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Why be so Critical? Nineteenth Century Mathematics and the Origins of Analysis

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One striking feature of nineteenth century mathematics, as contrasted with that of previous eras, is the higher degree of rigor and precision demanded by its practitioners. This tendency was especially noticeable in *analysis*, a field of mathematics that essentially began with the "invention" of calculus by Leibniz and Newton in the mid-17th century. Unlike the calculus studied in an undergraduate course today, however, the calculus of Newton, Leibniz and their immediate followers focused entirely on the study of geometric *curves*, using algebra (or "analysis") as an aid in their work. This situation changed dramatically in the 18th century when the focus of calculus shifted instead to the study of *functions*, a change due largely to the influence of the Swiss mathematician and physicist Leonhard Euler (1707–1783). In the hands of Euler and his contemporaries, functions became a powerful problem solving and modelling tool in physics, astronomy, and related mathematical fields such as differential equations and the calculus of variations. Why then, after nearly 200 years of success in the development and application of calculus techniques, did 19th-century mathematicians feel the need to bring a more critical perspective to the study of calculus? This project explores this question through selected excerpts from the writings of the 19th century mathematicians who led the initiative to raise the level of rigor in the field of analysis.

1 The Problem with Analysis: Bolzano, Cauchy and Dedekind

To begin to get a feel for what mathematicians felt was wrong with the state of analysis at the start of the 19th century, we will read excerpts from three well-known analysts of the time: Bernard Bolzano (1781–1848), Augustin-Louis Cauchy (1789–1857) and Richard Dedekind (1831–1916). In these excerpts, these mathematicians expressed their concerns about the relation of calculus (analysis) to geometry, and also about the state of calculus (analysis) in general. As you read what they each had to say, consider how their concerns seem to be the same or different. The project tasks that follow these excerpts will then ask you about these comparisons, and also direct your attention towards certain specific aspects of the excerpts.

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Bernard Bolzano, 1817[1]

In the study of equations there are two propositions for which it could be said until recently that a fully correct proof is unknown. One is the proposition: *Between any two values of the unknown quantity that give opposite [sign] results, there must always lie at least one real root of the equation*. The other states: *Every algebraic rational entire function*[2] *of a variable can be decomposed into real factors of the first or second degree*. — After unsuccessful attempts by *d'Alembert, Euler, de Foncenex, La Grange, La Place, Klügel* among others, *Mr. Gauss* finally provided us last year with a couple of proofs of the latter theorem that should no longer leave anything to be desired. Indeed as early as the year 1799 this excellent scholar provided us a proof of this theorem, though it still had a mistake as he admitted himself, in that he based the purely analytic truth on a geometric consideration. His two newest proofs are indeed completely free of this mistake, as the trigonometric functions that appear in the last one can and should be understood in a purely analytic sense.

The other proposition that we mentioned above does not belong to those which, until now, have occupied the reflection of scholars in an excellent way. In the meantime we find that mathematicians of high esteem are tackling this theorem and have already attempted several ways of proving it. To anyone who wants to be convinced of this, compare the various presentations of this theorem that, for instance, *Kästner, Clairaut, Lacroix, Metternich, Klügel, La Grange, Rösling*, and many others have given.

A thorough examination of these proof methods very readily shows that none of them could be considered sufficient. The most common proof method relies upon a truth borrowed from geometry: Namely, *that each continuous line of a simple curve, whose ordinates are first positive then negative (or vice versa), must necessarily cut the abscissa axis at a point somewhere between those ordinates*. There are no objections whatsoever to both the correctness of and the evidence for this geometric statement. But it is also just as obvious that it is an intolerable violation of good method to want to derive the truths of pure (or general) mathematics (i.e. Arithmetic,[3] Algebra, or Analysis) from considerations that belong to only an applied (or special) part of it, namely to Geometry. ...

Augustin-Louis Cauchy, 1821[4]

As for the methods [in this text], I have sought to give them all the rigour that is demanded in geometry, in such a way as never to refer to reasons drawn from the generality of algebra. ...... One should also note that [reasons drawn from the generality of algebra] tend to cause an indefinite validity to be attributed to the algebraic formulae, even though, in reality, the majority of these formulae hold only under certain conditions, and for certain values of the variables which they contain. By determining these conditions and values, and by fixing precisely the meaning of the notations of which I make use, I remove any uncertainty; ...

[1] The translation of this excerpt from *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reele Wurzel der Gleichung liege* (*Purely analytic Proof of the Theorem, that between any two Values that give opposite [sign] Results, there lies at least one real Root of the Equation*) was prepared by Michael Saclolo, St. Edward's University, 2023.

[2] Today, we would simply say "polynomial function" in place of "algebraic rational entire function."

[3] As was not uncommon in the nineteenth century, Bolzano's use of the word "Arithmetic" here referred to the mathematical discipline that is today called "number theory."

[4] The translation of this excerpt from *Cours d'Analyse* (*Course on Analysis*) was prepared by the project author.

Augustin-Louis Cauchy, 1823[5]

My principal aim has been to reconcile rigor, which I took as a law in my *Cours d'Analyse*, with the simplicity that results from the direct consideration of infinitesimals. For this reason, I believed I should reject the expansion of functions by infinite series whenever the series obtained was divergent; and I found myself forced to defer Taylor's formula until the integral calculus, [since] this formula can not be accepted as general except when the series it represents is reduced to a finite number of terms, and completed with [a remainder given by] a definite integral. I am aware that [Lagrange] used the formula in question as the basis of his theory of derivative functions. However, despite the respect commanded by such a high authority, most geometers[6] now recognize the uncertainty of results to which one can be led by the use of divergent series; and we add further that, in some cases, Taylor's theorem seems to furnish the expansion of a function by a convergent series, even though the sum of that series is essentially different from the given function.

Richard Dedekind, 1872[7]

My attention was first directed toward the considerations which form the subject of this pamphlet in the autumn of 1858. As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic.[8] In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question until I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis.

Task 1 In what way do the concerns of these three mathematicians about the relation of calculus (analysis) to geometry, and about the state of calculus (analysis) in general, seem to be the same/different?

[5]The translation of this excerpt from *Résumé des leçons sur le calcul infinitésimal* (*Summary of lessons on the infinitesimal calculus*) was prepared by the project author.

[6]The meaning of the word "geometer" also changed over time; in Cauchy's time, this word referred to any mathematician (and not just someone who worked in geometry).

[7]The translation of this excerpt from *Stetigkeit und irrationale Zahlen* (*Continuity of irrational numbers*) is taken from [Dedekind, 1901].

[8]Unlike Bolzano's use of the word "arithmetic" to mean "number theory," Dedekind's use of the expression "scientific foundation for arithmetic" was related to the set of real numbers and its underlying structure.

Task 2 This task looks at some of the mathematical results mentioned by Bolzano, Cauchy and Dedekind.

(a) Note that:

- Bolzano discussed two specific theorems; identify these theorems.
- Dedekind discussed one specific theorems; identify that theorem.
- Cauchy made reference to the Taylor formula and related results; look back to see what he had to say, and briefly describe his concerns.

(b) Which of the results in part (a) are familiar to you?
For each that is, try to state it in "modern" terms, or give its "modern name."

(c) Which of the results in part (a), if any, do you believe to be true (and why)?

2 Niels Abel: "Hold your laughter, friends!"

In this section, we will examine an excerpt from a letter written by young Norwegian mathematician Niels Abel (1802–1829) to his high school teacher, Bernt Michael Holmboe, on January 26, 1826. Abel is often remembered for his celebrated impossibility proof in the theory of equations in which he proved that a "quintic formula" for the general fifth degree polynomial equation (akin to the quadratic formula for second degree polynomial equations) does not exist — a proof that marked an important step in the mathematical quest for algebraic solutions to polynomial equations which began with the development of Babylonian procedures for solving quadratic equations in 1700 BCE. Abel is equally well known for his work in analysis, and especially the theory of elliptic functions. In his letter to Holmboe, written during a study-abroad trip to Paris and Berlin, Abel described some of his concerns about the state of analysis in general, and particularly about the use of infinite series. The **letter itself (in English translation[9]) appears after Tasks 3—6**; read through these tasks first in order to have them in mind while you read Abel's letter; then complete your responses to Tasks 3–6 below after you've finished reading the letter.

Task 3 Find at least two references in Abel's letter to infinite series as an important concept or issue in mathematics. To what degree do the concerns that Cauchy expressed about series agree with Abel's view of series?

Task 4 What was it that Abel thought was "extremely surprising" about the state of mathematics at the time? Be specific here! Do you agree with his reaction to that state of affairs? Explain.

[9]The translations of excerpts from Abel's letter in this project were taken from [Bottazzini, 1986, pp. 87–89], with minor changes made by the project author based on the original text of the letter (pp. 15–18 of Norwegian section) and its French translation (pp. 15–19 of French section) in [Holst et al., 1902]. The English translations in [Bekken, 2003] and French translations in [Abel, 1839, pp. 266–268] have also been consulted.

Task 5 Towards the end of the excerpt that we are reading from his letter, Abel remarked that a series of the following form can be convergent for "x smaller than 1," but divergent for $x = 1$:

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

(a) Provide an example in which this occurs, specifying both the series (by giving values for the coefficients $a_0, a_1, \dots$) and the function $\phi(x)$ to which that series converges for "x smaller than 1," (This doesn't really take much work, so don't make this harder than it is!)

(b) Notice that Abel went on to speculate that an even worse situation might occur. Namely, he proposed the possibility that a series $\phi(x) = a_0 + a_1 x + a_2 x^2 + \dots$ might be convergent for "x smaller than 1" *and* convergent for $x = 1$, but in such a way that $\lim_{x \to 1} \phi(x)$ is not equal to $\phi(1)$.

What mathematical concept is involved here? That is, if such a function ϕ does in fact exist, what function property is ϕ lacking?

Task 6 Consider the following series discussed by Abel at the end of this extract:

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \text{etc.}$$

(a) Describe how this series is different from a power series.

(b) Now complete Abel's arguments concerning the numerical aspects of this series by determining what is absurd about this formula for $x = \pi$.

(c) Next complete Abel's comments about the differential aspects of this series by differentiating the formula term by term in order to show what can go wrong when one "performs every kind of operation on infinite series, as if they were finite." [Be sure to say what is wrong with the differentiation results!]

Heinrik Abel, 1826, Letter to Holmboe

Another problem that has greatly concerned me is the summation of the series

$$\cos mx + m\cos(m-2)x + \frac{m(m-1)}{2}\cos(m-4)x + \dots.$$

If m is a positive integer, the sum of this series, as you know, is $(2\cos x)^m$, but if m is not an integer, this is no longer the case unless x is smaller than $\pi/2$.

There is no other problem that has occupied mathematicians as much as this in the recent past. Poisson, Poinsot, Plana, Crelle, and many others have sought to resolve it, and Poinsot is the first who has found an exact sum, but his reasoning is completely false, and no one as yet has been able to find out why. Happily, I have succeeded with complete rigor. A memoir about this will appear in the *Journal*, and I will soon send it to France to appear in Gergonne's *Annales de Mathematiques*.

[There follows a discussion, omitted here, of some results that Abel had found concerning the above series.]

Divergent series are on the whole devilish and it is a shame to base the slightest demonstration on them. You can get whatever you want when you use them, and they are what has produced so many failures and paradoxes. Can one think of anything more horrible than to say that

$$0 = 1 - 2^n + 3^n - 4^n + etc.$$

where n is a positive integer? *Risum teneatis amici!*[10] My eyes have at last opened up in a striking way because, except for cases of the most extreme simplicity, for example geometric series, there is hardly anywhere in the whole of mathematics a single infinite series whose sum is determined in a rigorous manner. In other words, that which is the most important in mathematics is without foundation. Most things are correct, this is true; and this is extremely surprising. I strive to find the reason for this. An exceedingly interesting subject.

I do not think you can show me many propositions where infinite series appear, where I cannot make fundamental objections against their demonstration. Do it, and I will answer you. Even the binomial formula is not yet rigorously demonstrated.

[There follows a discussion, omitted here, about the Binomial Series, about which Abel had derived certain results.]

In order to show by a general example (*sit venia verbo*)[11] how poorly one can reason and how it is necessary to be prudent, I will choose the following example:

Let

$$a_0 + a_1 + a_2 + a_3 + a_4 + \text{etc.}$$

be any infinite series. You know that a very common manner of finding the sum is to take the sum of

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + etc.$$

and then to let $x = 1$ in the result. This is probably correct, but it seems to me that we cannot accept it without demonstration, because if we prove that

$$\phi(x) = a_0 + a_1x + a_2x^2 + \dots$$

for all values of x smaller than 1, it does not follow that one can say the same for $x = 1$. It is very possible that the series $a_0 + a_1x + a_2x^2 + \dots$ approaches a completely different value than $a_0 + a_1 + a_2 + \dots$ as x approaches 1. This is clear in the general case where the series $a_0 + a_1 + a_2 + \dots$ is divergent, because then it does not have any sum. I have demonstrated that this is correct when the series is convergent. The following example shows how one can err. One can rigorously demonstrate that

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - etc.$$

for all values of x smaller than π. It seems that consequently the same formula must be true for $x = \pi$; but this will give [an absurdity] ...

[10] Latin for "Hold your laughter, friends!"

[11] Latin for "pardon the expression."

. . . .

One performs every kind of operation on infinite series, as if they were finite, but is it permissible? I do not think so. Where has it been demonstrated that one can obtain the derivative of an infinite series by taking the derivative of each term? It is easy to cite examples where this is not right, for example:

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - etc.$$

. . . .

3 Conclusion

The concerns expressed by Abel, Bolzano, Cauchy and Dedekind in the excerpts we have read in this project were emblematic of the state of analysis at the turn of the nineteenth century. Ultimately, mathematicians of that century responded to this set of concerns by moving to the requirement of *formal proof* as a way to certify knowledge via the *rigorous use of inequalities* intended to capture the notion of two real numbers "being close" that underlies the limit concept. Other factors that influenced this direction included new teaching and research situations, such as the École Polytechnique in Paris, that required mathematicians to think carefully about their ideas in order to explain them to others. Today, this nineteenth-century response remains at the core of the study and practice of real analysis. The final task in this project takes one last look at the motivations of those who led the way in formulating this response, as they expressed it in their own words.

Task 7 Look back at the excerpts from the works of Abel, Bolzano, Cauchy and Dedekind that we have read in this project. What questions or comments would you address to these mathematicians about aspects of their concerns that are not addressed in the earlier tasks? (Write at least one question and at least one comment, please!)

References

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S. B. Russ. A Translation of Bolzano's Paper on the Intermediate Value Theorem. *Historia Mathematica*, 5:156–185, 1980.

H. K. Sørensen. Exceptions and Counterexamples: Understanding Abel's comment on Cauchy's Theorem. *Historia Mathematica*, 32:453–480, 2005.

4 Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed for use in an Introductory Analysis course. It has also been used in History of Mathematics courses and Capstone Seminars for mathematics majors. Its goal is to provide context for the use of rigorous proofs and precise ϵ-inequalities that developed out of concerns about the state of analysis that first arose in the nineteenth century, but which remain defining characteristics of today's analysis. Both these tools of the current trade (i.e., rigorous proof, precise inequalities) offer challenges to students of introductory analysis, who have typically encountered calculus only as a procedural and applied discipline up to this point in their mathematical studies. By offering a glimpse into the problems that motivated nineteenth century mathematicians to shift towards a more formal and abstract study of the concepts underlying these procedures and applications, the readings in this PSP provide students with a context for making a similar shift in their own understanding of these concepts. Completing this PSP early in the course can also provide students and instructors with a basis for reflection on and discussion of current standards of proof and rigor throughout the course.

Student Prerequisites

The project assumes that students are familiar with fundamental concepts from a first year calculus course, including basic results about limits and power series. However, no prior study of analysis or experience with formal proof writing is needed.

PSP Design and Task Commentary

This project consists of two main parts. In Section 1, brief excerpts from works by Bolzano, Cauchy and Dedekind paint a general picture of the issues that motivated nineteenth century mathematicians to attempt to infuse greater rigor into the study of analysis. Section 2 then examines a letter written by Abel in which he discussed concerns about infinite series in particular. Tasks 5 and 6 in the Abel section are the most technical parts of the project, but are still reasonably straightforward to complete. (Setting all coefficients equal to 1 in the series in Task 5(a) yields, for instance, a geometric series with ratio x.) Nevertheless, these two tasks can seem baffling to students who have not studied infinite series recently. Reassuring them that they should not make the questions in these tasks overly complicated can be helpful, as can some well-timed Calculus 2 reminders.

Note that none of the excerpts or tasks in this project describe how the study of analysis changed as a result of the concerns expressed by Abel, Bolzano, Cauchy and Dedekind. Rather, the quotes from these mathematicians used in this project simply lay out the worries of the day. This is intentional, in that those changes (e.g., use of $\epsilon - \delta$ inequalities, the arithmetization of analysis, increased rigor and precision in definitions and proofs) are precisely what students will encounter (and wrestle with!) throughout their introductory analysis course. The "Summary Discussion Notes" in a later section of these Notes provide some additional details that instructors may find useful in helping to make the connection between the issues raised by Abel, Bolzano, Cauchy and Dedekind in the excerpts in this PSP, and how they and others responded to these issues helped to shape analysis in the nineteenth century.

Classroom Implementation Suggestions

Classroom implementation of this project can be accomplished by way of one of the two following basic approaches; hybrids of these two methods are, of course, also possible.

- *IMPLEMENTATION METHOD I*

  Students are assigned to read the entire PSP and respond (in writing) to the tasks therein prior to class discussion, where students are provided with a copy of the project that leaves blank space below each task where they can record their final responses. Typically, the author assigns this reading one week prior to a class discussion of it; other instructors have confirmed that sufficient time for careful advance reading is important for high quality in-class discussions. Students are encouraged to discuss the readings and PSP tasks with each other or with the instructor (outside of class time) before the assigned due date (provided their written responses are their own). While there is no prohibition against using additional resources to complete the PSP (e.g., a calculus text), it is important to assure students that there is no need to do any historical research in order to complete it.

  On the assignment due date, a whole class discussion of the reading is conducted by the instructor, with student responses to various PSP tasks elicited during that discussion. An instructor-prepared handout containing solutions to select tasks (especially Task 2) can be helpful during this discussion. The completed written work is typically collected at the close of that class period; however, the discussion could also be conducted after the instructor has collected and read students' written PSP work. The author does evaluate students' individual written work for a grade. That evaluation and grade is based primarily on completeness, but also takes into account both presentation (e.g., use of complete sentences) and accuracy (particularly with regard to the mathematical details in Tasks 2, 5, 6).

  A brief set of "Summary Discussion Notes" that could be used by an instructor during a whole class discussion of the PSP is offered in the Appendix to these Notes. Although some type of summarizing discussion is highly recommended, that discussion need not adhere to the notes provided here.

- *IMPLEMENTATION METHOD II*

  Students are assigned to read only the primary source excerpts in the project as preparation for small group work on project during class time. During class time, students then work together in small groups to write their answers to the PSP tasks, with the instructor circulating between groups to facilitate that work. The completed written work is then either collected from each group at the close of that class period (and possibly evaluated for a grade), or students can be asked to write formal responses to some or all of the tasks on an individual basis (again, possibly evaluated for a grade). Instructors opting for implementation in small groups may also wish to conduct a whole-group discussion, based on select portions of the "Summary Discussion Notes" included below, at one or more junctures during implementation.

  Depending on the course and the class period length, this implementation plan may take up to 2 full class days to complete; a sample schedule for accomplishing this is provided in the next section of these Notes.

Sample Implementation Schedule (based on a 50-minute class period)

The following sample schedule, based on Implementation Method II, offers several options to help instructors tailor this mode of implementation to their course goals and available class time. Depending on the exact combination of individual/small-group/whole-class work, this method of implementation requires 1.5–2 class days (based on 50-minute class periods).

- **Advance Preparation Work[12] (to be completed before class)**
  Read the project introduction and all of Section 1; prepare answers to Tasks 1–2 for class discussion. Also read the introduction to Section 2 and the complete Abel excerpt in that section; prepare answers to Tasks 3–4 for class discussion.
- **Day 1 of Class Work**
  - Optional: Mini-lecture by instructor (about 10 minutes) to provide overview of pre-nineteenth century calculus themes (based on table in second bullet of the "Summary Discussion Notes" below); this could also be saved for a Day 2 closing discussion.
  - Small-group discussion of Tasks 1–2 (about 20 minutes).
  - Whole-class summarizing discussion of Section 1, segueing into Section 2 by soliciting students' general comments and reactions to Abel's letter (about 10 minutes).
  - Whole-class discussion of Tasks 3–4 (about 10 minutes); those who prefer could instead have students discuss these tasks in small groups.
  - Time permitting, begin individual or small-group work on Task 5.
- **Advance Preparation Work for Day 2**
  Prepare answers to Tasks 5–6 for class discussion.
- **Day 2 of Class Work (30–50 minutes)**
  - Small-group discussion of Tasks 5–6 (15–20 minutes).
  - Whole-class discussion (15–30 minutes) of Section 2 and the PSP in general (including comments on the nineteenth-century response to the set of concerns raised in the PSP, per the final bullet of the "Summary Discussion Notes" below).
- **Homework**: A complete formal write-up of Tasks 2(a), 5, 6 and 7, to be due at a later date (e.g., one week after completion of the in-class work).

[12]The author's method of ensuring that advance reading takes place is to require student completion of "Reading Guides" (or "Entrance Tickets") for which students receive credit for completion, but with no penalty for errors in solutions. Students are asked to always strive to answer each question correctly, but to think of Reading Guides as preparatory work for class, not as a final product (e.g., formal polished write-ups are not expected). Students who arrive unprepared to discuss assignments on days when group work is conducted based on advance reading are not allowed to participate in those groups, but are allowed to complete the in-class work independently. Guides are collected at the end of each class period for instructor review and scoring prior to the next class period.

A typical guide will include "Classroom Preparation" exercises (generally drawn from the PSP Tasks) for students to complete prior to arriving in class, as well as "Discussion Questions" that ask students only to read a given task and jot down some notes in preparation for class work. Students are also encouraged to record any questions or comments they have about the assigned reading on their guide and are sometimes explicitly prompted to write 1–3 questions or comments about a particular primary source excerpt; their responses to such prompts are especially useful as starting points for in-class discussions. On occasion, tasks are also assigned as follow-up to a prior class discussion.

Experience has proven the value of reproducing the full text of any assigned project task on the guide itself, with blank space for students' responses deliberately left below each question. This not only makes it easier for students to jot down their thoughts as they read, but also makes their notes more readily available to them during in-class discussions. It also makes it easier for the instructor to efficiently review each guide for completeness (or to skim responses during class for a quick assessment of students' understanding), and allows students to make more effective use of their Reading Guide responses and instructor feedback on them at a later date.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be be completed in at most 2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from `https://digitalcommons.ursinus.edu/triumphs\_topology`; all other listed projects are available at `https://digitalcommons.ursinus.edu/triumphs\_analysis`.

- *Investigations into Bolzano's Bounded Set Theorem* (David Ruch)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)
  Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)
  A second version of this prjoect suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *Understanding Compactness: Early Work, Uniform Continuity to the Heine-Borel Theorem* (Naveen Somasunderam)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Mean Value Theorem*(David Ruch)
- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Heine Barnett)
- *Euler's Rediscovery of e** (David Ruch; sequence convergence, series & sequence expressions for e)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

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APPENDIX: Summary Discussion Notes: *Why be so Critical?*

- Caution that one of the difficulties with historical readings is that the meanings of words change over time; for example, 'geometer' referred to any mathematician (not just someone who worked with geometry)
- Overview of pre-nineteenth century calculus themes

Time Period	Focus *What objects should we study?*	Primary justification of "correctness" *How do we know our mathematics is "true"?*
17th century	Calculus of CURVES (using algebra as a tool)	New methods produce results that matched "old" (known) results (obtained from geometry)
18th century	Calculus of FUNCTIONS (with physics as primary motivation) NEW QUESTION: What is a function really? Related historical controversies: Fourier Series Convergence Vibrating String Problem	Methods produce correct predictions (in physics) NEW CONCERN: Is it valid to borrow "truths" from one domain (e.g., geometry, physics) to justify truths in another (e.g., mathematics)?

- Overview of the situation at the end of 18th/start of 19th century *(Four main points, I – IV)*

  I. Increasing mistrust of "geometric" intuition as valid proof method for "analytic" truths (and more general frustration that analytic "truths" are being verified by non-analytic 'proofs")
  Ask for evidence of this in the assigned reading.

  II. Concern that existing 'algebraic' proof methods lack adequate rigor
  Ask for evidence of this in the assigned reading; two subthemes to elicit here:
  - Euclid had long been a model of rigor; nineteenth century mathematicians express desire to bring back something like an axiomatic approach as a foundation for certain knowledge
  - algebra allows too much generality (e.g., unrestricted)
    Makes it too easy to assume that properties (e.g., continuity, rationality) that hold at all "lower" values will also hold in the limit (*elicit or mention Abel power series example here*)

  III. Use of power series (in particular) lacks firm foundation
  Ask for evidence of this in the assigned reading; two mathematical points to elicit in particular:
  - Discuss current views about $\sum_{n=1}^{\infty} x^n$ (converges for $-1 < x < 1$ but diverges for $x = \pm 1$)
    Discuss Abel's use of the phrase 'x less than 1' here (where today we would write '$|x| < 1$').
  - Abel mentions we could also have convergence for $|x| \leq 1$ with $\lim\limits_{x \to 1} \phi(x) \neq \phi(1)$.
    Ask students for their answers to Tasks 4 and 5 here.

  IV. General concerns about foundations: *If we don't base calculus on power series, what do we use instead?*
  - Some possibilities (and early proponents of each):
    Fluxions (Newton) ; Infinitesimals (Leibniz) ; **Limits** (d'Alembert) ← **The "winner"!**
  - Chosen option of 'limit' raises yet another new question: What is a limit really??

- Ultimate nineteenth century response to this set of concerns:

  Require $\underbrace{\text{FORMAL PROOFS}}_{\text{as way to certify knowledge}}$ via $\underbrace{\text{RIGOROUS use of INEQUALITIES}}_{\text{as way to talk about 'being close'}}$.

  - *Historical Aside:* Another factor that influenced this direction were new teaching & research situations (École Polytechnique) that required thinking carefully about ideas in order to explain them to others.
  - This nineteenth century response, which forms the basis of the work we will do together throughout this course, is often described as 'the arithmetization of analysis'.

An optional historical aside related to Discussion Item III

The use of series and power series itself was NOT new in the nineteenth century!

- Power series had been around well before the invention of calculus; they were also part of 'pre-calculus' in the sense that, at least through the eighteenth century, understanding power series was considered a *pre-requisite* to the study of calculus.
- Newton (and others) used power series extensively as infinite polynomials that are easy to integrate and differentiate.
- An infinite series example from the 18th century: $1 - 1 + 1 - 1 \ldots = \frac{1}{2}$
  - A first "proof":
    $(1-1) + (1-1) + \ldots = 0 \qquad ; \qquad 1 - (1-1) + (1-1) + \ldots = 1$
    Series value is the average: $\frac{0+1}{2} = \frac{1}{2}$.
  - A second "proof" (endorsed by Euler, among others): $\displaystyle\sum_{n=1}^{\infty}(-1)^n = \frac{1}{1-(-1)} = \frac{1}{2}$

  For more about this and other divergent series in the 17th century, see the June 2006 entry of the MAA Online series *How Euler Did It* by Ed Sandifer (available at `http://eulerarchive.maa.org/hedi/HEDI-2006-06.pdf`).

An optional historical aside related to nineteenth century mathematicians

Commenting on his experience during a visit to Paris, Abel wrote the following to Holmboe on October 24, 1826:

> Legendre is an extremely amiable man, but unfortunately "as old as stones." Cauchy is mad and there is no way to get anywhere with him, although at present he is the [only] mathematician who knows how to treat mathematics. His works are excellent, but he writes in a very confused manner. In the beginning, I understood almost nothing that he wrote, now that's going better. ... Cauchy is extremely Catholic and bigoted. A very strange thing for a mathematician. ... Poisson is a small man with a nice little belly. He carries himself with dignity. Likewise Fourier. Lacroix is terribly bald and remarkably old. ... Otherwise I do not like the French as much as the German: the French are extremely reserved with foreigners. It is very difficult to make their close acquaintance. And I dare not count on doing so. Everyone works for himself without caring about others. Everyone wants to teach and no one wants to learn. The most absolute egoism reigns everywhere. The only thing the French look for from foreigners is the practical; no one knows how to think except [the French] themselves. The French are the only ones who can produce something theoretical. Such are their thoughts, and you can well conceive that it is difficult to attract any attention, especially for a beginner.

Translation prepared by the project author based on original text of the letter (pp. 41–42 of Norwegian section) and its French translation (p. 45 of French section), both given in [Holst et al., 1902].