approaches zero for a particular sequence of divisions of (a, b) into increasingly small intervals (x_i, x_{i+1}) so that Cauchy's definition can be used. Darboux added that under the usual passage to a limit, $\overline{S}$ and $\underline{S}$ always determine two numbers, $\overline{\int} f(x)dx$, $\underline{\int} f(x)dx$, which are generally different and equal only when the Cauchy-Riemann integral exists.

From a logical point of view, these are very natural definitions, aren't they? Yet, from a practical point of view, we can say that they serve no purpose. That of Riemann, in particular, has the disadvantage of applying only rarely and, in a way, by chance.

This is because, in fact, while it is quite obvious that partitioning (a, b) into smaller and smaller intervals (x_i, x_{i+1}) renders the differences $\overline{f} - \underline{f}$ smaller and smaller when $f(x)$ is continuous, and by virtue of this same continuity, it is clear that this refinement process will again make $\overline{S} - \underline{S}$ approach zero when there are only a few points of discontinuity, yet there is no reason to hope that this will be the case even for an everywhere discontinuous functions. Thus, in effect, taking smaller and smaller intervals (x_i, x_{i+1}), that is to say, values of $f(x)$ for increasingly close values of x, in no way guarantees that we have values of $f(x)$ whose differences become smaller.

Task 5 This task compares Lebesgue's discussion of the Riemann integral to the presentation given for this concept in a current undergraduate textbook in analysis. (You can choose any such textbook for completion of this task.)

(a) How do the definitions of $\underline{S}$ and $\overline{S}$ relate to the corresponding concepts in the definition of the Riemann integral in the textbook you have selected? Compare both the definition given in that text, and the notation used therein.

[4]The different approaches to integration taken by Cauchy and Riemann are examined in detail in the project *The Definite Integrals of Cauchy and Riemann* [Ruch, 2017].

Task 5 - continued

(b) Directly after defining $\underline{S}$ and $\overline{S}$, Lebesgue mentioned a result about Riemann integration. Find a statement of this result in your selected textbook. (Depending on the textbook, it may be either a theorem or an exercise.) Identify it both by the name (or theorem/exercise number) used in that textbook and by the page number on which it appears. How is the textbook's version the same as or different from that given by Lebesgue?

(c) Who did Lebesgue credit for being the first to recognize that $\underline{S}$ and $\overline{S}$ "always determine two numbers"? What other theorem(s) are attributed to this same individual in your selected textbook? [Give the name/theorem number, the page number and a full statement].

(d) In the final paragraph of this excerpt, Lebesgue commented that "this refinement process will again make $\overline{S} - \underline{S}$ approach zero when there are only a few points of discontinuity." Find an example in your selected textbook of a function f that has infinitely many discontinuities but for which the refinement process also makes $\overline{S} - \underline{S}$ approach zero. That is, find a function f that has infinitely many discontinuities but is still Riemann integrable. In what sense does this function have "only a few points of discontinuity?"

(e) Explain how the Dirichlet function defined in Task 2(c) illustrates Lebesgue's comment in the final sentence of this excerpt about why "there is no reason to hope that this will be the case even for an everywhere discontinuous functions."

3 Enter Lebesgue!

We now look at the initial discussion in Lebesgue's 1927 paper of the key idea behind his new approach to integration.

Let us therefore be guided by the goal to be achieved: to bring together, to group values of $f(x)$ whose differences are small. It is clear then that we should partition not (a, b), but the interval $(\underline{f}, \overline{f})$, bounded by the lower and upper bounds of $f(x)$ in (a, b). Let us do this using numbers y_i that are less than a distance of ϵ of each other; we are led, for example, to consider the values of $f(x)$ defined by

$$y_i < f(x) < y_{i+i}.$$

The corresponding values of x form a set E_i. In the example shown in Figure 2, this set E_i is made up of four intervals; with some continuous functions $f(x)$, it could be made up of infinitely many intervals; with an arbitrary function, it could be very complicated. But this matters little; it is this set E_i that plays a role analogous to that of the interval (x_i, x_{i+1}) in the [Cauchy-Riemann] definition of the integral of continuous functions, since it makes known to us the values of x which give values of $f(x)$ whose differences are small.

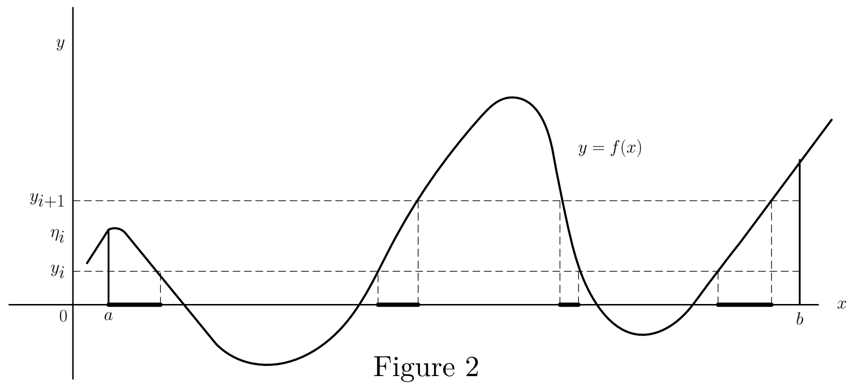

Figure 2

If η_i is any number between y_i and y_{+1}, the values of $f(x)$ for points lying in E_i differ from η_i by less than ϵ. η_i will play the role that $f(\xi_i)$ played in formula (1). As for the role of the length or measurement $x_{i+1} - x_i$ of the interval (x_i, x_{i+1}), it will be played by a measure $m(E_i)$ that we will assign to the set E_i in a moment. We will then form the sum

$$S = \sum \eta_i \, m(E_i). \tag{2}$$

But first, let's take a good look at what we have done and, to understand it better, repeat it in different terms.

Seventeenth-century geometers[5] considered the integral of $f(x)$ — the word "integral" was not yet invented, but that matters little — to be the sum of infinitely many indivisibles, each of which was the ordinate, positive or negative, of $f(x)$. Very well! We have simply grouped together the indivisibles of comparable size; as is said in algebra, we have collected similar terms. We could again say that, with Riemann's process, one was trying to sum the indivisibles by taking them in the order given by variations in x. One was thus operating like a non-methodical merchant who counts coins and bills at random in the order in which they come to hand, while we operate like a methodical merchant who says:[6]

I have $m(E_1)$ nickels worth $5 \cdot m(E_1)$,
I have $m(E_2)$ quarters worth $25 \cdot m(E_2)$,
I have $m(E_3)$ dollar bills worth $100 \cdot m(E_3)$, etc.

I thus have in all

$$S = 5 \cdot m(E_1) + 25 \cdot m(E_2) + 100 \cdot m(E_3) + \cdots .$$

The two procedures will, of course, lead the merchant to the same result because, no matter how rich he is, he has only a finite number of bills to count. But, for us, who have to add infinitely many indivisibles, the difference between the two ways of doing this is of great importance.

[5] In the seventeenth century, the word "geometer" referred to anyone who did mathematics (and not just someone who worked in geometry).

[6] Lebesgue himself used Danish krones (the official currency of the country in which he gave the talk) in this example.

Let's pause to consider what Lebesgue had done so far, before we continue our reading of [Lebesgue, 1927].

Task 6 (a) Note that Lebesgue has partitioned the range of the function $f(x)$, using sets of the form $\{y_0, y_1, \ldots, y_n\}$ with $y_i - y_{i-1} < \epsilon$ for each $i \in \{1, 2, \ldots, n\}$ and $\epsilon > 0$. How is this similar to what happens with the Riemann integral? How is it different?

(b) As you examine equation (2) in the previous excerpt, note that S is a number that depends on the values of η_i chosen to represent each set E_i. Also note that the sets E_i in turn depend on the partition $\{y_0, y_1, \ldots, y_n\}$ chosen. Thus, for a given function f on a given interval $[a, b]$, we get a large collection of numbers S (one for each possible partition and each choice of η_i), not just a single number S. How is this similar to what happens with the Riemann integral? How is it different? In particular, does the Riemann integral involve a similar collection of values?

Task 7 In terms of the money-counting analogy, how did Lebesgue describe the difference between the Riemann-Cauchy definition for integrals and Lebesgue's idea for defining this concept? How does this relate to the different types of partitioning that is involved in these two types of integral?

The next excerpt picks up where the last one left off, and includes a closer look at the general notion of the *measure of a set* that Lebesgue used to complete the definition of his integral. As you read this, keep in mind that he omitted some technical details from the paper we are reading. Accordingly, you should read for the general feel of what Lebesgue was doing, and not be too concerned about all the technical details.

Let us now take care of the definition of the number $m(E_i)$ attached to E_i. The analogy between this measure and a length, or even a number of banknotes, naturally leads us to say that in the example of Figure (2), $m(E_i)$ will be the sum of the lengths of the four intervals constituting E_i, and that, in an example where E_i would be formed of an infinity of intervals, $m(E_i)$ would be the sum of the lengths of all these intervals. In the general case, this analogy leads us to proceed as follows. We cover E_i by intervals, either finite or countably infinite in number, letting $l_1, l_2, \ldots$ be the lengths of these intervals. We obviously want to have:

$$m(E_i) \leq l_1 + l_2 + \cdots .$$

If we take the lower bound of the right-hand side of this inequality for all possible collections of intervals that cover E_i, that bound will therefore be an upper bound for $m(E_i)$. For this reason, we denote it as $\overline{m(E_i)}$ and we obviously have

$$m(E_i) \leq \overline{m(E_i)}. \tag{3}$$

If C is the set of points of (a, b) that do not belong to E_i, we similarly have

$$m(C) < \overline{m(C)}.$$

However, we obviously want to have

$$m(E_i) + m(C) = m[(a, b)] = b - a;$$

Therefore, we must have

$$m(E_i) \geq b - a - \overline{m(C)}. \tag{4}$$

The inequalities (3) and (4) therefore give us the lower and upper bounds for $m(E_i)$. It is easy to see that the two are never contradictory. When these lower and upper bounds are equal, $m(E_i)$ is defined and we say that E_i is measurable.

A function $f(x)$ for which the sets E_i are measurable for all choices of y_i is called measurable. For such a function, formula (2) defines a sum S. It is easy to show that, when the choice of y_i's are varied in such a way that ϵ approaches zero, then S approaches a determinate limit which is, by definition, $\int_a^b f(x)dx$.

Task 8 This task looks at the Lebesgue integral for the Dirichlet function.

Using the definition of set measure given by Lebesgue in the last excerpt, it can be shown that $m(A) = 0$ for any set A that is either finite or countably infinite.

(a) Use the measure facts given above to explain why $m(\mathbb{Q} \cap [0, 1]) = 0$ and $m((\mathbb{R} - \mathbb{Q}) \cap [0, 1]) = 1$.

(b) Use the measure facts stated in part (a) of this task to determine the value of the Lebesgue integral $\int_0^1 f(x)dx$ for the Dirichlet function (defined in Task 2). Explain your reasoning.

(c) Comment on how the value of the Lebesgue integral for the Dirichlet function differs from the situation with the Riemann integral for this same function.

(d) Which of these integrals (Lebesgue versus Riemann) do you feel captures the notion of "area" under the Dirichlet function more accurately, and why?

(e) Now look at the function sequence (f_n) defined in Task 2. Use the measure facts from part (a) of this task to determine the value of the Lebesgue integral $\int_0^1 f_n(x)dx$ for each $n \in \mathbb{Z}^+$.

(f) Recall (from Task 2) that the following equation does not hold when Riemann integration is used.

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

Does it hold when Lebesgue integration is used? Explain why or why not.

We end our reading of Lebesgue's 1927 paper with one final excerpt in which he discussed two variations of his basic idea for how to approach integration.

This first extension of the notion of a definite integral leads to many others. Assuming it is a question of integrating a function $f(x,y)$ of two variables, we will proceed exactly as before: we will assign to it sets E_i which will now be sets of points in the plane and no longer points on a line. To these sets, it will be necessary to attribute a surface measure; this measure can be deduced from the area of rectangles

$$\alpha \leq x \leq \beta; \qquad \gamma \leq y \leq \delta$$

in exactly the same manner that the linear measure is deduced from the length of intervals. That measure defined, formula (2) will give the sums S from which the integral will be deduced by passage to the limit.

The definition that we have considered thus immediately extends to functions of several variables; here is another which would apply equally regardless of the number of variables and which I explain only for the case where it is a question of integrating $f(x)$ on (a,b).

I have said that integration is a question of finding the sum of indivisibles represented by the various ordinates, $y = f(x)$, of points x; we have, a little while ago, grouped these indivisibles according to their sizes. Let us now confine ourselves to grouping them according to their signs; we will consider the surface set of points [in the plane] for which the ordinates are positive, E_p, and the set, E_n, of points with negative ordinates. As I recalled at the beginning, for the simple case where $f(x)$ is continuous, one put, before Cauchy,

$$\int_a^b f(x)\,dx = \text{area}(E_p) - \text{area}(E_n);$$

this leads us to now formulate

$$\int_a^b f(x)\,dx = m_s(E_p) - m_s(E_n),$$

m_s denoting a surface measure. This new definition is equivalent to the preceding one; it brings us back to the intuitive pre-Cauchy method, but the definition of measure gives it solid logical foundation.

Task 9 This task includes some closing questions about Lebesgue's approach to integration.

(a) At the very end of the final paragraph above, Lebesgue made the interesting assertion that his definition captures the pre-Cauchy intuitive idea about integrals, while placing this intuitive idea on a "solid logical foundation." Do you agree that his definition accomplishes these two goals? Why or why not?

(b) Lebesgue's primary reason for generalizing the Cauchy-Riemann definition was to handle certain kinds of functions that the earlier definition could not deal with. (He commented on this in several places in the excerpts provided in this project.) What types of functions could Lebesgue handle with his definition of an integral that the earlier definition could not?

4 Epilogue

What *classes* of functions are *integrable*? For example, are all derivatives integrable? Although these are now standard questions to consider in analysis, it would not have occurred to mathematicians prior to the late nineteenth century to ask them. As Lebesgue has explained, its answer also depends on the type of integration used. In the seventeenth and eighteenth centuries, the integral was just an antiderivative, so that all derivatives were integrable, but nothing else was. With the Riemann integral, some non-derivatives are integrable; for example, any function with a single jump discontinuity is easily seen to be Riemann integrable, but can not be a derivative since it fails to satisfy the Intermediate Value Property[7]. *(You should be able to prove both these facts about functions with a single jump discontinuity, using results from an undergraduate textbook on analysis!)*

On the other hand, some derivatives have too many discontinuities to be Riemann integrable. In fact, Lebesgue proved the following in his doctoral dissertation:

> **Lebesgue's Criterion of Riemann Integrabilty.** f is Riemann integrable if and only if the set D_f of all discontinuities of f has measure zero.

As noted earlier (in Task 8), all finite and countably infinite sets have measure 0 — but so do some uncountably infinite sets. This means that the cardinality of the set of discontinuities D_f is not important for Riemann integrability of f, since only the the *measure* of D_f matters. For instance, if $D_f = C$, where C is the Cantor set,[8] then f will be Riemann integrable, since $m(C) = 0$, even though C is uncountable!

[7]The fact that all derivatives have the Intermediate Value Property is today known as *Darboux's Theorem*. Darboux's original proof of that theorem is presented in the project *Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis*" [Barnett, 2016].

[8]The Cantor set C is typically constructed by starting with the unit interval $[0, 1]$, and removing its middle third, then removing the middle third of each of the two remaining sections, and so on ad infinitum; C is then the set of all points remaining in the end. C can also be described as the set of all real numbers with a ternary (or base-3) expansion that contains only the digits 0 and 2. This set is named after the famous German mathematician Georg Cantor (1845–1918), who mentioned it in an 1883 paper as an example of a set with certain special topological properties (e.g., perfect, but nowhere dense). It also appeared in an earlier 1874 paper on the integration of discontinuous functions, written by the less well-known Irish mathematician Henry John Stephen Smith (1826–1883). For more about Smith's work, see the project *The Cantor Set Before Cantor* [Scoville, 2016].

Returning now to the issue raised by Lebesgue in the very first excerpt in this project, there are also derivatives f' for which the set of discontinuities $D_{f'}$ is not of measure zero; thus, by Lebesgue's Criterion, such derivatives f' are *not* Riemann integrable. This means that the well-beloved Evaluation Version of the Fundamental Theorem of Calculus $\left[\int_a^b f' = f(b) - f(a)\right]$ might not hold, since $\int_a^b f'$ might not even exist!

As it turns out, not all derivatives are Lebesgue integrable either. However, the class of Lebesgue integrable functions is larger than the class of Riemann integrable functions, as the example of the Dirichlet function demonstrates. Importantly, if f is Riemann integrable, then f is also Lebesgue integrable, and both integrals will have the same value. For these and other reasons, the Lebesgue integral is the current standard in graduate courses and mathematical research — at least for the time being!

Task 10 This task includes some closing reflection questions about the concept of integration based on our work in this project.

(a) What questions or comments do you have about the excerpts we have read from Lebesgue that have not been addressed in the tasks in this project? Write at least one mathematical question and at least one mathematical comment.

(b) What questions or comments do you have about the concept of integration in general as a result of working this project? Write at least one mathematical question and at least one mathematical comment.

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H. Lebesgue. Sur le développement de la notion d'intégrale (On the development of the integral concept). *Revue de Métaphysique et de Morale*, 34(2):149–167, 1927.

D. Ruch. *The Definite Integrals of Cauchy and Riemann*. 2017. Available at `https://digitalcommons.ursinus.edu/triumphs_analysis/11`.

N. A. Scoville. *The Cantor Set Before Cantor*. 2016. Available at `https://digitalcommons.ursinus.edu/triumphs_topology/2`.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed for use in an introductory course in real analysis. It could also be used in a history of mathematics course as an example of an advanced twentieth-century topic, especially within a course focused on the development of calculus. The project's primary goal is to consolidate students' understanding of the Riemann integral, and its relative strengths and weaknesses. This is accomplished by contrasting the Riemann integral with the Lebesgue integral, as these were described by Lebesgue himself in a relatively non-technical paper published in 1927. A second mathematical goal of this PSP is to introduce the important concept of the Lebesgue integral, which is rarely discussed in an undergraduate course on real analysis. Additionally, by offering an overview of the evolution of the integral concept, students are exposed to the ways in which mathematicians hone various tools of their trade (e.g., definitions, theorems).

Student Prerequisites

It is assumed that students have studied the rigorous definition of the Riemann integral as it is presented in an undergraduate textbook on analysis. Additionally, familiarity with the Dirichlet function is useful (but not required) for Task 2 and Task 8. These two tasks also refer to pointwise convergence of function sequences, but no prior familiarity with function sequences is required.

PSP Design and Task Commentary

In support of its primary goal, three tasks in this PSP rely exclusively on the definition of and theorems about Riemann integration. These include Tasks 1 and 2 in Section 1, both of which are also essential to the comparison of the Riemann and Lebesgue integrals that takes place later in the project. Task 5 in Section 2, which asks students to compare certain comments made by Lebesgue about the Riemann integral with today's standard textbook treatment of that integral, further supports the goal of consolidating students' understanding of Riemann integration.

Because introducing students to the concept of the Lebesgue integral is only a secondary focus of this PSP, certain technical details related to Lebesgue integration are intentionally glossed over. This is especially the case with the discussion of the definition of measure in the excerpt that immediately precedes Task 8 in Section 3. Instructors who wish to study these ideas in more detail could develop additional tasks for students to consider, or discuss the definition of measure with students in a whole-class discussion. This would naturally require additional class time. Because Task 8 itself is essential to drawing the comparison of the Riemann and Lebesgue integrals that is set up in Task 2 of Section 1, the measure-related facts that are needed to complete it are simply provided to students without proof.

In addition to addressing certain aspects of the integration theory, this project also touches on issues related to the tensions between "logical rigor" and "geometrical intuition" as guiding principles in mathematics. In fact, Lebesgue explicitly described his new definition of the integral as an effort to reconcile these two desirable but sometimes conflicting aspects of mathematics. Tasks 3 and 4 in Section 2 prompt students to reflect on this theme. Task 4 in particular requires a careful reading of Lebesgue's commentary about the desirability of working purely within arithmetized analysis (i.e., the integral as a numerical limit of numerical sums) without reference to geometry (i.e., the integral as an area, volume, or length). Instructors who choose not to pursue this theme in great depth could omit that task altogether, or limit the amount of class time spent on its discussion. Those who do choose to assign Task 4 may wish to share some additional historical background with students about the motivations and concerns that led nineteenth-century mathematicians to pursue the "arithmetization of analysis." One source of information about this earlier history is the PSP *Why Be So Critical? Nineteenth-Century Mathematics and the Origins of Analysis*, available at `https://digitalcommons.ursinus.edu/triumphs_analysis/1/`.

Suggestions for Classroom Implementation

Classroom implementation of this PSP can be carried out in a number of different ways.

The author has often used this PSP as a culminating class project on Riemann integration by having students read the entire PSP and prepare written responses to the Tasks therein. This assignment is made about a week prior to its due date, during which time students are encouraged to discuss the reading and PSP tasks with each other or with the instructor outside of class (with the sole provision that their final written responses must be their own). While there is no prohibition against using additional resources to complete the PSP, it is important to assure students that there is no need to do any historical research in order to complete it. On the assignment due date, a whole class discussion (45–50 minutes) of the reading is conducted by the instructor, with student responses to various PSP tasks elicited during that discussion. (This discussion could also be conducted after the instructor has collected and read students' written PSP work.) Students' completed PSP write-ups are evaluated and assigned a score that is included in the computation of their course grade.

Alternatively, the majority of tasks in this PSP are well suited to completion by students in small groups during class time (supplemented by whole-class discussion at key points in the PSP to consolidate student understanding), while certain tasks work well as individual homework assigned after those discussions. To reap the full mathematical benefits offered by the reading of primary sources, students should be required in some way to read assigned sections in advance of any in-class work; advance preparation by students of (perhaps preliminary) responses to tasks that will be discussed during in-class work is also recommended.[9] Depending on the exact combination of individual/small-group/whole-class work, this method of implementation requires 2–3 class days (based on 50-minute class periods). A sample schedule that offers some options to help instructors tailor this mode of implementation to their course goals and available class time is outlined in the next subsection of these Notes.

Yet another implementation alternative that has been used with this PSP requires complete individual write-ups of all PSP tasks (evaluated as part of students' course grades) following four half-days of small-group and whole-class discussions spread out over the course of a month. In advance of each half-day of in-class work, students prepare draft responses to specific PSP

[9]The author's method of ensuring that advance reading takes place is to require student completion of "Reading Guides" (or "Entrance Tickets") for which students receive credit for completion, but with no penalty for errors in solutions. See the Appendix to these Notes for a sample guide based on this particular PSP and more detail about their general design.

tasks. They then revise their responses based on in-class discussions before submitting second-draft write-ups for instructor feedback, with final corrections of all PSP tasks due about a week after that instructor feedback is returned.

LaTeX code of this PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

For instructors who choose to implement this PSP via a combination of small-group and whole-class discussions, the following sample schedule offers several options to help instructors tailor that mode of implementation to their course goals and available class time. Depending on the exact combination of individual/small-group/whole-class work, 2–3 class days will be required.

- **Preparation for Day 1.** All instructors should have students read the project introduction, all of Section 1 and the first excerpt of Section 2; students should also complete Tasks 1–3 for class discussion.

  - Instructors pursuing the logical rigor/geometrical intuition theme (described in the "PSP Design and Task Commentary" section above) should also ask students to prepare preliminary notes and questions about Task 4.
  - Instructors not pursuing that theme should have students skip Task 4 altogether, and instead assign advance reading of all of Section 2 and completion of Task 5.

- **Day 1.** Small-group discussion of the following, supplemented by whole-class discussion as needed:

  - Section 1: Quick review of answers to advance preparation work on Task 1, and more detailed discussion of Task 2.
  - Section 2: Quick review of answers to advance preparation work on Task 3; parts (c)–(d) of that task are especially relevant to the theme of logical rigor/geometrical intuition.
    - If Task 4 was assigned for advance preparation and time permits, discussion of that task can also begin (but may need to continue to Day 2). This task is best suited for whole-class discussion, perhaps following some initial discussion in small groups.
    - If Task 5 was assigned for advance preparation and time permits, discussion of that task can begin. If time runs short for a full discussion, students' advance preparation write-ups can simply be collected and reviewed by the instructor prior to the next class period to determine whether a follow-up discussion on Day 2 would be helpful.

  Homework. A complete formal write-up of Tasks 1 and 2, to be due at a later date (e.g., one week after completion of the in-class work).

- **Preparation for Day 2.** All instructors should have students read Section 3 and complete Tasks 6, 7, and 8(a)–(d) for class discussion.

  - Instructors pursuing the logical rigor/geometrical intuition theme should also have students complete the reading of Section 2 and Task 5 in preparation for class discussion.
  - Instructors not pursuing that theme should instead assign advance reading of all of Section 3 and completion of Task 9 for class discussion (in addition to the tasks listed above). Those who wish to complete in-class implementation in just 2 days should also assign advance reading of Section 4 and completion of Task 10 for class discussion.

- **Day 2.** Begin with Section 2 follow-up:

  - Instructors pursuing the logical rigor/geometrical intuition theme may wish to continue or follow up on the discussion of Task 4 from Day 1. Small-group or whole-class discussion of Task 5 can also take place prior to moving to in-class work on Section 3; alteratively, students' advance preparation write-ups for that task can simply be collected and reviewed by the instructor prior to the next class period to determine whether a follow-up discussion on Day 3 would be helpful.
  - Instructors not pursuing that theme may wish to quickly follow up on Task 5, especially if there was limited time for discussion of that task on Day 1.

  Continue to Section 3, with

  - Whole-class discussion of Tasks 6 and 7; this should be relatively quick, but is important to ensuring students appreciate Lebesgue's approach before continuing to the later tasks in this section.
  - Small-group discussion (supplemented as desired by whole-class discussion) of the following:
    * Task 8(a)–(f). Note that part (d) is especially suited to whole-class discussion.
    * If Task 9 was assigned for advance preparation and time permits, this task can also be discussed. If time runs short for a full discussion, students' advance preparation write-ups can simply be collected and reviewed by the instructor prior to the next class period to determine whether a follow-up discussion on Day 3 would be helpful.

  If advance reading of Section 4 was assigned for Day 2, 10–20 minutes should be reserved for a closing whole-class discussion of the PSP with a focus on the commentary in Section 4. During this closing discussion, students could be asked to share their answers to Task 10; alternatively, students' advance preparation write-ups for that task can simply be collected and reviewed by the instructor prior to the next class period to determine if any final clarification of the ideas in the project seems necessary.

  Homework. A complete formal write-up of Task 8, to be due at a later date (e.g., one week after completion of the in-class work).

- **Preparation for Day 3 (if not following the 2-day plan).**
  All instructors should have students read Section 4 and complete Task 10.

  - Instructors pursuing the logical rigor/geometrical intuition theme should also have students complete the reading of Section 3 and complete Task 9 as advance preparation for class discussion.

- **Day 3 (10–50 minutes).** Begin with Section 3 follow-up:

  - Instructors pursuing the logical rigor/geometrical intuition theme may wish to have students quickly discuss their answers to Task 9 in small groups; alternatively, their answers to this task could be worked into a closing whole-class discussion of the PSP.
  - Instructors not pursuing that theme may wish to quickly follow up on Task 9, especially if there was limited time for discussion of that task on Day 2.

  Moving to Section 4, a closing whole-class discussion of the PSP, with a focus on the commentary in Section 4, could vary from 10–50 minutes, depending on instructor's goals and how students' work on the project has gone on Days 1–2. During this closing discussion, students could be asked to share their answers to Task 10; alternatively, students' advance preparation write-ups for that task can simply be collected and reviewed by the instructor prior to the next class period to determine if any final clarification of the ideas in the project seems necessary.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be completed in at most 2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from `https://digitalcommons.ursinus.edu/triumphs\_topology`; all other listed projects are available at `https://digitalcommons.ursinus.edu/triumphs\_analysis`.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Heine Barnett)
- *Investigations into Bolzano's Bounded Set Theorem* (David Ruch)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)
  Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)
  A second version of this project suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *Understanding Compactness: Early Work, Uniform Continuity to the Heine-Borel Theorem* (Naveen Somasunderam)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Mean Value Theorem*(David Ruch)

- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Euler's Rediscovery of e** (David Ruch; sequence convergence, series/sequence expressions for e)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

Recommendations for Further Reading

Instructors who wish to know more about the history of integration in the nineteenth and early twentieth centuries will find the article [Hochkirchen, 2003] of interest. See the reference list of the student portion of this PSP for bibliographic details.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.

APPENDIX

This appendix provides a 'Sample Reading Guide' that illustrates the author's method for assigning advance preparation work in connection with classroom implementation of primary source projects. As described in the subsection "Suggestions for Classroom Implementation" of the Notes to Instructors for this project, students receive credit for completion of these guides, but with no penalty for errors in solutions. Students are asked to strive to answer each question correctly, but to think of Reading Guides as preparatory work for class, not as a final product (e.g., formal polished write-ups are not expected). Students who arrive unprepared to discuss assignments on days when group work is conducted based on advance reading are not allowed to participate in those groups, but are allowed to complete the in-class work independently. Guides are collected at the end of each class period for instructor review and scoring prior to the next class period.

A typical guide (such as the one that follows) will include "Classroom Preparation" exercises (generally drawn from the PSP Tasks) for students to complete prior to arriving in class, as well as "Discussion Questions" that ask students only to read a given task and jot down some notes in preparation for class work. Students are also encouraged to record any questions or comments they have about the assigned reading on their guide and are sometimes explicitly prompted to write 1–3 questions or comments about a particular primary source excerpt; their responses to such prompts are especially useful as starting points for in-class discussions. On occasion, tasks are also assigned as follow-up to a prior class discussion.

Experience has proven the value of reproducing the full text of any assigned project task on the guide itself, with blank space for students' responses deliberately left below each question. This not only makes it easier for students to jot down their thoughts as they read, but also makes their notes more readily available to them during in-class discussions. It also makes it easier for the instructor to efficiently review each guide for completeness (or to skim responses during class for a quick assessment of students' understanding), and allows students to make more effective use of their Reading Guide responses and instructor feedback on them at a later date.

The primary goal of the reading and tasks assigned in this particular 4-page reading guide is to familiarize students with the historical and mathematical background of this project, and to prepare them for in-class small-group work on Tasks 1–4. The final question also sets up the possibility of beginning class discussion of Task 5, should time permit.

Day 1 Reading Guide: *Henri Lebesgue and the Development of the Integral Concept*
Reading Assignment: Pages 1–5

1. Read the Introduction.

   Questions or comments?

2. In Section 1, read the first excerpt from Lebesgue (top of page 2):

   Write at least one comment OR one question about this excerpt:

3. **Complete Task 1,** reproduced below for your convenience.

   Recall that the following theorem holds for the Riemann integral (as was first rigorously proven by Darboux):

   If f is continuous at x_0, then $F(x) = \int_a^x f(y)dy$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

   Explain how this solves the problem of finding a function with a given derivative in the case where the given derivative is a continuous function.

4. **Answer the following questions from Task 2,** reproduced below for your convenience. The footnotes to this task given in the project may also be helpful to look back at.

Consider the sequence of functions (f_n) where for each $n \in \mathbb{Z}^+$, $f_n : [0,1] \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \end{cases},$$

where the set A_n is defined by $A_n = \{\frac{p}{q} \,|\, p, q \in \mathbb{Z}^+ \wedge \gcd(p,q) = 1 \wedge q \leq n\} \cup \{0\}$.

(a) Use theorems about Riemann integrals to explain why each of the individual functions f_n is Riemann integrable on $[0,1]$. (Feel free to use a modern textbook as needed to remind yourself about these theorems.)

(b) What is the value of each of the individual Riemann integrals $\int_0^1 f_n(x)dx$? Explain.

(c) Given $x \in [0,1]$, explain why $\lim_{n\to\infty} f_n(x) = f(x)$, where f is the Dirichlet function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(d) Use the definition of the Riemann integral to explain why f is NOT Riemann integrable on $[0,1]$.

(e) Finally, explain why the following equation fails to hold when Riemann integration is used:

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

5. Continue your reading with the first Lebesgue excerpt in Section 2.

   Write at least one comment OR one question about this excerpt:

6. **Answer the following questions from Task 3,** reproduced below for your convenience.

   According to Lebesgue's description of the early history of the integral:

   (a) How was the integral defined before Cauchy?

   (b) What was Cauchy's motivation for providing a definition of the integral?
   Do you agree with Cauchy that this was an important reason to give a definition?

   (c) What new difficulties arose because of Cauchy's new approach to defining the integral? Identify at least two such difficulties. Of these, which do you think is the greater obstacle for someone who might try to learn about integration starting with Cauchy's definition of the integral, and why?

   (d) What progress did Cauchy's approach make possible? Be specific!

   Do you agree with Lebesgue that this was progress? Why or why not?

7. **Prepare some notes for discussion of Task 4,** reproduced below for your convenience. You will probably find it necessary to re-read the two paragraphs in question a few times.

   In the last paragraph of the excerpt just above Task 3, Lebesgue discussed the question

   > And, now, should we confine ourselves to doing pure analysis?

   What did Lebesgue seem to mean by this question, and how did he answer it?

   To answer these questions, it will also be useful to look back at the two paragraphs immediately preceding the last paragraph of this excerpt (starting with "This is the price of ..." and "For the reduction of geometries to be complete ...," respectively).

8. Do a preliminary reading of the next excerpt from Lebesgue's paper (just below Task 4).

   Questions or comments about this excerpt, or about the project so far?