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Henri Lebesgue and the Development of the Integral Concept

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In an important text published in 1853, the celebrated German mathematician Bernhard Riemann (1826–1866) presented the approach to integration that is still known by his name today. In fact, Riemann devoted only a small portion (5–6 pages) of his text to the question of how to define the integral. Over two decades later, the French mathematician Gaston Darboux (1842–1917), an admirer of Riemann's ideas, provided the rigorous reformulation of the Riemann integral which is learned in most undergraduate level analysis courses in his publication *Mémoire sur les fonctions discontinues* [Darboux, 1875]. Using the precise definitions in the reformulation, Darboux also provided rigorous proofs of the fundamental properties of Riemann integrable functions, including the following:

- Every continuous function is integrable.
- If f is integrable, then $F(x) = \int_a^x f(y)dy$ is continuous in x.
- If f is continuous at x_0, then $F(x) = \int_a^x f(y)dy$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

Despite possessing these useful properties, Riemann's version of integration was not perfect. Just over twenty five years later, the French mathematician Henri Lebesgue (1875–1941) formulated a new integral concept with the goal of addressing certain weaknesses of Riemann's version. Lebesgue began his work on integration immediately after he finished his undergraduate work at the age of 22 and completed his doctoral dissertation, *Intégrale, Longueur, Aire* (*Integral, Length, Area*) [Lebesgue, 1902], just five years later. In this project, we will examine excerpts from a later paper, "Sur le développement de la notion d'intégrale" ("On the development of the integral concept") [Lebesgue, 1927], in which Lebesgue used somewhat less technical terms to describe the essential idea of what is now called the 'Lebesgue integral.' Our primary goals in studying this particular paper will be to gain insight into the Riemann integral and its relative strengths and weaknesses, and to examine how the underlying idea of the Lebesgue integral differs from that of the Riemann integral.

1 A first glimpse at what goes wrong with the Riemann integral

We begin with an excerpt from the introduction to Lebesgue's doctoral thesis (as quoted in [Hochkirchen, 2003, pp. 271-272]):

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It is known that there are derivatives that are not integrable, if one accepts Riemann's definition of the integral; the kind of integration as defined by Riemann does not allow in all cases to solve the fundamental problem of calculus:

Find a function with a given derivative.

It thus seems natural to search for a definition of the integral which makes integration the inverse operation of differentiation in as large a range as possible.

Notice that the problem of finding a function with a given derivative can be rephrased as follows: given a function f, can we find an antiderivative (also called a 'primitive function') F such that $F' = f$? Task 1 gives a reminder about why the Riemann integral *does* solve this problem for a certain special class of functions.

Task 1 Recall that the following theorem holds for the Riemann integral (as was first rigorously proven by Darboux):

If f is continuous at x_0, then $F(x) = \int_a^x f(y)dy$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

Explain how this solves the problem of finding a function with a given derivative in the case where the given derivative is a continuous function.

Taking Task 1 into account, we see that every continuous function is indeed antidifferentiable. Thus, a function that is Riemann integrable but not antidifferentiable (i.e., not itself a derivative) must necessarily be discontinuous. Although the construction of a discontinuous function that is Riemann-integrable but not antidifferentiable is beyond the scope of this project, Task 2 gives us a glimpse into a related difficulty with the Riemann integral.

Task 2 Consider the sequence of functions (f_n) where for each $n \in \mathbb{Z}^+$, $f_n : [0,1] \to \mathbb{R}$ is defined by[1]

$$f_n(x) = \begin{cases} 1 & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \end{cases},$$

where the set A_n is defined by $A_n = \{\frac{p}{q} \,|\, p, q \in \mathbb{Z}^+ \wedge \gcd(p,q) = 1 \wedge q \leq n\} \cup \{0\}$.[2]

(a) Use theorems about Riemann integrals to explain why each of the individual functions f_n is Riemann integrable on $[0,1]$. (Feel free to use a modern textbook as needed to remind yourself about these theorems.)

[1] Alternatively, we could use the fact that the set of rational numbers $\mathbb{Q}$ is countable to enumerate the elements of $\mathbb{Q} \cap [0,1]$ as $\{x_k \,|\, k \in \mathbb{Z}^+\}$ and then define a different sequence of functions $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \begin{cases} 1 & \text{if } x \in \{x_1, x_2, \ldots, x_n\} \\ 0 & \text{otherwise} \end{cases}$.

[2] For example, $A_1 = \{0\}$, $A_4 = \{0, 1/2, 1/3, 2/3, 1/4, 3/4\}$, $A_6 = \{0, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, 5/6\}$.

(b) What is the value of each of the individual Riemann integrals $\int_0^1 f_n(x)dx$? Explain.

(c) Given $x \in [0,1]$, explain why $\lim_{n\to\infty} f_n(x) = f(x)$, where f is the Dirichlet function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

[In other words, show that (f_n) converges pointwise to f.]

(d) Use the definition of the Riemann integral to explain why f is NOT Riemann integrable on $[0,1]$.

(e) Finally, explain why the following equation fails to hold when Riemann integration is used:

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

2 The History of the Integral Concept according to Lebesgue

We now turn to our reading of Lebesgue's 1927 paper "Intégrale, longueur, aire" ("Integral, Length, Area"). Lebesgue began this paper with a discussion of the pre-history of his notion of integration.

Leaving aside all technicalities, we are going to examine the successive modifications and enrichments of the concept of the integral and the appearance of other notions used in recent research on functions of a real variable.

Before Cauchy there was no definition of the integral in the modern meaning of the word "definition". One merely said which areas had to be added or subtracted in order to obtain the integral $\int_a^b f(x)dx$.

For Cauchy a definition was necessary, because with him there appeared the concern for rigor which is characteristic of modern mathematics. Cauchy defined continuous functions and their integrals in about the same way as we do today. In order to arrive at the integral of $f(x)$ it suffices to form the sums (Fig. 132.1),

$$S = \sum f(\xi_i)(x_{i+1} - x_i), \tag{1}$$

which surveyors and mathematicians have always used to approximate area, and then deduce the integral $\int_a^b f(x)\,dx$ by passage to the limit.

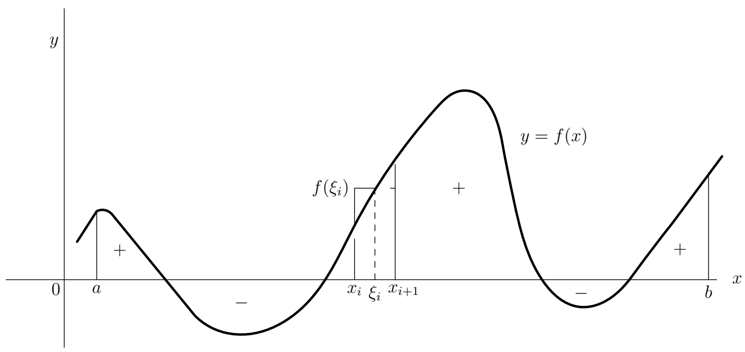

Figure 132.1

Although the legitimacy of such a passage to the limit was evident for one who thought in terms of area, Cauchy had to demonstrate that S actually tended to a limit in the conditions he considered. A similar necessity appears every time one replaces an experimental notion by a purely logical definition. One should add that the interest of the defined object is no longer obvious, it can be developed only from a study of the properties following from the definition. This is the price of logical progress.

What Cauchy did is so substantial that it has a kind of philosophic sweep. It is often said that Descartes reduced geometry to algebra. I would say more willingly that by the use of coordinates he reduced all geometries to that of the straight line, and that the straight line, in giving us the notions of continuity and irrational number, has permitted algebra to attain its present scope.

In order to achieve the reduction of all geometries to that of the straight line, it was necessary to eliminate a certain number of concepts related to geometries of several dimensions, such as the length of a curve, the area of a surface, and the volume of a body. The progress realized by Cauchy lies precisely here. After him, in order to complete the arithmetization of mathematics it was sufficient for the arithmeticians to construct the linear continuum from the natural numbers.

And now, should we limit ourselves to doing analysis? No. Certainly, everything that we do can be translated into arithmetical language, but if we renounce direct, geometrical, and intuitive views, if we are reduced to pure logic which does not permit a choice among things that are correct, then we would hardly think of many questions and certain concepts, for example, most of the ideas that we are going to examine here today, would escape us completely.

Task 3 According to Lebesgue's description of the early history of the integral in the previous excerpt:

(a) How was the integral defined before Cauchy?

(b) What was Cauchy's motivation for providing a definition of the integral?
Do you agree with Cauchy that this was an important reason to give a definition?

Task 3 – continued

(c) What new difficulties arose because of Cauchy's new approach to defining the integral? Identify at least two such difficulties. Of these, which do you think is the greater obstacle for someone who might try to learn about integration starting with Cauchy's definition of the integral, and why?

(d) What progress did Cauchy's approach make possible? Be specific!
Do you agree with Lebesgue that this was progress? Why or why not?

Task 4 In the last paragraph of the preceding excerpt, Lebesgue discussed the question

> And now, should we limit ourselves to doing analysis?'

What did Lebesgue seem to mean by this question, and how did he answer it?
To answer these questions, it will also be useful to look back at the two paragraphs immediately preceding last paragraph of the preceding excerpt(starting with " 'What Cauchy did was so substantial that …" and "In order to achieve the reduction " respectively).

Let's return now to our reading of Lebesgue's discussion of the history of integration, which he continued by looking at Riemann's approach.

For a long time certain discontinuous functions have been integrated. Cauchy's definition still applies to these integrals, but it is natural to examine, as did Riemann, the exact capacity of this definition.

If $\underline{f_i}$ and $\overline{f_i}$ represent the upper and lower bounds of $f(x)$ in (x_i, x_{i+1}), then S lies between

$$\underline{S} = \sum \underline{f_i}(x_{i+1} - x_i) \quad \text{and} \quad \overline{S} = \sum \overline{f_i}(x_{i+1} - x_i).$$

Riemann showed that for the definition of Cauchy to apply it is sufficient that

$$\overline{S} - \underline{S} = \sum (\overline{f} - \underline{f})(x_{i+1} - x_i)$$

tends toward zero for a particular sequence of partitions of the interval from a to b into smaller and smaller subdivisions (x_i, x_{i+1}). Darboux added that under the usual operation of passage to the limit $\underline{S}$ and $\overline{S}$ always give two definite numbers

$$\underline{\int_a^b} f(x)\,dx \quad \text{and} \quad \overline{\int_a^b} f(x)\,dx.$$

From a logical point of view, these are very natural definitions, aren't they? However, one can say that from a practical point of view they have been useless. In particular, Riemann's definition has the drawback of applying only rarely and in a sense by chance.

It is evident that breaking up the interval (a, b) into smaller and smaller subintervals (x_i, x_{i+1}) makes the differences $\overline{f_i} - \underline{f_i}$ smaller and smaller if $f(x)$ is continuous, and that the continued refinement of the subdivision will make $\overline{S} - \underline{S}$ tend toward zero if there are only a few points of discontinuity. But we have no reason to hope that the same thing will happen for a function that is discontinuous everywhere. To take smaller intervals (x_i, x_{i+1}), that is to say values of $f(x)$ corresponding to values of x closer together, does not in any way guarantee that one takes values of $f(x)$ whose differences become smaller.

Task 5 This task compares Lebesgue's discussion of the Riemann integral to the presentation given for this concept in a current undergraduate textbook in analysis. (You can choose any such textbook for completion of this task.)

(a) How do the definitions of $\underline{S}$ and $\overline{S}$ relate to the corresponding concepts in the definition of the Riemann integral in the textbook you have selected? Compare both the definition given in that text, and the notation used therein.

(b) Directly after defining $\underline{S}$ and $\overline{S}$, Lebesgue mentioned a result about Riemann integration. Find a statement of this result in your selected textbook. (Depending on the textbook, it may be either a theorem or an exercise.) Identify it both by the name (or theorem/exercise number) used in that textbook and by page number on which it appears. How is the textbook's version the same/different from that given by Lebesgue?

(c) Who did Lebesgue credit for being the first to recognize that $\underline{S}$ and $\overline{S}$ "always give two definite numbers"? What other theorem(s) are attributed to this same individual in your selected textbook? [Give the name/theorem number, page number and a full statement].

(d) In the final pargraph of this excerpt, Lebesgue commented that "the continued refinement of the subdivision will make $\overline{S} - \underline{S}$ tend toward zero if there are only a few points of discontinuity." Find an example in your selected textbook of a function f that has infinitely many discontinuities but for which "the continued refinement of the subdivision will make $\overline{S} - \underline{S}$ tend toward zero." That is, find a function f that has infinitely many discontinuities but is still Riemann integrable. In what sense does this function have "only a few points of discontinuity?"

(e) Explain how the Dirichlet function in Task 2(c) illustrates Lebesgue's comment in the final sentence of this excerpt about why "we have no reason to hope that the same thing will happen for a function that is discontinuous everywhere."

3 Enter Lebesgue!

We now look at the initial discussion in Lebesgue's 1927 paper of the key idea behind his new approach to integration

Let us be guided by the goal to be attained—to collect approximately equal values of $f(x)$. It is clear then that we must break up not (a, b), but the interval $(\underline{f}, \overline{f})$ bounded by the lower and upper bounds of $f(x)$ in (a, b). Let us do this with the aid of numbers y_i differing among themselves by less than ϵ. We are led to consider the values of $f(x)$ defined by

$$y_i \leq f(x) \leq y_{i+1}.$$

The corresponding values of x form a set E_i. In Figure 132.2 this set E_i consists of four intervals. With some continuous functions it might consist of an infinity of intervals. For an arbitrary function it might be very complicated. But this matters little. It is this set E_i which plays the role analogous to the interval (x_i, x_{i+1}) in the usual definition of the integral of continuous functions, since it tells us the values of x which give to $f(x)$ approximately equal values.

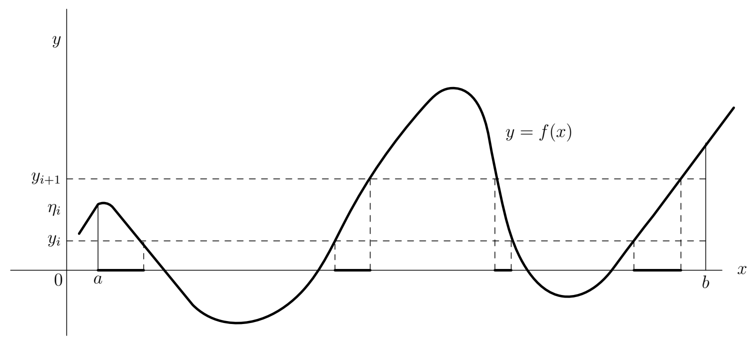

Figure 132.2

If η_i is any number whatever taken between y_i and y_{i+1}, $y_i \leq \eta_i \leq y_{i+1}$, the values of $f(x)$ for points of E_i differ from η_i by less than ϵ. The number η_i is going to play the role which $f(\xi_i)$ played in formula (1)[3]. As to the role of the length or measure $x_{i+1} - x_i$ of the interval (x_i, x_{i+1}), it will be played by a measure $m(E_i)$ which we shall assign to the set E_i in a moment. In this way we form the sum

$$S = \sum \eta_i \, m(E_i). \tag{2}$$

Let us look closely at what we have just done and, in order to understand it better, repeat it in other terms.

The geometers of the seventeenth century considered the integral of $f(x)$—the word "integral" had not been invented, but that does not matter—as the sum of an infinity of indivisibles, each of which was the ordinate, positive or negative, of $f(x)$. Very well! We have simply grouped together the indivisibles of comparable size. We have, as one says in algebra, collected similar terms. One could say that, according to Riemann's procedure, one tried to add the indivisibles by taking them in the order in which they were furnished by the

[3]Lebesgue's formula (1) is stated in the earlier excerpt describing Cauchy's view of integration, on page 3 of this project.

variation in x, like an unsystematic merchant who counts coins and bills at random in the order in which they came to hand, while we operate like a methodical merchant who says:

I have $m(E_1)$ pennies which are worth $1 \cdot m(E_1)$,
I have $m(E_2)$ nickels worth $5 \cdot m(E_2)$,
I have $m(E_3)$ dimes worth $10 \cdot m(E_3)$, etc.

Altogether then I have

$$S = 1 \cdot m(E_1) + 5 \cdot m(E_2) + 10 \cdot m(E_3) + \cdots$$

The two procedures will certainly lead the merchant to the same result because no matter how much money he has there is only a finite number of coins or bills to count. But for us who must add an infinite number of indivisibles the difference between the two methods is of capital importance.

We now consider the definition of the number $m(E_i)$ attached to E_i. The analogy of this measure to length, or even to a number of coins, leads us naturally to say that, in the example of Fig. 132.2, $m(E_i)$ will be the sum of the lengths of the four intervals that make up E_i, and that, in an example where E_i is formed from an infinity of intervals, $m(E_i)$ will be the sum of the length of all these intervals. ...

Let's pause to consider what Lebesgue had done so far, before we continue our reading of [Lebesgue, 1927].

Task 6 (a) Note that Lebesgue partitioned the range of the function, using sets $\{y_0, y_1, \ldots, y_n\}$ for which $y_i - y_{i-1} < \epsilon$ for each $i \in \{1, 2, \ldots, n\}$ and $\epsilon > 0$.

How is this similar to what happens with the Riemann integral? How is it different?

(b) As you examine equation (2) in the previous excerpt, note that S is a number that depends on the values of η_i chosen to 'represent' each set E_i. Also note that the sets E_i in turn depend on the partition $\{y_0, y_1, \ldots, y_n\}$ chosen. Thus, for a given function f on a given interval $[a, b]$, we get a large collection of numbers S (one for each possible partition and each choice of η_i), not just a single number S.

How is this similar to what happens with the Riemann integral? How is it different? In particular, does the Riemann integral involve a similar collection of values?

(c) In terms of the 'money-counting' analogy, how did Lebesgue describe the difference between the Riemann-Cauchy definition for integrals and Lebesgue's idea for defining this concept? How does this relate to the different types of partitioning that is involved in these two types of integral?

The next excerpt picks up where the last one left off, and includes a closer look at the general notion of the 'measure of a set' that Lebesgue used to complete the definition of his integral. As you read this, keep in mind that he omitted some technical details from the paper we are reading. Accordingly, you should read for the general feel of what Lebesgue was doing, and not be too concerned about all the technical details.

. . . In the general case it [i.e., the analogy of this measure to length, or even to a number of coins] leads us to proceed as follows. Enclose E_i in a finite or denumerably infinite number of intervals, and let $l_1, l_2, \ldots$ be the length of these intervals. We obviously wish to have

$$m(E_i) \leq l_1 + l_2 + \cdots .$$

If we look for the greatest lower bound of the second member[4] for all possible systems of intervals that cover E_i, this bound will be an upper bound of $m(E_i)$. For this reason we represent it by $\overline{m(E_i)}$, and we have

$$m(E_i) \leq \overline{m(E_i)}. \tag{3}$$

If C is the set of points of the interval (a, b) that do not belong to E_i, we have similarly

$$m(C) \leq \overline{m(C)}.$$

Now we certainly wish to have

$$m(E_i) + M(C) = m[(a, b)] = b - a,$$

and hence we must have
$$m(E_i) \geq b - a - \overline{m(C)}. \tag{4}$$

The inequalities (3) and (4) give us upper and lower bounds for $m(E_i)$. One can easily see that these two inequalities are never contradictory. When the lower and upper bounds for E_i are equal, $m(E_i)$ is defined, and we say then that E_i is measurable.

A function $f(x)$ for which the sets E_i are measurable for all choices of y_i is called measurable. For such a function formula (2) defines a sum S. It is easy to prove that when the y_i vary so that ϵ tends toward zero, the S tend toward a definite limit which is, by definition,$\int_a^b f(x)\,dx$.

Task 7 This task looks at the Lebesgue integral for the Dirichlet function.

Using the definition of 'measure of a set' given by Lebesgue in the last excerpt, it can be shown that $m(A) = 0$ for any set A that is either finite or countably infinite.

(a) Use the measure facts given above to explain why $m(\mathbb{Q} \cap [0, 1]) = 0$ and $m(\mathbb{I} \cap [0, 1]) = 1$.

(b) Use the measure facts stated in part (a) of this task to determine the value of the Lebesgue integral $\int_0^1 f(x)dx$ for the Dirichlet function (defined in Task 2). Explain your reasoning.

(c) Comment on how the value of the Lebesgue integral for the Dirichlet function differs from the situation with the Riemann integral for this same function.

(d) Which of these integrals (Lebesgue vs. Riemann) do you feel captures the notion of 'area' under the Dirichlet function more 'accurately', and why?

[4]The phrase second member' here refers to what we would call the right-hand side of the inequality.

Task 7 – continued

(e) Now look at the function sequence (f_n) defined in Task 2. Use the measure facts from part (a) of this task to determine the value of the Lebesgue integral $\int_0^1 f_n(x)dx$ for each $n \in \mathbb{Z}^+$.

(f) Recall (from Task 2) that the following equation does not hold when Riemann integration is used. Does it hold when Lebesgue integration is used? Explain why or why not.

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

We end our reading of Lebesgue's 1927 paper with one final excerpt in which he discussed two extensions of his basic idea for how to approach integration.

This first extension of the notion of the definite integral led to many others. Let us suppose that it is a question of integrating a function $f(x, y)$ of two variables. Proceeding exactly as before, we construct sets E_i which are now sets of points in the plane and no longer on a line. To these sets we must now attribute a plane measure, and this measure is deduced from the area of rectangles

$$\alpha \leq x \leq \beta; \qquad \gamma \leq y \leq \delta$$

in exactly the same way as the linear measure was derived from the length of intervals. Once measure is defined, formula (2) gives the sums S from which the integral is obtained by passage to the limit. Hence the definition that we have considered extends immediately to functions of several variables.

Here is another extension which applies equally well, regardless of the number of variables, but which I explain only in the case where it is a question of integrating $f(x)$ in the interval (a, b). I have said that it is a question of summing indivisibles represented by the various ordinates at points $x, y = f(x)$. A moment ago, we collected these indivisibles according to their sizes. Now let us merely group them according to their signs. We will have to consider then the set E_p of points in the plane whose ordinates are positive, and the set E_n of points whose ordinates are negative. As I recalled at the beginning of my lecture, for the simple case where $f(x)$ is continuous, even before Cauchy's time one wrote

$$\int_a^b f(x)\,dx = \text{area}(E_p) - \text{area}(E_n).$$

This leads us to assert

$$\int_a^b f(x)\,dx = m_s(E_p) - m_s(E_n),$$

where m_s stands for a plane measure. This new definition is equivalent to the preceding one. It brings us back to the intuitive method before Cauchy, but the definition of measure puts it on a solid logical foundation.

Task 8 This task includes some closing questions about Lebesgue's approach to integration.

(a) At the very end of final paragraph, Lebesgue made the interesting assertion that his definition captures the pre-Cauchy intuitive idea about integrals, while placing this intuitive idea on a 'solid logical foundation'. Do you agree that his definition (the systematic merchant idea) accomplishes these two goals? Why or why not?

(b) Lebesgue's primary reason for generalizing the Cauchy-Riemann definition was to handle certain kinds of functions that the earlier definition could not deal with. (He commented on this in several places in the excerpts provided in this project.)

What types of functions could Lebesgue 'handle' with his definition of an integral that the earlier definition could not?

4 Epilogue

What *classes* of functions are *integrable*? For example, are all derivatives integrable? Although these are now standard questions to consider in analysis, it would not have occurred to mathematicians prior to the late 19th century to ask them. As Lebesgue has explained, its answer also depends on the type of integration used. In the 17th and 18th centuries, the integral was just an antiderivative, so that all derivatives were integrable, but nothing else was. With the Riemann integral, some non-derivatives are integrable; for example, any function with a single jump discontinuity is easily seen to be Riemann integrable, but can not be a derivative since it fails to satisfy the Intermediate Value Property. *(You should be able to prove both these facts about functions with a single jump discontinuity, using results from an undergraduate textbook on analysis!)*

On the other hand, some derivatives have too many discontinuities to be Riemann integrable. In fact, Lebesgue proved the following in his doctoral dissertation:

Lebesgue's Criterion of Riemann Integrabilty
f is Riemann integrable iff the set D_f of all discontinuities of f has measure zero.

As noted earlier (in Task 7), all finite and countably infinite sets have measure 0 — but so do *some* uncountably infinite sets. This means that the cardinality of the set of discontinuities D_f is not important for Riemann integrability of f, since only the the *measure* of D_f matters. For instance, if $D_f = C$, where C is the Cantor set,[5] then f will be Riemann integrable, since $m(C) = 0$, even though C is uncountable!

Returning now to the issue raised by Lebesgue in the very first excerpt in this project, there are also DERIVATIVES f' for which the set of discontinuities $D_{f'}$ is not of measure zero; thus, by

[5]The Cantor set C is typically constructed by starting with the unit interval $[0, 1]$, and removing its middle third, then removing the middle third of each of the two remaining sections, then removing the middle third of the remaining four sections, and so on ad infinitum; C is then the set of all points remaining in the end. (More formally, C is the intersection of the sequence of nested sets defined by the 'remove middle thirds' process described above.) The Cantor set can also be described as the set of all real numbers with a ternary (or base-3) expansion that contains only the digits 0 and 2. This set is named after the famous German mathematician and set theorist Georg Cantor (1845–1918), who mentioned it in an 1883 paper [Cantor, 1883] as an example of a more general type of set with certain topological properties (e.g., perfect, but nowhere dense). The Cantor set also appeared in an earlier paper [Smith, 1874] concerning the integration of discontinuous functions, written by the less well-known Irish mathematician Henry John Stephen Smith (1826–1883). For more about Smith's work, see the primary source project *The Cantor Set Before Cantor*, written by Nicholas A. Scoville and available at `https://digitalcommons.ursinus.edu/triumphs_topology/2/`.

Lebesgue's Criterion, such derivatives f' are NOT Riemann integrable! This means that the well-beloved Evaluation Version of the Fundamental Theorem of Calculus $\left[\int_a^b f' = f(b) - f(a)\right]$ might not hold, since $\int_a^b f'$ might not even exist!

As it turns out, not all derivatives are Lebesgue integrable either. However, the class of Lebesgue integrable functions is larger than the class of Riemann integrable functions, as the example of the Dirichlet function demonstrates. Importantly, if f is Riemann integrable, then f is also Lebesgue integrable, and both integrals will have the same value. For these and other reasons, the Lebesgue integral is the current standard in graduate courses and mathematical research — at least for the time being!

Task 9 This task includes some closing reflection questions about the concept of integration based on our work in this project.

(a) What questions or comments do you have about the excerpts we have read from Lebesgue that have not been addressed in the tasks in this project? Write at least one MATHEMATICAL question and at least one MATHEMATICAL comment.

(b) What questions or comments do you have about the concept of integration in general as a result of working this project? Write at least one MATHEMATICAL question and at least one MATHEMATICAL comment.

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H. Lebesgue. Sur le développement de la notion d'intégrale (On the development of the integral concept). *Revue de Métaphysique et de Morale*, 34(2):149–167, 1927. English translation by Kenneth O. May in R. Calinger, editor, *Classics of Mathematics*, pages 762–765, Prentice-Hall, New Jersey, 1995.

H. J. S. Smith. On the Integration of Discontinuous Functions. *Proceedings of the London Mathematical Society*, 6(1):140–153, 1874.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed for use in an Introductory Analysis course. It could also be used in a History of Mathematics course as an example of an advanced 20th-century topic, especially within a course focused on the development of calculus. The project's primary goal is to consolidate students' understanding of the Riemann integral, and its relative strengths and weaknesses. This is accomplished by contrasting the Riemann integral with the Lebesgue integral, as described by Lebesgue himself in a relatively non-technical paper published in 1927. A second mathematical goal of this PSP is to introduce the important concept of the Lebesgue integral, which is rarely discussed in an undergraduate course on analysis. Additionally, by offering an overview of the evolution of the integral concept, students are exposed to the ways in which mathematicians hone various tools of their trade (e.g., definitions, theorems).

Student Prerequisites

It is assumed that students have studied the rigorous definition of the Riemann integral as it is presented in an undergraduate textbook on analysis. Additionally, familiarity with the Dirichlet function is useful (but not required) for Task 2 and Task 7. These two tasks also refer to pointwise convergence of function sequences, but no prior familiarity with function sequences is required.

PSP Design and Task Commentary

In support of its primary goal, three tasks in this PSP rely exclusively on the definition of and theorems about Riemann integration. These include Tasks 1 and 2 in Section 1, both of which are also essential to the comparison of the Riemann and Lebesgue integrals that takes place later in the project. Task 5 in Section 2, which asks students to compare certain comments made by Lebesgue about the Riemann integral with today's standard textbook treatment of that integral, further supports the goal of consolidating students' understanding of Riemann integration.

Because introducing students to the concept of the Lebesgue integral is only a secondary focus of this PSP, certain technical details related to Lebesgue integration are intentionally glossed over. This is especially the case with the discussion of the definition of measure in the excerpt that immediately precedes Task 7 in Section 3. Instructors who wish to study these ideas in more detail could develop additional tasks for students to consider, or discuss the definition of measure with students in a whole class discussion. This would naturally require additional class time. Because Task 7 itself is essential to drawing the comparison of the Riemann and Lebesgue integrals that is set up in Task 2 of Section 1, the measure-related facts that are needed to complete it are simply provided to students without proof.

In addition to addressing certain technical aspects of the integral, this project also touches on issues related to the tensions between "logical rigor" and "geometrical intuition" as guiding principles in mathematics. In fact, Lebesgue explicitly described his new definition of the integral as an effort to reconcile these two desirable but conflicting aspects of mathematics. Tasks 3 and 4 in Section 2 prompt students to reflect on this theme. Task 4 in particular requires a careful reading of Lebesgue's commentary about the desirability of working purely within arithmetized analysis (i.e., the integral as a numerical limit of numerical sums) without reference to geometry (i.e., the integral as an area, volume, or length). Instructors who choose not to pursue this theme in great depth could omit that task altogether, or limit the amount of class time spent on its discussion. Those who do choose to assign Task 4 may wish to share some additional historical background with students about the motivations and concerns that led nineteenth century mathematicians to pursue the 'arithmetization of analysis.' One source of information about this earlier history is the author's primary source-based project *Why be so Critical? Nineteenth Century Mathematical and the Origins of Analysis*, available at `https://digitalcommons.ursinus.edu/triumphs_analysis/1/`.

Suggestions for Classroom Implementation

Classroom implementation of this PSP can be carried out in a number of different ways.

The author has often used this PSP as a culminating class project on Riemann integration by having students read the entire PSP and prepare written responses to the Tasks therein. This assignment is made about a week prior to its due date, during which time students are encouraged to discuss the reading and PSP tasks with each other or with the instructor outside of class (with the sole provision that their final written responses must be their own). While there is no prohibition against using additional resources to complete the PSP, it is important to assure students that there is no need to do any historical research in order to complete it. On the assignment due date, a whole class discussion (45–50 minutes) of the reading is conducted by the instructor, with student responses to various PSP tasks elicited during that discussion. (This discussion could also be conducted after the instructor has collected and read students' written PSP work.) Students' completed PSP write-ups are evaluated and assigned a score that is included in the computation of their course grade.

Alternatively, the majority of tasks in this PSP are well suited to completion by students in small groups during class time (supplemented by whole-class discussion at key points in the PSP to consolidate student understanding), while certain tasks work well as individual homework assigned after those discussions. To reap the full mathematical benefits offered by the reading of primary sources, students should be required in some way to read assigned sections in advance of any in-class work; advance preparation by students of (perhaps preliminary) responses to tasks that will be discussed during in-class work is also recommended.[6] Depending on the exact combination of individual/small-group/whole-class work, this method of implementation requires 2–3 class days (based on 50-minute class periods). A sample schedule that offers some options to help instructors tailor this mode of implementation to their course goals and available class time is outlined in the next subsection of these Notes.

Yet another implementation alternative that has been used with this PSP combines aspects of the two options described above, with complete individual write-ups of all PSP tasks required (and evaluated as part of students' course grades) following four half-days of small-group and whole-class discussions spread out over the course of a month. In advance of each half-day of in-class work, students prepare draft responses to specific PSP tasks. They then revised their responses based on in-class discussions before submitting second-draft write-ups for instructor feedback, with final corrections of all PSP tasks due about a week after that instructor feedback was returned.

LaTeX code of this PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

[6] The author's method of ensuring that advance reading takes place is to require student completion of "Reading Guides" (or "Entrance Tickets") for which students receive credit for completion, but with no penalty for errors in solutions. See the Appendix to these Notes for a sample guide based on this particular PSP and more detail about their general design.

Sample Implementation Schedule (based on a 50-minute class period)

For instructors who choose to implement this PSP via a combination of small-group and whole-class discussions, the following schedule options allow for completion in 2–3 class periods.

- **Advance Preparation Work for Day 1** (to be completed before class)
  All instructors should have students read the Introduction, all of Section 1 and the first excerpt of Section 2; students should also complete Tasks 1–3 for class discussion.
  - Instructors pursuing the logical rigor/geometrical intuition theme (described in the "PSP Design, and Task Commentary" subsection of these Notes) should also ask students to prepare preliminary notes and questions about Task 4.
  - Instructors not pursuing that theme should have students skip Task 4 altogether, and instead assign advance reading of all of Section 2 and completion of Task 5.
- **Day 1 of In-Class Work**
  - Small-group discussion of the following, supplemented by whole-class discussion as needed:
    - Section 1: Quick review of answers to advance preparation work on Task 1, and more detailed discussion of Task 2.
    - Section 2: Quick review of answers to advance preparation work on Task 3; parts (c)–(d) of that Task are especially relevant to the logical rigor/geometrical intuition theme.
      - If Task 4 was assigned for advance preparation and time permits, discussion of that Task can also begin (may need to continue to Day 2). This task is best suited for whole-class discussion, perhaps following some initial discussion in small groups.
      - If Task 5 was assigned for advance preparation and time permits, discussion of that Task can begin. If time runs short for a full discussion, students' advance preparation write-ups can simply be collected and reviewed by the instructor prior to the next class period to determine whether a follow-up discussion on Day 2 would be helpful.
  - **Homework**: A complete formal write-up of Tasks 1 and 2, to be due at a later date (e.g., one week after completion of the in-class work).
- **Advance Preparation Work for Day 2**
  All instructors should have students read Section 3 and complete Task 6 and Task 7(a)–(d) for class discussion.
  - Instructors pursuing the logical rigor/geometrical intuition theme should also have students complete the reading of Section 2 and complete Task 5 in preparation for class discussion.
  - Instructors not pursuing that theme should instead assign advance reading of all of Section 3 and completion of Task 8 for class discussion. Those who wish to complete in-class implementation in just 2 days should also assign advance reading of Section 4 and completion of Task 9 for class discussion.

- **Day 2 of In-Class Work**
  - Section 2 Follow-up:
    * Instructors pursuing the logical rigor/geometrical intuition theme may wish to continue or follow up on the discussion of Task 4 from Day 1. Small-group or whole-class discussion of Task 5 can also take place prior to moving to in-class work on Section 3; alteratively, students' advance preparation write-ups for that Task can simply be collected and reviewed by the instructor prior to the next class period to determine whether a follow-up discussion on Day 3 would be helpful.
    * Instructors not pursuing that theme may wish to quickly follow up on Task 5, especially if there was limited or no time for discussion of that Task on Day 1.
  - Section 3:
    * Whole-class discussion of Task 6; this should be relatively quick, but is important to ensuring students appreciate Lebesgue's approach before continuing to the later Tasks in this section.
    * Small-group discussion (supplemented as desired by whole-class discussion) of the following:
      · Task 7 parts (a)–(f). Note that part (d) is especially suited to whole-class discussion.
      · If Task 8 was assigned for advance preparation and time permits, this Task can also be discussed. If time runs short for a full discussion, students' advance preparation write-ups can simply be collected and reviewed by the instructor prior to the next class period to determine whether a follow-up discussion on Day 3 would be helpful.
  - Section 4 (only if advance reading assigned for Day 2): 10–20 minutes should be reserved for a closing whole-class discussion of the PSP with a focus on the commentary in Section 4. During this closing discussion, students could be asked to share their answers to Task 9; alternatively, students' advance preparation write-ups for that Task can simply be collected and reviewed by the instructor prior to the next class period to determine if any final clarification of the ideas in the project seems necessary.
  - **Homework**: A complete formal write-up of Task 7, to be due at a later date (e.g., one week after completion of the in-class work).

- **Advance Preparation Work for Day 3 (if not following the 2-day plan)**
  All instructors should have students read Section 4 and complete Task 9.
  - Instructors pursuing the logical rigor/geometrical intuition theme should also have students complete the reading of Section 3 and complete Task 8 as advance preparation for class discussion.

- **Day 3 of In-Class Work (10–50 minutes)**
  - Section 3 Follow-up:
    * Instructors pursuing the logical rigor/geometrical intuition theme may wish to have students quickly discuss their answers to Task 8 in small groups; alternatively, their answers to this Task could be worked into a closing whole-class discussion of the PSP.
    * Instructors not pursuing that theme may also wish to quickly follow up on Task 8, especially if there was limited or no time for discussion of that Task on Day 2.
  - Section 4: Closing whole-class discussion of the PSP, with a focus on the commentary in Section 4. The time needed for this could vary from 10–50 minutes, depending on instructor's goals and how students' work on the project has gone on Days 1–2. During this closing discussion, students could be asked to share their answers to Task 9; alternatively, students' advance preparation write-ups for that Task can simply be collected and reviewed by the instructor prior to the next class period to determine if any final clarification of the ideas in the project seems necessary.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be completed in at most 2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from `https://digitalcommons.ursinus.edu/triumphs\_topology`; all other listed projects are available at `https://digitalcommons.ursinus.edu/triumphs\_analysis`.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Heine Barnett)
- *Investigations into Bolzano's Bounded Set Theorem* (David Ruch)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)
  Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)
  A second version of this project suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Mean Value Theorem*(David Ruch)
- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Euler's Rediscovery of e** (David Ruch; sequence convergence, series & sequence expressions for e)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

Recommendations for Further Reading

Instructors who wish to know more about the history of integration in the nineteenth and early twentieth centuries will find the article [Hochkirchen, 2003] of interest. See the reference list of the student portion of this PSP for bibliographic details.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.

APPENDIX

This appendix provides a 'Sample Reading Guide' that illustrates the author's method for assigning advance preparation work in connection with classroom implementation of primary source projects. As described in the subsection "Suggestions for Classroom Implementation" of the Notes to Instructors for this project, students receive credit for completion of these guides, but with no penalty for errors in solutions. Students are asked to strive to answer each question correctly, but to think of Reading Guides as preparatory work for class, not as a final product (e.g., formal polished write-ups are not expected). Students who arrive unprepared to discuss assignments on days when group work is conducted based on advance reading are not allowed to participate in those groups, but are allowed to complete the in-class work independently. Guides are collected at the end of each class period for instructor review and scoring prior to the next class period.

A typical guide (such as the one that follows) will include "Classroom Preparation" exercises (generally drawn from the PSP Tasks) for students to complete prior to arriving in class, as well as "Discussion Questions" that ask students only to read a given task and jot down some notes in preparation for class work. Students are also encouraged to record any questions or comments they have about the assigned reading on their guide and are sometimes explicitly prompted to write 1–3 questions or comments about a particular primary source excerpt; their responses to such prompts are especially useful as starting points for in-class discussions. On occasion, tasks are also assigned as follow-up to a prior class discussion.

Experience has proven the value of reproducing the full text of any assigned project task on the guide itself, with blank space for students' responses deliberately left below each question. This not only makes it easier for students to jot down their thoughts as they read, but also makes their notes more readily available to them during in-class discussions. It also makes it easier for the instructor to efficiently review each guide for completeness (or to skim responses during class for a quick assessment of students' understanding), and allows students to make more effective use of their Reading Guide responses and instructor feedback on them at a later date.

The primary goal of the reading and tasks assigned in this particular 4-page reading guide is to familiarize students with the historical and mathematical background of this project, and to prepare them for in-class small-group work on Tasks 1–4. The final question also sets up the possibility of beginning class discussion of Task 5, should time permit.

Day 1 Reading Guide: *Henri Lebesgue and the Development of the Integral Concept*
Reading Assignment: pp. 1–6 ending just above Task 5.

1. Read the Introduction, page 1.

   Questions or comments?

2. Read the start of Section 1, including the excerpt at the top of page 2 of the project:

   Write at least one comment OR one question about this excerpt:

3. **Complete Task 1,** reproduced below for your convenience.

   Recall that the following theorem holds for the Riemann integral (as was first rigorously proven by Darboux):

   If f is continuous at x_0, then $F(x) = \int_a^x f(y)dy$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

   Explain how this solves the problem of finding a function with a given derivative in the case where the given derivative is a continuous function.

4. **Answer the following questions from Task 2,** reproduced below for your convenience. The footnotes to this task on page 2 of the project may also be helpful to look back at.

Consider the sequence of functions (f_n) where for each $n \in \mathbb{Z}^+$, $f_n : [0,1] \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \end{cases},$$

where the set A_n is defined by $A_n = \{\frac{p}{q} \,|\, p, q \in \mathbb{Z}^+ \wedge \gcd(p,q) = 1 \wedge q \leq n\} \cup \{0\}$.

(a) Use theorems about Riemann integrals to explain why each of the individual functions f_n is Riemann integrable on $[0,1]$. (Feel free to use a modern textbook as needed to remind yourself about these theorems.)

(b) What is the value of each of the individual Riemann integrals $\int_0^1 f_n(x)dx$? Explain.

(c) Given $x \in [0,1]$, explain why $\lim_{n\to\infty} f_n(x) = f(x)$, where f is the Dirichlet function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(d) Use the definition of the Riemann integral to explain why f is NOT Riemann integrable on $[0,1]$.

(e) Finally, explain why the following equation fails to hold when Riemann integration is used:

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

5. Continue your reading with the first excerpt in Section 3.

   Write at least one comment OR one question about this excerpt:

6. **Answer the following questions from Task 3,** reproduced below for your convenience.

   According to Lebesgue's description of the early history of the integral (in the excerpt preceding Task 3):

   (a) How was the integral defined before Cauchy?

   (b) What was Cauchy's motivation for providing a definition of the integral?
   Do you agree with Cauchy that this was an important reason to give a definition?

   (c) What new difficulties arose because of Cauchy's new approach to defining the integral? Identify at least two such difficulties. Of these, which do you think is the greater obstacle for someone who might try to learn about integration starting with Cauchy's definition of the integral, and why?

   (d) What progress did Cauchy's approach make possible? Be specific!

   Do you agree with Lebesgue that this was progress? Why or why not?

7. **Prepare some notes for discussion of Task 4,** reproduced below for your convenience. You will probably find it necessary to re-read the two paragraphs in question a few times.

   In the last paragraph of the excerpt just above Task 3, Lebesgue discussed the question

   > And now, should we limit ourselves to doing analysis?'

   What did Lebesgue seem to mean by this question, and how did he answer it?

   To answer these questions, it will also be useful to look back at the two paragraphs immediately preceding last paragraph of the preceding excerpt(starting with "What Cauchy did was so substantial that ..." and "In order to achieve the reduction " respectively).

8. Do a preliminary reading of the next excerpt from Lebesgue's paper (just below Task 4).

   Questions or comments about this excerpt, or about the project so far?