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Jerry Lodder
New Mexico State University, jlodder@nmsu.edu

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The Exigency of the Euclidean Parallel Postulate and the Pythagorean Theorem

Jerry Lodder*

January 29, 2022

1 Euclid's *Elements*

Euclid's *Elements of Geometry* has been circulated more widely than any other mathematical work, in various translations, reproductions, and editions. Written as a model of deductive reasoning, it has been used to teach geometry for over two millennia. Euclid lived sometime around 300 BCE likely in Alexandra, Egypt, during the reign of Ptolemy I. The *Elements* was originally written on 13 rolls of papyrus in ancient Greek, comprising 13 books or chapters [12, p. 16]. Although only a few fragments of papyri containing text from the *Elements* remain today, the text was copied and recopied by scribes and scholars in antiquity through the Middle Ages, some adding material, some slightly editing the original. One of the oldest surviving Greek versions of the *Elements* dates back to 888 CE, housed today at the Bodleian Library in Oxford, England. Perhaps older than this and closer to Euclid's original text is a copy in the Vatican Library, cataloged as Greek manuscript 190, dating to about 850 CE. The *Elements* was translated into Arabic around 800 CE in Bagdad, and into Latin during the twelfth century in England. With the advent of the printing press in Europe, an elaborate printed edition of the *Elements* in Latin appeared in 1482 in Venice. Jesuit priests brought the text to China, where it was translated into Chinese around 1607. A Greek version was compiled by the Danish historian and philologist Johan Ludvig Heiberg (1854–1928) during the years 1883–1888 based on existing Greek copies, such as manuscript 190. There have been numerous translations into other languages. The most widely circulated English edition today is a translation by Sir Thomas L. Heath (1861–1940) [7], based on Heiberg's work. Heath's translation of all 13 books of the *Elements* is available in one volume, edited by Dana Densmore [4].

Book I of the *Elements* comprises some 48 propositions, ending with the Pythagorean Theorem and its converse. This theorem states that in a right triangle, the square on the side opposite the right angle is equal to the sum of the squares on the other two sides. The proof of the Pythagorean Theorem found in the *Elements* is an area argument that rests on recognizing when two parallelograms or two triangles have the same area. Certainly no algebraic

*Mathematical Sciences; Dept. 3MB, Box 30001; New Mexico State University; Las Cruces, NM 88003; jlodder@nmsu.edu.

formulas for area or any other quantities were used in ancient Greece. Two parallelograms have the same area when they rest on the same base with their opposite sides lying along a line that is parallel to the base, although the opposite sides of the two parallelograms need not coincide. Since the diagonal of a parallelogram divides the figure into two congruent triangles, a corresponding statement about the area of triangles can be made. These area results from the *Elements* are studied in a separate section of this project. It remains a discovery exercise (later in the project) to see how the square on the hypotenuse of a right triangle can divided into smaller figures (parallelograms or rectangles) that have areas matching the areas of the squares on the other two sides. The Pythagorean Theorem has found many uses in modern mathematics, such as an expression for the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the (Euclidean) xy-plane as $\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, or in the trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$ for angles θ.

Since the area results above rely on the use of parallel lines, a study of the angles formed when one line falls on two other possibly parallel lines (called a transversal today) is presented from the *Elements*. The reader might wish to identify all transversals formed by a parallelogram, as in Exercise (1.8). Congruent angles formed by a transversal can be identified by the alternate interior angle and its converse, both presented in this project. Proposition 29 from Book I, the converse of the alternate interior angle theorem, states: "A straight line falling on parallel straight lines makes the alternate angles equal to one another." This is the first proposition in Book I that requires for its proof a rather lengthy postulate, the fifth postulate in Heath's translation, known today as the (Euclidean) parallel postulate. This postulate states: "If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

The logical status of a postulate is a statement that is accepted without proof, since it appears self-evident or represents a property that is held to be true without justification. Since the time of Euclid, geometers have tried, and failed, to prove the fifth postulate from Euclid's first four postulates (and common notions or definitions), which reflects a belief that the parallel postulate is not self-evident, and should be a separate proposition [6, p. 23] [8, p. 24]. Certainly many results in the *Elements* depend on the parallel postulate, beginning with the converse of the alternate interior angle theorem. The construction of a parallelogram, a rectangle, or even a square depends on the parallel postulate. The area results cited above require the fifth postulate for their validity as well. The Pythagorean Theorem and with it the usual distance formula between points would fail without the parallel postulate. The usual notion of similar triangles and ratios defining the sine and cosine would no longer be valid. Without the Euclidean parallel postulate, the angle sum of a triangle is a variable, depending on the triangle, and is never 180°. With so many results depending on the parallel postulate, how could it fail? Let us then admit to the exigency of the parallel postulate and as Euclid accept this as a separate postulate of our geometry.

The proof of the Pythagorean Theorem also relies on many of the previous propositions from Book I, either directly or indirectly. This project does not reproduce all of them, but only five key preliminary propositions along with the postulates and common notions from Book I. The reader may wish to have a copy of the *Elements* on hand for reference [4, 7]. The further sections of this project are:

- The Postulates and Common Notions of Book I
- Construction of Perpendiculars
- The Alternate Interior Angle Theorem
- The Converse of the Alternate Interior Angle Theorem
- On the Construction of a Square
- The Ancient Greek View of Area
- The Pythagorean Theorem
- An Ancient Chinese "Hypotenuse Diagram"

Each section presenting a preliminary proposition begins with a discussion of the prerequisite material needed for its proof. This is followed by an outline of the proof strategy with the final proof requested in the exercises of each section.

There are many proofs of the Pythagorean Theorem, each requiring the Euclidean parallel postulate. There are several proofs that are variations of the area argument found in the *Elements* as well as proofs using similar triangles. The equality of ratios of corresponding sides in similar triangles, not presented until Book VI of the *Elements*, rests eventually on an area argument, and, of course, requires the parallel postulate for its proof. See for example [10]. This project offers an optional final section with a "Hypotenuse Diagram" from ancient China, ascribed to the sixth-century CE commentator Zhen Luan [3, p. 222] that shows the relation between the base, height and diagonal of what today would be called a right triangle. When modern algebraic symbols are associated with the these lengths and the area of the hypotenuse diagram is computed in two different ways, another proof of the Pythagorean Theorem results. It remains an exercise to identify all places in this argument where the parallel postulate is used.

Here are some warm-up exercises before beginning a study of the mathematics of Book I.

Exercise 1.1. Report on the following events and the people involved:

(a) the production of the copy of the *Elements* from 888 CE housed today at the Bodleian Library in Oxford.

(b) the translation of the *Elements* into Arabic around 800 CE in Baghdad;

(c) the translation of the *Elements* into Latin in the twelfth century in England;

(d) the printing of the *Elements* in 1482 in Venice;

(e) the translation of the *Elements* into Chinese in the early 1600s;

(f) Heiberg's work on a compilation of a Greek edition of the *Elements*;

(g) Heath's English translation of the *Elements*.

See, for example, Wardhaugh [12].

Exercise 1.2. Report on the contents of Book II of the *Elements*.

Exercise 1.3. What is meant by the *converse* of a theorem? Suppose that a hypothetical theorem states "If event A happens, then event B follows." State the converse of this hypothetical theorem.

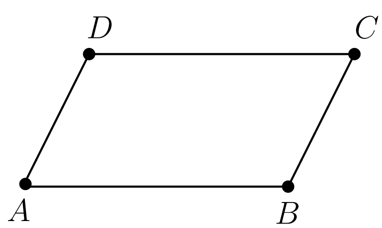
Exercise 1.4. What is the *hypotenuse* of a right triangle?

Exercise 1.5. What is meant for two lines in the same plane to be *parallel*?

Exercise 1.6. What is meant for two triangles to be *congruent*?

Exercise 1.7. What is meant for two triangles to be *similar*?

Exercise 1.8. Find all transversals in parallelogram $ABCD$ below. For this exercise a transversal is formed when one line falls on two other (possibly parallel) lines, and any line segment can be extended to form a line.



Exercise 1.9. Find the distance between the points $P(3,\, 2)$ and $Q(6,\, 6)$ in the Euclidean xy-plane.

2 The Postulates and Common Notions of Book I

The following are the postulates, common notions, and some of the definitions from the *Elements*, Book I, [4, pp. 1–2] [7, Vol. I, pp. 153–155]. Not all twenty-three definitions that begin Book I are not reproduced here, although definitions needed in the later sections are included below.

Definitions.

1. A *point* is that which has no part.
2. A *line* is breathless length.
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.
5. A *surface* is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A *plane surface* is a surface which lies evenly with the straight lines on itself.
8. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called *rectilineal*.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called *perpendicular* to that on which it stands. ...

19. *Rectilineal figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.
20. Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles* triangle that which has two of its sides alone equal, and a *scalene* triangle that which has its three sides unequal.
21. Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, ...
22. Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; .... .
23. *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet in either direction.

Postulates.

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

Common Notions.

1. Things which are equal to the same thing are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Exercise 2.1. For each of the following definitions given in Book I of the *Elements*, state whether in your opinion enough information is given to sufficiently define the term in question and why.

(a) a *point*;

(b) a *straight line*;

(c) a *plane surface*;

(d) a *right angle*;

(e) an *equilateral triangle*;

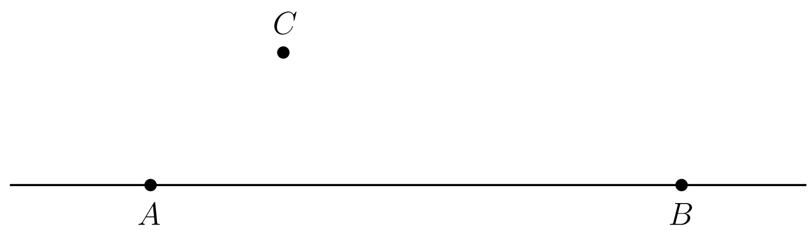
(f) an *isosceles triangle*;

(g) *parallel* straight lines.

Exercise 2.2. Is the term *parallelogram* defined in the list of definitions that begin Book I of the *Elements*? What is the modern mathematical definition of a *parallelogram*?

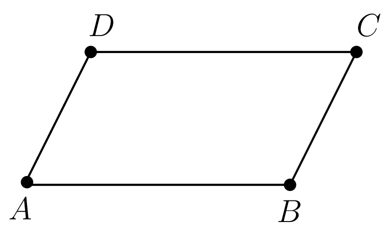
Exercise 2.3. Cite a postulate or common notion to justify each of the following statements.

(a) Given a (straight) line $\overleftrightarrow{AB}$ and a point C not on $\overleftrightarrow{AB}$, there is a line through C meeting line $\overleftrightarrow{AB}$ at the point A.

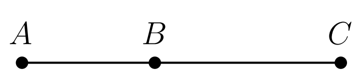


(b) If the length of line segment AB equals the length of line segment CD, and the length of line segment EF equals the length of line segment CD, then the length of segment AB equals the length of segment EF.

(c) Given parallelogram $ABCD$ below, the side AB can be extended to form a line.

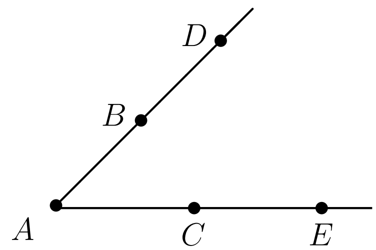


(d) Suppose that segment AB is extended to segment AC as below. Then (the length of) segment AC is greater than (the length of) segment AB.



(e) Suppose that segment AB is extended to segment AC as in part (d). Then (the length of) segment AB added to (the length of) segment BC equals (the length of) segment AC.

(f) Suppose that segment AB is extended to segment AD and segment AC is extended to segement AE as below. Then (the measure of) angle $\angle BAC$ equals (the measure of) angle $\angle DAE$.



Exercise 2.4. Look up the *transitive porperity of equality*. Which common notion corresponds to what is called transitivity today?

Exercise 2.5. Write the second common notion as an algebraic equation. Be sure to explain your work.

Exercise 2.6. Write the third common notion as an algebraic equation. Be sure to explain your work.

Exercise 2.7. Why in your opinion is the fourth postulate necessary?

Exercise 2.8. Draw a picture representing the fifth postulate. Be sure to explain your work.

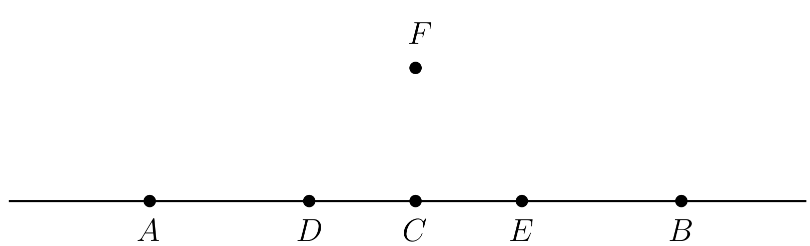
3 Construction of Perpendiculars

Purpose: The goal of this series of mini-projects is to gain insight into the proof of the Pythagorean Theorem found in Euclid's *Elements* [7, Vol. I]. This proof requires the literal construction of squares on the three sides of a right triangle. As a first step in the construction of a square, we examine the construction of a line perpendicular to a given line, found in Proposition 11 of the *Elements*.

Prerequisite Material: The *Elements* is written in a logically precise manner with one proposition building on the previous ones. The reader may use, without proof, the following results already proven in the *Elements*: segment duplication, construction of an equilateral triangle with a given side, the basic congruency theorems for triangles, such as side-side-side and side-angle-side. To establish the perpendicularity of two lines, it must be shown that the lines intersect at right angles. According to Euclid, "When a straight line set up on a straight line makes the adjacent angles equal to another, each of the equal angles is *right*" [7, Vol. I, p. 153].

Proposition 11: To draw a straight line at right angles to a given straight line from a given point on it.

Strategy of Proof: Given line $\overleftrightarrow{AB}$ and a point C on $\overleftrightarrow{AB}$, find some line through C perpendicular to $\overleftrightarrow{AB}$. To meet Euclid's definition of a right angle, we must find some line $\mathcal{L}$ through point C so that the adjacent angles formed by $\mathcal{L}$ and $\overleftrightarrow{AB}$ (on one side of $\overleftrightarrow{AB}$) are equal (or congruent). The equality of these angles will follow from the congruency of two triangles. But what two triangles? Consider another point D on $\overleftrightarrow{AB}$, $D \neq C$. Construct a line segment CE on line $\overleftrightarrow{AB}$ so that $CE = DC$ in length. Then form an equilateral triangle $\triangle DEF$ with one side being segment DE. Draw line segment FC. Can you find two congruent triangles? Can you prove that segment FC is perpendicular to line $\overleftrightarrow{AB}$?

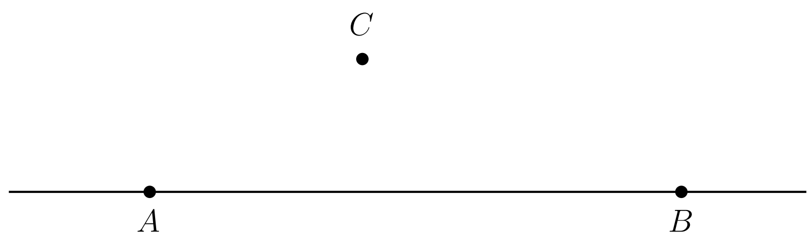


Exercise 1. In a step-by-step argument, prove that given line $\overleftrightarrow{AB}$ and a point C on $\overleftrightarrow{AB}$, there is some line segment FC perpendicular to $\overleftrightarrow{AB}$. Be sure to offer a reason why each step

holds. You may cite any previous proposition, definition, postulate or common notion from the *Elements* as a valid reason.

Question 1. Does the construction of a perpendicular line require the parallel postulate (Postulate 5 of the *Elements*)?

Exercise 2. In a step-by-step argument, prove that given line $\overleftrightarrow{AB}$ and a point C **not** on $\overleftrightarrow{AB}$, there is a line segment through point C perpendicular to $\overleftrightarrow{AB}$. Be sure to offer a reason why each step holds. You may cite any previous proposition, definition, postulate or common notion from the *Elements* as a valid reason. You may also cite Proposition 9, angle bisection, or Proposition 10, segment bisection, as reasons.



Question 2. Does the construction in Exercise 2 rely on the parallel postulate?

4 The Alternate Interior Angle Theorem

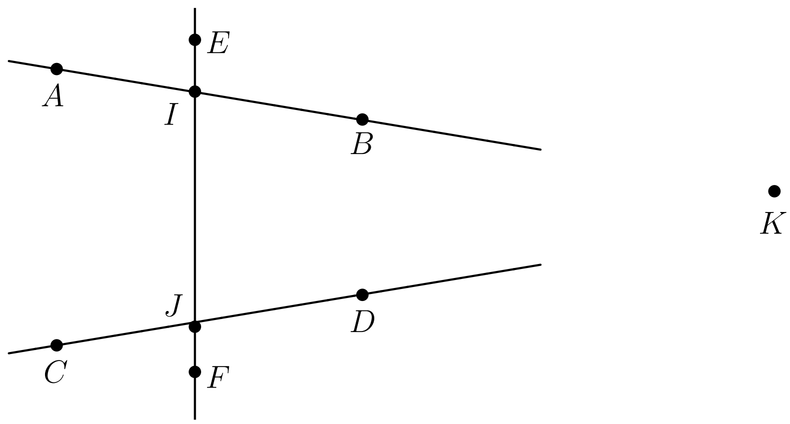
Purpose: Recall that the proof of the Pythagorean Theorem found in Euclid's *Elements* [7, Vol. I] requires the literal construction of squares on the three sides of a right triangle. In a previous mini-project we saw how to construct a line perpendicular to a given line, which is a first step in the construction of a square. Of course, opposite sides of a square must form parallel lines, a topic of this mini-project. Today, when one line intersects (falls on) two other lines, the three lines are said to form a transversal. In this mini-project we prove that if the alternate interior angles of a transversal are congruent, then two of the lines are parallel. We use Euclid's definition [7, Vol. I, p. 154] that "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet another in either direction." We then prove Proposition 27 of the *Elements*, known today as the Alternate Interior Angle Theorem.

Prerequisite Material: The reader may use the postulates and common notions of the *Elements* as well as the exterior angle theorem for triangles, which states that the exterior angle of a triangle is greater than either of the opposite interior angles. Also, the vertical angle theorem, the construction of perpendiculars, and angle duplication may be used in the exercises.

Proposition 27: If a straight line falling on two straight lines make the alternate angles equal to one anther, the straight lines will be parallel to another.

Strategy of Proof: Given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, suppose that line $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at point I and line $\overleftrightarrow{EF}$ intersects line $\overleftrightarrow{CD}$ at point J. Given that $\angle AIJ \simeq \angle IJD$, prove that line $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$. By Euclid's definition of parallel, we must show that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do not meet. Using an indirect proof, assume that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet at some point K on the same side of $\overleftrightarrow{EF}$ as point B (or point D). Then consider triangle $\triangle IKJ$. Is $\angle AIJ$ an exterior angle to triangle $\triangle IKJ$? Can a contradiction be reached? What if lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at some point L on the same side of $\overleftrightarrow{EF}$ as point A (or point C). What is your final conclusion?

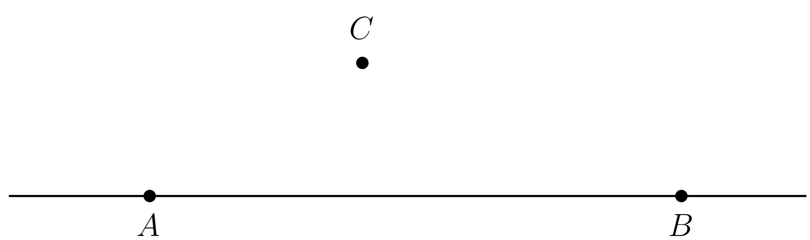
Exercise 1. Given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, suppose that line $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at point I and line $\overleftrightarrow{EF}$ intersects line $\overleftrightarrow{CD}$ at point J. Given that $\angle AIJ \simeq \angle IJD$, in a step-by-step argument, prove that line $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$. Use an indirect proof and provide the details of two cases, where the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect on (i) the same side of $\overleftrightarrow{EF}$ as point B and (ii) the same side of $\overleftrightarrow{EF}$ as point A. Be sure to offer a reason why each step holds. You may cite any previous proposition, definition, postulate or common notion from the *Elements* as a valid reason.



Question 1. Does the proof of the Alternate Interior Angle Theorem (Exercise 1) require the parallel postulate (Postulate 5 of the *Elements*)?

Exercise 2. [The Corresponding Angle Theorem] Given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, suppose that line $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at point I and line $\overleftrightarrow{EF}$ intersects line $\overleftrightarrow{CD}$ at point J. Given that $\angle EIB \simeq \angle IJD$, in a step-by-step argument, prove that line $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$.

Exercise 3. In a step-by-step argument, prove that given line $\overleftrightarrow{AB}$ and a point C not on $\overleftrightarrow{AB}$, there is some line through C parallel to line $\overleftrightarrow{AB}$. Be sure to offer a reason why each step holds.



Question 2. Does Exercise 3 (The Existence of Parallels) rely on the parallel postulate?

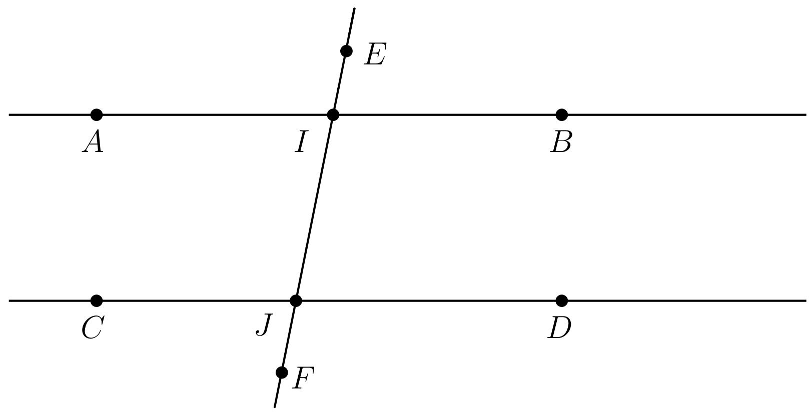
5 The Converse of the Alternate Interior Angle Theorem

Purpose: We continue with a discussion of the propositions necessary for the construction of a square in Euclidean geometry, where Euclid's fifth postulate, the so-called parallel postulate holds. A basic stepping stone in this direction is Proposition 29 of the *Elements* [7, Vol. I], the converse of the alternate interior angle theorem. First recall Euclid's formulation of the fifth postulate [7, Vol. I, p. 155]: "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." Although wordy by today's standards, the above formulation of the parallel postulate states the precise conditions needed for the proof of Proposition 29. In the sequel we will use a more streamlined version of the parallel postulate stated by the Scottish mathematician John Playfair (1748–1819) [11]: "Two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another." Given a line $\overleftrightarrow{AB}$ and a point C not on $\overleftrightarrow{AB}$, Playfair's axiom asserts the existence of a unique line through C parallel to $\overleftrightarrow{AB}$. Recall that in the previous mini-project the existence of some line through C parallel to line $\overleftrightarrow{AB}$ was proven without use of the parallel postulate. Thus, the main impact of Euclid's fifth postulate is that the line through point C parallel to $\overleftrightarrow{AB}$ is *unique*. In this mini-project we outline Euclid's proof of Proposition 29 and discuss the logical equivalence between Euclid's fifth postulate and Playfair's postulate (stated above).

Prerequisite Material: The reader may use the postulates and common notions of the *Elements* as well as Proposition 13, which states that the angle sum around a line is 180°. Also, keep in mind Euclid's version of the fifth postulate and Playfair's version of the parallel postulate, written above. For the exercises, the vertical angle theorem and the existence of parallels may be used.

Proposition 29 [abbreviated]: A straight line falling on parallel straight lines makes the alternate angles equal to one another . . . .

Strategy of Proof: Given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, suppose that line $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at point I and line $\overleftrightarrow{EF}$ intersects line $\overleftrightarrow{CD}$ at point J. Given that $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$, prove that $\angle AIJ \simeq \angle IJD$. Using an indirect proof, assume that $\angle AIJ$ is not congruent to $\angle IJD$. Then in measure, either (i) $\angle AIJ > \angle IJD$ or (ii) $\angle AIJ < \angle IJD$. For case (i), consider $\angle AIJ + \angle BIJ$. If $\angle AIJ > \angle IJD$, what is true about $\angle BIJ + \angle IJD$? Can Euclid's fifth postulate now be applied? What contradiction is reached? Then consider case (ii), where $\angle AIJ < \angle IJD$.



Exercise 1. Given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, suppose that line $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at point I and line $\overleftrightarrow{EF}$ intersects line $\overleftrightarrow{CD}$ at point J. Given that line $\overleftrightarrow{AB}$ is parallel to line $\overleftrightarrow{CD}$, in a step-by-step argument prove that $\angle AIJ \simeq \angle IJD$. Use an indirect proof and provide the details of two cases, where (i) $\angle AIJ > \angle IJD$ and (ii) $\angle AIJ < \angle IJD$. Be sure to offer a reason why each step holds. You may cite any previous proposition, definition, postulate or common notion from the *Elements* as a valid reason.

Exercise 2. [Converse of the Corresponding Angle Theorem] Given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, suppose that line $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at point I and line $\overleftrightarrow{EF}$ intersects line $\overleftrightarrow{CD}$ at point J. Given that line $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$, in a step-by-step argument, prove that $\angle EIB \simeq \angle IJD$. Be sure to offer a reason why each step holds.

Exercise 3. Given Playfair's parallel postulate, prove Euclid's fifth postulate. Be sure to offer a reason why each step holds.

Exercise 4. Given Euclid's fifth postulate, prove Playfair's parallel postulate. Be sure to offer a reason why each step holds.

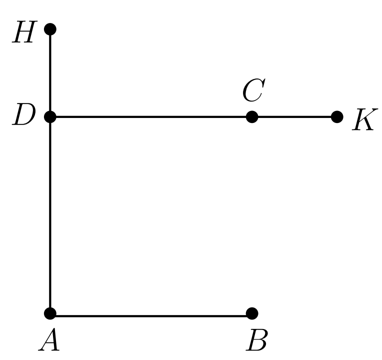
6 On the Construction of a Square

Purpose: In this mini-project we examine the construction of a square, needed later for the Pythagorean theorem. Once Euclid's fifth postulate (the parallel postulate) is accepted, then there are several possible strategies to prove the existence of a square. We outline one strategy which will have particular consequences in hyperbolic geometry, where unique parallels fail to exist.

Prerequisite Material: The reader may use all postulates and common notions of the *Elements* [7, Vol. I] as well as segment duplication, the existence of perpendicular lines, the alternate interior angle theorem and its converse. The triangle congruency theorems may also be used.

Proposition 46: On a given straight line to describe a square.

Strategy of Proof: Given line segment AB, we must find a quadrilateral (four-sided figure) $ABCD$ that is a square, i.e., all four sides are congruent and all four angles are right angles. Construct a segment AH perpendicular to AB. Then on segment AH find a segment AD so that $AD \simeq AB$. From point D, construct a segment DK so that DK is perpendicular to AD and point K is on the same side of line $\overleftrightarrow{AD}$ as point B. On segment DK construct segment DC so that $DC \simeq AB$. Connect points B and C with a line segment. Must figure $ABCD$ form a square? We must show that both angles $\angle DCB$ and $\angle ABC$ are right angles, and we must show that segment BC is congruent to AB (or AD). Are lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ parallel? Is $\angle DCA \simeq \angle BAC$? Why or why not? Is $\triangle DCA \simeq \triangle BAC$? Why or why not? What can be concluded about $\angle ABC$? What can be concluded about segment BC? Finally, find an argument showing that $\angle BCD$ is a right angle.



Exercise 1. Given segment AB, in a step-by-step argument, prove the existence of a square $ABCD$ on AB. Provide a reason why each step holds.

Question 1. Where is Exercise 1 is the parallel postulate being used?

Exercise 2. Given parallelogram $ABCD$, in a step-by-step argument, prove that

(i) $\triangle ABC \simeq \triangle CDA$.

(ii) $AD \simeq BC$

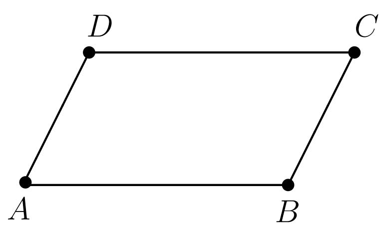
(iii) $AB \simeq DC$

(iv) $\angle DAB \simeq \angle BCD$

(v) $\angle ADC \simeq \angle CBA$

(vi) $\triangle DAB \simeq \triangle BCD$.

Be sure to offer a reason why each step holds.



7 The Ancient Greek View of Area

Purpose: At the heart of the proof of the Pythagorean Theorem in Euclid's *Elements* [7, Vol. I] is an area argument. Given a right triangle, then the area of the square on the hypotenuse (the side opposite the right angle) is shown to be the sum of the area of the squares on the other two sides. How is area treated in the *Elements* given that algebra had not yet been developed in antiquity, precluding the use of any formula for area? Moreover, Euclid avoids the use of numbers in the *Elements* so an expression of area in terms of a numerical value is not given, even verbally. The key feature of figures (parallelograms or triangles) with equal area is that they are constructed on the same base and are contained between the same parallel lines. Once this view of area is understood, then the proof of the Pythagorean Theorem reduces to a geometric puzzle where certain figures (parallelograms or triangles) can be found on the same base and contained between the same parallel lines.

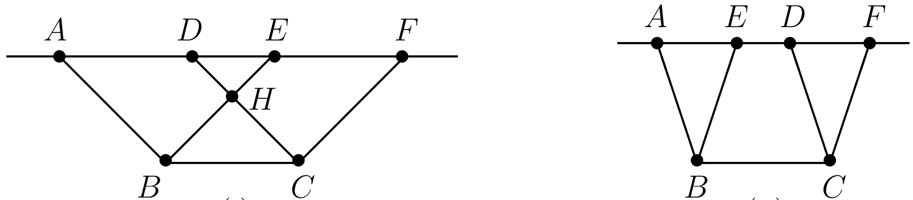
Prerequisite Material: The reader may use all postulates and common notions of the *Elements*. The triangle congruency theorems may be used as well as results about parallelograms asserting the congruency of opposite sides of a parallelogram as well as opposite angles of a parallelogram. Also, that a diagonal divides a parallelogram into two congruent triangles may be used without proof. Of course, congruent figures have equal area, a fact that Euclid uses liberally. In fact, Euclid writes that figures "are equal" in place of "have equal area."

Proposition 35: Parallelograms which are on the same base and in the same parallels are equal to one another.

Strategy of Proof: Consider parallelograms $ABCD$ and $EBCF$, both on base BC such that the opposite sides AD and EF are contained in the same line that is parallel to line $\overleftrightarrow{BC}$. Consider two cases, where (i) point D falls between points A and E, and (ii) where point E falls between points A and D. For case (i), argue why $\triangle ABE \simeq \triangle DCF$. Suppose that segment BE intersects segment CD at point H in case (i). Then add the area of $\triangle BCH$ to the area of $\triangle ABE$ and compare this to the sum of the areas of $\triangle BCH$ and $\triangle DCF$. Now subtract the area of $\triangle HDE$ from both of these. What results? How much of the argument for case (i) can be applied to case (ii)?

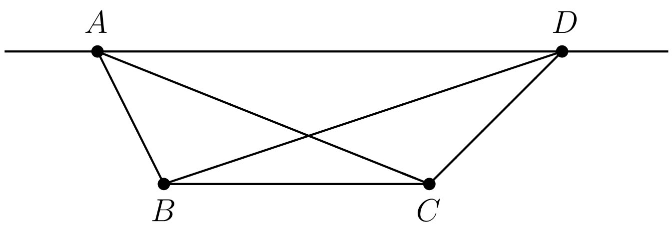
Exercise 1. In a step-by-step argument, prove the following. Consider parallelograms $ABCD$ and $EBCF$, both on base BC such that the opposite sides AD and EF are contained in the same line that is parallel to line $\overleftrightarrow{BC}$. Then parallelograms $ABCD$ and $EBCF$ have the same area. Be sure to provide the details of both cases above, and provide a reason why each step holds.

Question 1. Does the proof of Exercise 1 rely on the parallel postulate? How?

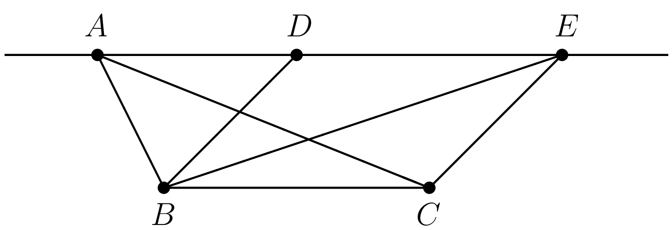


(i) (ii)

Exercise 2. Consider triangles $\triangle ABC$ and $\triangle DBC$, both on base BC with point A different from point D. Suppose that line $\overleftrightarrow{AD}$ is parallel to line $\overleftrightarrow{BC}$. In a step-by-step argument, prove that the area of $\triangle ABC$ is equal to the area of $\triangle DBC$. Be sure to offer a reason why each step holds.



Exercise 3. Consider triangle $\triangle ABC$ and parallelogram $DBCE$, both on base BC with points A, D and E on the same line that is parallel to $\overleftrightarrow{BC}$. In a step-by-step argument, prove that the area of parallelogram $DBCE$ is double the area of $\triangle ABC$. Be sure to offer a reason why each step holds.

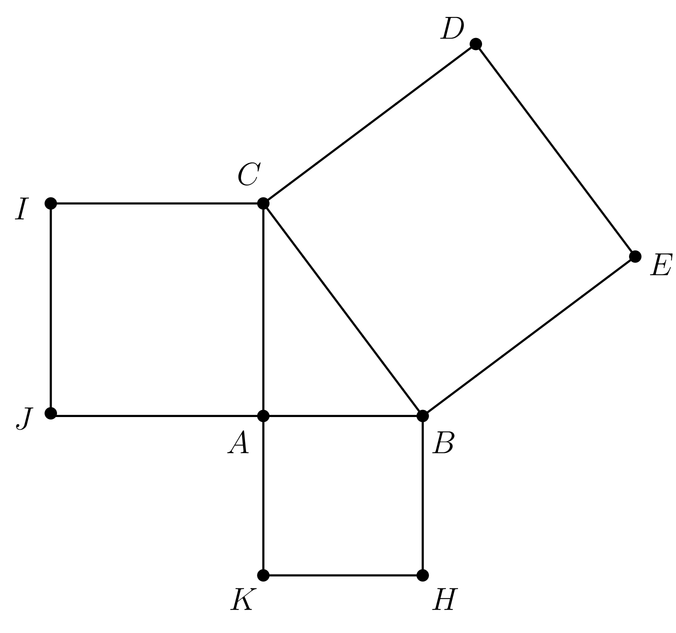


8 The Pythagorean Theorem

Purpose: Book I of the *Elements* [7, Vol. I] ends with the proof of the Pythagorean Theorem and its converse. In the millennia following the *Elements*, this fundamental theorem has become the basis of the distance formula between two points in Euclidean geometry as well as many trigonometric identities such as $\cos^2(\theta) + \sin^2(\theta) = 1$ for angles in Euclidean geometry. In this mini-project we examine the proof of the Pythagorean Theorem found in the *Elements*, based entirely on geometric squares and their area.

Prerequisite Material: The reader may use all postulates and common notions of the *Elements* as well as the triangle congruency theorems, the existence of a square on a given line segment, the existence of a line through a given point parallel to a given line, and results comparing the area of a triangle to a parallelogram. In particular, if a parallelogram and a triangle have the same base and be contained between the same parallel lines, then the area of the parallelogram is double that of the triangle. Also, if line segments AB and BC have angle measure 180° (two right angles), then points A, B and C lie on the same line.

Proposition 47: In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.



Strategy of Proof: Given right triangle $\triangle ABC$ with $\angle CAB$ a right angle, first construct square $BCDE$ on side BC. Then construct square $ABHK$ on side AB. Then construct square $ACIJ$ on side AC. We wish to show that the areas of squares $ABHK$ and $ACIJ$ add to the area of square $BCDE$. Are segments CA and AK on the same line? Why or why not? Are segments JA and AB on the same line? Why or why not? Consider lines $\overleftrightarrow{CK}$ and $\overleftrightarrow{BH}$ as a pair of parallel lines and form $\triangle CBH$ with base BH contained between these two parallels. How does the area of this triangle compare with the area of square $ABHK$? Find a new triangle with base BE congruent to $\triangle CBH$. Between what two parallels is this new triangle contained if one of the lines must be $\overleftrightarrow{BE}$? How does the area of this new triangle compare to (part of) square $BCDE$. Now repeat the argument with square $ACIJ$. What is your final conclusion?

Exercise 1. In a step-by-step argument, prove that given right triangle $\triangle ABC$ with $\angle CAB$ a right angle, then the area of the square on side BC is the sum of the area of the squares on sides AB and AC. Be sure to offer a reason why each step holds.

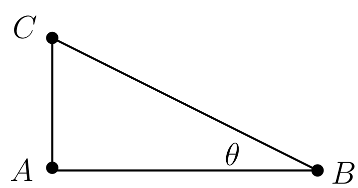
Question 1. Does the proof of Exercise 1 rely on the Euclidean parallel postulate? How?

Exercise 2. Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are points in the (Euclidean) xy-plane. Using modern algebraic techniques, show that the distance between P and Q is given by

$$\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Exercise 3. Given triangle $\triangle ABC$ with right angle $\angle CAB$, let $x = \overline{AB}$, $y = \overline{AC}$ and $r = \overline{BC}$. Let $\angle ABC$ have measure θ. Then $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. Using modern algebraic techniques, show that

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$



Epilogue: We have seen that the modern distance formula between two points is a direct consequence of the Pythagorean Theorem as is the basic trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$. The Pythagorean Theorem itself depends on the Euclidean parallel postulate. With such basic results relying on the parallel postulate, how could it fail? What would replace the distance formula? What would replace the trigonometric functions? We must wait for the pioneers of non-Euclidean or hyperbolic geometries.

9 An Ancient Chinese "Hypotenuse Diagram"

This optional final section offers a method to prove the Pythagorean Theorem using modern algebraic techniques applied to a "hypotenuse diagram" from ancient China, ascribed to the sixth-century CE commentator Zhen Luan [2] [3, p. 222] that shows the relation between the base, height and diagonal of what today would be called a right triangle.

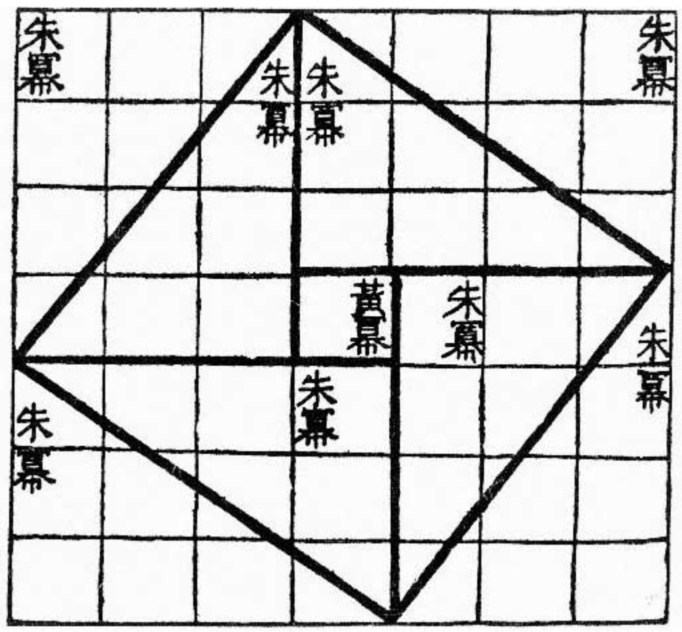


Figure 1: A "Hypotenuse Diagram."

Diagrams similar to Figure (1) may have appeared as early as 100 BCE in the Chinese text *Zhou Bi Suan Jing* (*Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*), available in English translation in *Astronomy and Mathematics in Ancient China: The 'Zhou Bi Suan Jing'* [1]. The area of the square on the hypotenuse of a right triangle in Figure (1) can be found by the applying the so-called "in-out principle" from ancient Chinese mathematics. The right triangles in the four corners of the figure can be removed or left out of the digram, leaving an inscribed square askew in the middle of the figure. See Exercise (9.1). Modern algebraic techniques can be used to formalize the "in-out principle" by cancelling like terms on both sides of an equation as in Exercise (9.2).

Exercise 9.1. For this exercise, the totality of Figure (1) is said to form a "big square." The grid lines in the figure divide the big square into 7×7 "small squares". The "inscribed square" (the *xian* or "hypotenuse" square [3, p. 282]) has its four vertices on the edges of the big square and sits askew in the big square.

(a) Find eight right triangles in Figure (1) with legs of length 3 and 4. Recall that the legs of a right triangle meet at a vertex forming a right angle. Use the grid lines in the diagram to measure the lengths of the legs.

(b) What is the total number of small squares formed by all the grid lines in Figure (1), i.e, in the big square?

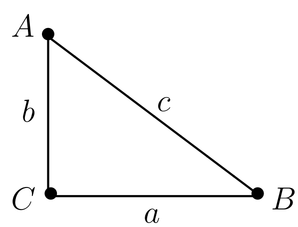
(c) What is the number of small squares formed by the grid lines that comprise the four right triangles with legs of length 3 and 4 in the corners of the big square? Hint: Two right triangles with legs of length 3 and 4 from a rectangle with adjacent sides of length 3 and 4.

(d) What is the number of small squares formed by the grid lines after the four triangles in part (c) are removed from the big square?

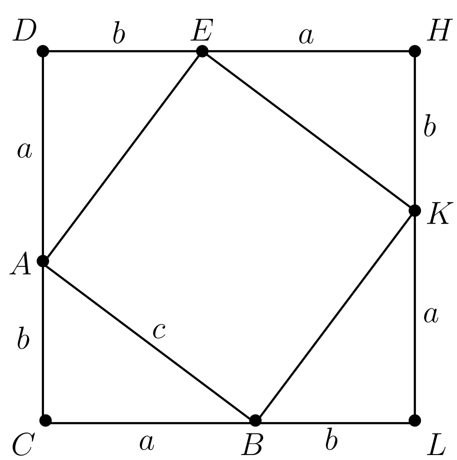
(e) Use part (d) to determine the number of small squares in the inscribed square.

(f) Using small squares to measure area, what can be concluded about the area of the square on the hypotenuse of a right triangle with legs of length 3 and 4? Compare this to the sum of the areas of the squares on the legs of this right triangle.

Exercise 9.2. Consider right triangle $\triangle ABC$ with right angle at vertex C and side lengths $a = \overline{CB}$, $b = \overline{CA}$, $c = \overline{AB}$, pictured below.



(a) In a step-by-step argument, construct a square with side length $a + b$ so that a triangle congruent to $\triangle ABC$ appears in each of the four corners of the square with the right angles of the triangles coinciding with the right angles of the square, pictured below. How is the Euclidean parallel postulate used in this construction?



(b) Prove that quadrilateral $ABKE$ is a square of side length c. Proposition 32 from Book I of the *Elements* may be useful, which states in part that "the three interior angles of [a] triangle are equal to two right angles." Is $\angle ABK$ is a right angle?

(c) Prove that the angle sum of a triangle is 180°. Consult Book I, Proposition 32, if necessary. How does this result rely on the (Euclidean) parallel postulate?

(d) Using modern algebraic techniques, compute the area of square $DCLH$ by squaring the length $a + b$, i.e., compute $(a + b)^2$.

(e) Using modern algebraic techniques, compute the area of square $DCLH$ by adding the area of square $ABKE$ to the areas of triangles $\triangle ACB$, $\triangle BLK$, $\triangle KHE$, $\triangle EDA$. Use that the area of a triangle is one-half its base times the height.

(f) Equate the two expressions for the area of square $DCLH$ found in parts (d) and (e). Cancel common and terms and express the final result as an equality for c^2.

Notes to the Instructor

In this Primary Source Project (PSP) we examine the use of the Euclidean parallel postulate (the fifth postulate of Book I of Euclid's *Elements* [4] [7, Vol. I]) in the proof of the Pythagorean Theorem, essential today for computing the distance between two points in Euclidean space. The proof of the Pythagorean Theorem found in *The Elements* relies on the literal construction of squares on the sides of a right triangle. The construction of a square itself is contingent upon the Euclidean parallel postulate, as are many properties of parallelograms used in results leading to the Pythagorean Theorem. These properties follow from a basic result about parallel lines, namely the converse of the alternate interior angle theorem, which states that "A straight line falling on parallel straight lines makes the alternate angles equal to one another," [7, I.29], a result that is logically equivalent to the Euclidean parallel postulate. With so many results relying on the parallel postulate, how could it fail? What would replace the Pythagorean Theorem and the modern distance formula between two points?

Let us admit then to the exigency of the Euclidean parallel postulate, and examine the proof of the Pythagorean Theorem in detail. Euclid's writing is a model of deductive reasoning, with the logical dependencies of one proposition on previous propositions or postulates clearly stated. A careful reader can easily discern which results depend on the fifth postulate (the parallel postulate). Aside from a rigorous logical structure and the development of basic geometric results, why is the Pythagorean Theorem true? An understanding of the ancient Greek view of area as used in Book I of *The Elements* is essential to answer this question. At the heart of the Pythagorean Theorem is an area argument. The sum of the areas of the squares on the two legs of a right triangle is the area of the square on the hypotenuse. Euclid's view of area is different from the modern point of view. First of all, no algebraic expressions for the area of even the simplest geometric figures were in use. Secondly, Euclid avoids numerical values, so a verbal expression of the area of figure in terms of a number in not given. Instead Euclid proves results stating when certain figures (parallelograms or triangles) have the same area. Parallelograms on the same base and contained between the same parallel lines have the same area. Likewise, triangles on the same base and contained between the same parallel lines have the same area. Not surprisingly, these area results depend on the Euclidean parallel postulate as well. The reader will recognize that the perpendicular distance between the two parallel lines containing the figure serves as the height of the figure, although this point of view is a bit anachronistic, and, in fact, counter-productive for an understanding of the Pythagorean Theorem. To understand Euclid on his own terms, draw literal squares on the sides of a right triangle. Then identify certain figures (triangles, parallelograms or rectangles) that are contained between two parallel lines. Drawing a line through a vertex of the triangle parallel to one of the sides of a square may be useful. The uniqueness of this parallel is, of course, equivalent to Euclid's fifth postulate.

Many high school students may not have seen a proof of the Pythagorean Theorem, although its proof builds on material often taught in high school geometry courses, such as results about transversals and theorems on triangle congruency. Euclid marshals these results to a significant conclusion in Book I of the *Elements*, namely the Pythagorean Theorem. This theorem can be presented as a discovery exercise in finding how to divide the square on the hypotenuse of a right triangle into figures matching the areas of the squares on the legs. Euclid's presentation is highly geometric, visual, and constructive. Modern treatments of geometry emphasize a study of all axioms necessary for the development of the subject, such as completeness of the number line (the real numbers), while such properties are taken for granted in the *Elements*. Euclid lists only five postulates at the beginning of Book I. By contrast, in 1961 the School Mathematics Study Group listed twenty-two postulates for inclusion in high school geometry texts [9, p. 379].

A prerequisite for this Primary Source Project (PSP) is a one-semester course in high school geometry, although three different implementation guides are offered below, depending on the students' background. The first follows the list of topics as presented in this PSP and assumes familiarity with such topics as segment duplication, angle duplication, and the triangle congruency theorems such as angle-side-angle and side-side-side. This is recommended for students with a good recall of topics in high school geometry. The second implementation guide is for students with a weak background and lists all propositions from Book I needed for an understanding of the Pythagorean Theorem. The third implementation guide is for students with a strong grasp of the Euclidean parallel postulate and its use in the proof of the converse of the alternate interior angle theorem. Each implementation schedule is based on three 50-minute class sessions per week. Each section of this PSP begins on a separate page for easy inclusion, should the instructor wish to pick and choose which sections to cover. A recommended companion text for an undergraduate course in geometry is Heath's translation of the *Elements*, presented in one complete volume edited by Dana Densmore [4]. With this text in hand, propositions from the *Elements* not presented in this PSP could be covered in class at the instructor's discretion.

Sections of this PSP presenting a proposition from the *Elements* begin with a statement of purpose, briefly describing how the result fits into the proof of the Pythagorean Theorem. This is followed by the prerequisite material needed for the constructions of the result. The prerequisite material may be used without proof, such as the congruency theorems for triangles. Also, the common notions and postulates of Book I may be used throughout the proofs. Next follows the statement of the proposition (or theorem) from *The Elements* that forms the core of the section. Instead of reproducing Euclid's polished proof of the result, the mini-projects outline the strategy of the proof, allowing the reader to discover the key logical connections. These proofs could form the material for class discussion or class presentations. This is followed by exercises or questions that could be assigned as homework problems. Ideally, the instructor would ask the students to read a section in advance and come to class with particular questions.

Sample Schedule I: Three-Week Implementation

Day 1. Ask the students to read Section 1, "Euclid's *Elements*," before class and prepare questions about the reading. Perhaps ask students to prepare answers to some of the easier questions at the end of the section before class. The instructor may wish to lead discussion about the *Elements* and the significance of the Pythagorean Theorem, perhaps supplementing discussion with material from Wardhaugh [12]. Ask students to submit solutions to some of the more open-ended questions at the end of Section 1 for the next class session.

Day 2. Ask students to read Section 2, "The Postulates and Common Notions of Book I," before class and prepare questions about the reading. The instructor may wish to discuss the meaning of the postulates and common notions, one by one, in class. Why are the postulates necessary? Can a geometric construction be drawn for each postulate? Do the common notions have modern counterparts? What exactly is the meaning of the fifth postulate? Time permitting, the instructor may wish to list all of the definitions of terms at the beginning of Book I [4, pp. 1–2]. Ask students to submit solutions to some of the questions at the end of Section 2 for the next class session.

Day 3. Ask students to read Section 3, "Construction of Perpendiculars," in advance and prepare questions about the reading. Discuss the proof strategy for Proposition 11, the construction of perpendiculars, in class. Assign the actual proofs, requested in the exercises, for the next class session.

Day 4. Ask students to read Section 4, "The Alternate Interior Angle Theorem," before class and prepare questions about the reading. Discuss the proof strategy for Proposition 27, the alternate interior angle theorem, in class, perhaps asking students what the steps of the proof would be and what reason would justify a particular step. Assign the actual proofs, requested in the exercises, for the next class session.

Day 5. Ask students to read Section 5, "The Converse of the Alternate Interior Angle Theorem," before class and prepare questions about the reading. Discuss the proof strategy for Proposition 29 in class paying careful attention to the Euclidean parallel postulate and its reformulation by Playfair. Assign the actual proofs, requested in the exercises, for the next class session.

Day 6. Ask students to read Section 6, "On the Construction of a Square." before class and prepare questions about the reading. Discuss a proof strategy for Proposition 46 in class, noting how the construction of a square requires the parallel postulate. The section also contains results about parallelograms needed later in the PSP. Assign the actual proofs, requested in the exercises, for the next class session. Without Euclid's fifth postulate, squares do not exist, nor do parallelograms.

Day 7. Ask students to read Section 7, "The Ancient Greek View of Area," before class and prepare questions about the reading. The ancient Greeks did not have an algebraic formula for the area of any figure, but instead identified when two parallelograms have the same area in terms of the parallelograms having a common base with their opposite sides along a line parallel to the base. Discuss a proof strategy for this result, Proposition 35, in class. Assign the actual proofs, requested in the exercises, for the next class session.

Day 8. Ask students to read Section 8, "The Pythagorean Theorem," before class and prepare questions about this epic theorem. Discuss the proof strategy for Proposition 47 in class. Allow one or two class sessions before the complete proof is due along with solutions to any of the companion exercises about the Euclidean distance formula or the modern trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$. These latter exercises could be assigned as in-class presentations.

Day 9. Section 9, "An Ancient Chinese 'Hypotenuse Diagram,' " is an optional section, which the instructor may wish to present in class or assign as in-class presentations. It offers an alternative proof of the Pythagorean Theorem using modern algebraic techniques applied to an ancient Chinese diagram attributed to the sixth-century CE commentator Zhen Luan.

The above implementation may be a bit ambitious, and more time could be spent on any section at the instructor's discretion.

Sample Schedule II: Five-Week Implementation

This implementation is for students who need a review of high school geometry. The propositions from the *Elements* not presented in the PSP can be covered from Heath's translation [4] [7, Vol. I]. For each day, the instructor may wish to assign advance reading of the topic and ask for the preparation of questions about the reading. For topics covered in this PSP, see Sample Schedule I above for specific suggestions.

Day 1. Cover Section 1 of this PSP, "Euclid's *Elements*." See Sample Schedule I.

Day 2. Cover Section 2 of this PSP, "The Postulates and Common Notions of Book I." See Sample Schedule I.

Day 3. Cover segment duplication, segment addition, and segment subtraction. See Book I, Proposition 2 of the *Elements*. If time, cover the construction of an equilateral triangle, Book I, Proposition 1.

Day 4. Cover the side-angle-side and side-side-side triangle congruency theorem as in Book I, Propositions 4, 7, 8. Note that modern mathematics treats the side-angle-side statement as a separate postulate of geometry.

Day 5. Cover segment bisection and angle bisection as in Book I, Propositions 9, 10.

Day 6. Cover Section 3, "Construction of Perpendiculars" of this PSP. See Sample Schedule I.

Day 7. Cover the angle sum around a line, Book I, Proposition 13, and its converse.

Day 8. Cover the exterior angle theorem as stated in Book I, Proposition 16.

Day 9. Cover the angle-side-angle and side-angle-angle congruency theorems, Book I, Proposition 26.

Day 10. Cover Section 4 of this PSP, "The Alternate Interior Angle Theorem." See Sample Schedule I.

Day 11. Cover Section 5 of this PSP, "The Converse of the Alternate Interior Theorem." See Sample Schedule I.

Day 12. Cover Section 6 of this PSP, "On the Construction of a Square." See Sample Schedule I.

Day 13. Cover Section 7 of this PSP, "The Ancient Greek View of Area." See Sample Schedule I.

Day 14. Cover Section 8 of this PSP, "The Pythagorean Theorem." See Sample Schedule I.

Day 15. Cover Section 9 of this PSP, "An Ancient Chinese 'Hypotenuse Diagram.' " See Sample Schedule I.

Sample Schedule III: Two-Week Implementation

This implementation is for students with a strong background in high school geometry.

Day 1. Cover Section 1 of this PSP, "Euclid's *Elements*." See Sample Schedule I. For advanced students, the instructor may wish to discuss the various results of geometry that depend on the Euclidean parallel postulate and how this occurs.

Day 2. Cover Section 2 of this PSP, "The Postulates and Common Notions of Book I." See Sample Schedule I. The instructor may wish to discuss the notion of independence of an axiom within a logical system, and ideas of a model for a system of axioms. What are the models for Euclidean geometry?

Day 3. Cover Section 7 of this PSP, "The Ancient Greek View of Area." See Sample Schedule I. Most students will not be familiar with the ancient Greek view of determining when two parallelograms have the same area.

Day 4. Cover Section 8 of this PSP, "The Pythagorean Theorem." See Sample Schedule I.

Day 5. Cover Section 9 of this PSP, "An Ancient Chinese 'Hypotenuse Diagram.' " See Sample Schedule I.

Historical projects in the TRIMUPHs series that would be of interest after completing this PSP include: "Failure of the Parallel Postulate," "A Look at Desargues' Theorem from Dual Perspectives," and "Generating Pythagorean Triples via Gnomons," available at `https://nscoville.github.io/website/TRIUMPHS/OurStudentProjects/Projects/projects.html`. Every Pythagorean triple with whole number values can be used to construct a Chinese "Hypotenuse Diagram."

Instructors seeking more historical information about the *Elements* might wish to adopt *Encounters with Euclid* [12] as a text, or place this book on course reserve. Readers interested in a further study of Greek geometry might wish to cover Book II of the *Elements* [4], noting how this second book ends with a geometric version of what today can be identified with the law of cosines. A natural stepping stone after finishing this PSP is to cover the next PSP in the TRIUMPHs geometry series, namely "Failure of the Parallel Postulate," cited above, that introduces the work of Lobachevski. Of course, there are many texts covering hyperbolic geometry, such as Gray [5] or Greenberg [6], to mention only a few.

The LaTeX code of this entire Primary Source Project (PSP) is available from the author by request (`jlodder@nmsu.edu`). The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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`https://blogs.ursinus.edu/triumphs/`

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