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Mathematical Modeling and Optimal Control of Alternative Pest Management for Alfalfa Agroecosystems

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Contents

Chapter 0

Abstract

This project develops mathematical models and computer simulations for cost-effective and environmentally-safe strategies to minimize plant damage from pests with optimal biodiversity levels. The desired goals are to identify tradeoffs between costs, impacts, and outcomes using the enemies hypothesis and polyculture in farming. A mathematical model including twelve size- and time-dependent parameters was created using a system of non-linear differential equations. It was shown to accurately fit results from open-field experiments and thus predict outcomes for scenarios not covered by these experiments.

The focus is on the application to alfalfa agroecosystems where field experiments and data were conducted and provided by Dr. Cory Straub of Ursinus College. Alfalfa is Pennsylvania's second-most important crop and the most cultivated forage legume in the world. Predator and plant diversity can control potato leafhopper (PLH) damage to the host-plant alfalfa. The pest damage is costly and chemical pesticides are unsafe. Ultimately, the

framework provides polyculture planting strategies for farmers that are cost-effective and environmentally-safe to minimize the alfalfa damage while maximizing farmer profits.

After the validation and sensitivity analysis of the model, the project focused on designing control strategies. Steady state solutions were determined and a sensitivity analysis established the relative importance of each parameter to reduce the plant damage. Optimal control theory led to designing practical strategies regarding diversity levels to minimize the plant damage while preserving plant nutritional efficacy and minimizing production costs. More specifically, Bang-Bang controls were investigated using switching functions to produce practical discontinuous functionals for the control parameters. Tools used include non-linear systems of differential equations, Hamiltonians, adjoints, Pontryagin's Maximum Principle, and computer simulations.

Chapter 1

Introduction

1.1 Mathematical Modeling

Mathematical modeling is a method of simulating real-life situations with mathematical equations to predict future behavior. In the last decade, it has gained momentum in biology due to advances in dynamical systems, computing capabilities, and access to data. Biologists and mathematicians have collaborated to better understand complex biology and ecosystems, allowing mathematical models to guide expensive and extensive field experiments. In comparison to the experimental biologists' conduct, a mathematician's model built using data from an experiment could provide useful information and analysis in a fraction of the time and costs, without harming the environment. Predictions can be drawn from mathematical models to guide field experiments to gain better and more efficient results.

In our mathematical models, the relationships between different variables and parameters are presented through a system of differential equations. Analysis and computer simulations can then highlight the effects of the variables and parameter interactions as well as the impact each parameter will have on the entirety of the modeled ecosystem. Scientists and mathematicians utilize models for deeper comprehension and insight of a complex system to provide informed predictions, utilize multidimensional data, and reveal characteristics which would otherwise remain unknown, leading to better designs for future experiments and breakthrough results.

1.2 The Ecological Problem

The alfalfa plant is Pennsylvania's second-most important crop and the most cultivated forage legume in the world [10]. The crop is a valuable source of nutrition for cows and other farm animals. The problem begins when the potato leafhopper (PLH) pest eats the leaves of the alfalfa plant. Their consumption causes the alfalfa to lose nutrition and turn its usual green leaf pallets to yellow; a changed appearance called "hopper burn". Reduction in alfalfa produced and loss in nutrition mean farmers cannot benefit from, sell, or use as much of their crop, a reduction in cows' milk production, and a shorter lifespan of alfalfa before needing to plant a new crop. Thus, farmers must endure severe monetary loss each year. The damsel bug, *Nabis sp.*, is a natural predator of the PLH, and can be used to control the population of PLH and thus control the damage to the alfalfa. Other control methods include the use of potentially dangerous pesticides to reduce the amount of pests present in

their fields which is ecologically harmful, expensive, and can lead to a lower milk quality and cost in revenue [4], [10].

By creating accurate mathematical models, it is possible to identify management strategies that can reduce this plant damage while taking into consideration many factors, such as the cost effectiveness, the impact on the environment, and the overall revenues for farmers.

1.3 The Resource Concentration and Enemies Hypotheses

1.3.1 Background

In agriculture, farmers use monoculture fields, those that contain only one type of plant species. However, pest outbreaks are common in this setting. One setting in which these outbreaks are not usually observed is in a polyculture field, one that contains more than one type of plant, including grasses. Polyculture fields are most commonly found in nature [4].

Farmers prefer monoculture fields because they yield more product than that of the polyculture fields due to the extraneous plants intermingled with the main crop. It may be beneficial to use well-designed polyculture as a farming technique to minimize pest damage. However, there are obvious limitations on the use of polyculture by farmers, including predicting economical and workload factors. Nevertheless, using polyculture as a planting technique may

allow farmers to be able to lessen or eliminate the use of pesticides, prolong the lifespan of

the alfalfa plant field, reduce farming costs, and increase revenue through 'organic' labels for

the plants and milk produced.

1.3.2   The Resource Concentration Hypothesis (RCH)

The Resource Concentration Hypothesis (RCH) states that increased diversity in crops can

reduce herbivorous pest outbreaks. According to this theory, alfalfa planted in a monocul-

ture crop will endure more damage than alfalfa planted in a polyculture environment. The

diversity in the field should be able to control the population of the pest as the RCH states

that pests' host plants should be more difficult to locate when intermingled with non-host

plants. A monoculture crop would cause an ever-increasing amount of pests, while a poly-

culture field would be able to control the population. With fewer pests, there should be less

damage [9].

1.3.3   The Enemies Hypothesis

The Enemies Hypothesis states that predators are more abundant and effective in areas of

higher plant diversity, thus controlling pest outbreaks. In other words, predators are more

effective in preying upon pests in polyculture environments rather than monoculture. There

are two parts to the theory: (a) predators kill herbivores at higher rates in polyculture fields

than in monoculture crops, and (b) the higher mortality rates in polyculture fields signifi-

cantly reduce herbivore populations. Similar to the Resource Concentration Hypothesis, the Enemies Hypothesis looks to the reduction or control of the pest population for the reduction of plant damage [9].

1.4 Data Overview

Dr. Cory Straub of the Ursinus College Biology Department investigated the Resource Concentration and Enemies hypotheses in field experiments. A mesocosm experiment was conducted in which there were four experimental treatments, all with PLH present: one monoculture without *Nabis*, one monoculture with *Nabis*, one polyculture without *Nabis*, and one polyculture with *Nabis*. The polyculture contained alfalfa and orchardgrass. The experimental results were consistent with the hypotheses that increasing plant diversity can reduce herbivore abundance and protect plants from herbivores. In addition, *Nabis* did provide greater protection to the plants in polyculture than in monoculture.

In summer 2011, the sample size was increased into twelve larger open-field plots. These plots were planted in alternating monoculture and polyculture fields. The 2011 data are the focus of this project. For more information on the actual experiments, see [2], [3].

Chapter 2

Open-Field Experiment

2.1   Model, Variables, and Parameters

In this project, deterministic models were utilized. A deterministic model is a mathematical model in which outcomes are precisely established through known relationships among states and events without any room for random variation. When changes of the variables are part of the relationships, one of the best ways to create a model is to use a system of differential equations.

In order to create the model, one must identify the variables and parameters. A variable is a quantity that changes over time, while a parameter is a quantity that usually remains constant over time. Next, the relationship between the different variables and parameters must be found and described. A flow diagram illustrates how each variable affects its own

dynamics and those of other variables. The dynamics of a system is the pattern of changes that occurs over time. After the model is drawn, a system of differential equations is derived and solved. The system of differential equations is non-linear and cannot be solved analytically. Numerical solutions can be produced and visualized using computer simulations. Simulations are then compared to the actual graphs from the field experiments.

Initial assumptions of this mathematical model include:

1. Temperatures surrounding and within the system are ideal for the regular growth of all three actors: *Nabis*, PLH, and alfalfa.

2. Neither the pest nor the predator migrates in or out of the system; all insects only leave the system through death. This is consistent with the field experiments where the data were collected in the middle of the field, far from the border.

3. For *Nabis*, eggs and nymphs will proceed to the next life stage at rates following natural evolutionary and resources factors.

4. For the PLH, eggs and nymphs will mostly die because of predator consumption and will not face mortality due to other causes.

5. Adult *Nabis* consume more adult PLH than *Nabis* nymphs do.

These assumptions are very reasonable and do not significantly affect the given results.

The three variables used in the model shown in figure 2.1 are the adult predator (Y), adult pest (X), and plant damage (Z). The twelve parameters are included below in table 2.1.
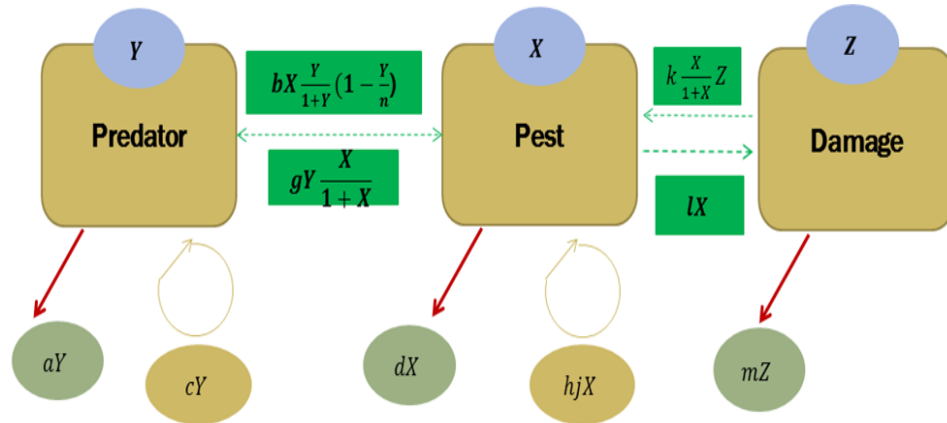
Figure 2.1: Open Field Experiment Model - Flow Diagram

For more information regarding how the parameters were calculated, see [5].

Table 2.1: Open-Field Experiment Parameters

Parameter	Description	Monoculture Base Case	Polyculture Base Case
a	predator adult mortality rate	0.25	0.25
b	predator adult consumption rate of PLH adults	0.24	0.24
c	predator adult population gain from nymphs	0.12	0.12
d	PLH adult mortality rate	0.385	0.3
g	PLH adult mortality rate by adult Nabis harm	0.3	0.3
j	PLH adult population gain from nymphs	0.5	0.512
k	PLH adult benefit from alfalfa	0.0001	0.0001
l	PLH adult consumption rate of alfalfa	0.0027	0.003
H	Shannon-Weiner Diversity Index	0	$\ln(2)$
m	recovery coefficient of alfalfa in absence of damage	0.0001	0.0001
n	carrying capacity	4	4
q	actual plant diversity	0	varies

2.2   System of Differential Equations

Using modeling approaches, the following system of differential equations is derived:

$$\begin{cases} \frac{dY}{dt} & = -aY + bX(\frac{Y}{1+Y})(1 - \frac{Y}{n}) + cY \\[2mm] \frac{dX}{dt} & = -dX - gY(\frac{X}{1+X}) + f(H)jX + k(X\frac{Z}{1+X}) \\[2mm] \frac{dZ}{dt} & = lX - mZ \end{cases} \tag{2.1}$$

This system and model is based on those developed by M. Yahdi et al. in [5] and is the focus

of the project.

The model implicitly incorporates age-structure. This implicit age structure includes the im-

pact of plant diversity on the life cycles for both the predator *Nabis* and the herbivore-prey

PLH, especially through the impact of the plant diversity, $f(H)$. Other modeling techniques

used to create this system of differential equations included logistic, scramble competition,

and the Allee effect. Scramble competition states that the survival and reproduction of indi-

vidual organisms declines as the density of the population rises. The Allee effect declines the

positive correlation between population density and individual fitness. The logistic approach

takes into consideration the limited resources and the impacts of the interactions on changes

of each population class. These modeling approaches allowed the models to accurately rep-

resent the dynamics between populations of the PLH and the *Nabis* as well as the alfalfa

damage and plant diversity.

For the models created, it is assumed that the only way to leave the system is through death.

No migration was considered in particular because sampling was used in the middle of the

crop in the open field experiments. For the PLH life cycle, only death due to predator con-

sumption is considered because of the shorter experiment time period. For more information

on how the model was created and additional models, see [5].

2.3 Plant Diversity Index

The plant diversity index is a mathematical measure of plant diversity. This index is used to develop functions that measure the impact of that index on the interactions between the pest PLH, predator *Nabis*, and the plants in the ecological systems of alfalfa fields.

The Shannon-Weiner Diversity Index H is used in this model. The measure of the diversity is based on species richness, the number present, species evenness, and the distribution of the number of organisms per species. The index H is defined by

$$H = -\sum_{i=1}^{S} p_i \ln(p_i)$$

where S is the total number of species in the community and p_i is the proportion of S made up of the ith species.

Since $S = 1$ for monoculture, H is zero. With $S = 2$ for our polyculture plot, alfalfa and orchardgrass, with the minimum of 50% alfalfa, $H = \ln(2)$. For more information regarding the calculation and use of the diversity index, see [5].

Chapter 3

Steady State Analysis

3.1 Background

Long-term behavior studies the steady states of the system. When simulations such as those in Appendix A.1 are run, the system shows stable behavior over long periods of time. In reality, the simulation shows the average value of these percentages so the actual values fluctuate around these averages. However, since we are looking into the big picture of the trends, these fluctuations are relatively small and thus the percentage values can be regarded as accurate representations of the trends.

Studying the steady states of Y, X, and Z means each derivative is zero. This leads to the study of equilibria of the system of differential equations, by solving for the derivatives equal to 0 [8]. Therefore, the system of differential equations in (2.1) can be transformed into the

system shown below:

$$
\begin{cases}
0 & = -aY + bX\left(\frac{Y}{1+Y}\right)\left(1 - \frac{Y}{n}\right) + cY \\
0 & = -dX - gY\left(\frac{X}{1+X}\right) + f(H)jX + k\left(X\frac{Z}{1+X}\right) \\
0 & = lX - mZ
\end{cases}
\tag{3.1}
$$

This system of equations (3.1) is solved numerically and typical non-zero solutions for a given set of parameters. One of the immediate goals of studying the long-term behavior is to control the plant damage and to determine which parameters have the largest effect on reducing plant damage and pest outbreaks.

3.2 Results

There are three possible steady state solutions, but only one makes sense in context. In the first case, $Y = X = Z = 0$, predator, pest, and damage all equal 0. In this case, nothing is happening in the system and it is not interesting to analyze. In the second case, both pest and damage exist in system, but the predator is not present. This could be a possible case in theory, but when the parameter values are substituted into the steady state solution, the values are negative. Therefore, this is not considered as a possible case.

Finally, the third case, which is the only possible non-zero steady state, allows predator, pest, and damage to cooperate. When specific monoculture and polyculture parameters are used, the steady states solutions are given in figures 3.1, 3.2, and 3.3.
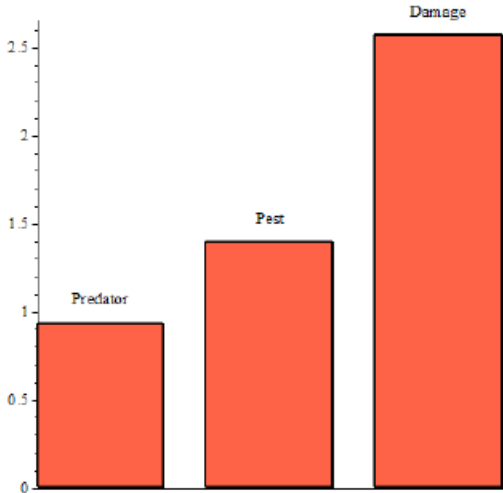
Figure 3.1: Monoculture - Steady State Solutions with monoculture base case parameters from Table 2.1. The values represent the magnitude of the respective variable.
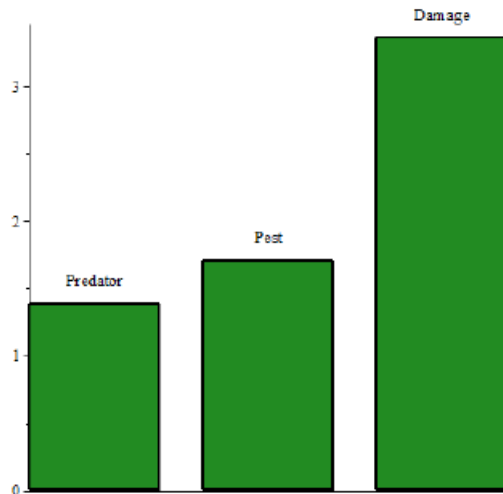
Figure 3.2: Polyculture - Steady State Solutions with polyculture base case parameters from Table 2.1. The values represent the magnitude of the respective variable.
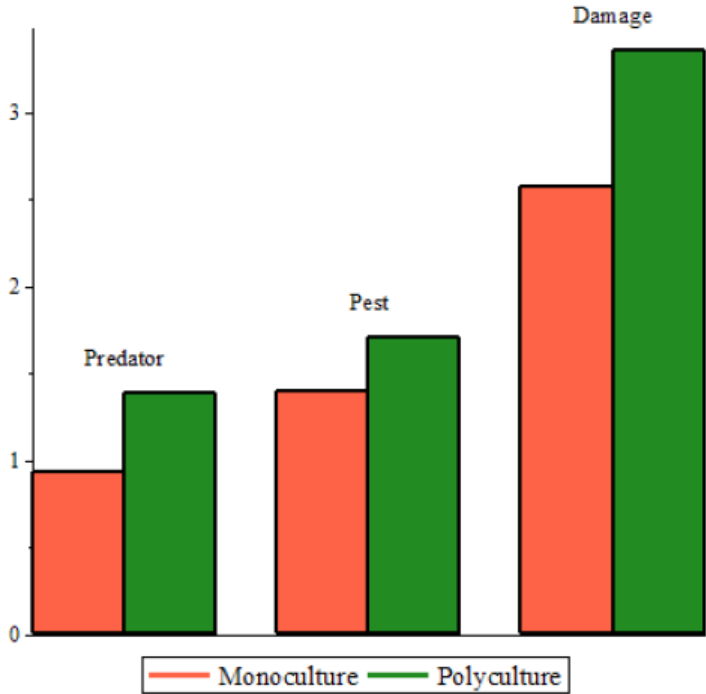
Figure 3.3: Monoculture and Polyculture Comparison - Steady State Solutions (with parameter values used in Figures 3.1 and 3.2). The values represent the magnitude of the respective variable.

It is unexpected that the damage is greater in polyculture than in monoculture due to the increase in predators in a polyculture environment. However, if we take into consideration the relative numbers of pests, predators, and damage, we find the following interesting results:

- There is a 6% decrease in the amount of damage per pest from monoculture to poly-culture.

- There is a 17% increase in the number of predators per pest from monoculture to polyculture.

The reduction in alfalfa damage per potato leafhopper is expected under the Resource Con-centration Hypothesis. While Figure 3.3 shows that there is more damage in polyculture, the pests are less efficient at causing damage. Also, the steady state analysis supports the Enemies Hypothesis that the impact of the predator *Nabis* in reducing the alfalfa damage is significantly more important and greater (about three times) compared to the PLH pest in polyculture fields. A polyculture field is able to attract more predators for each pest.

For more information regarding the calculations behind the steady state analysis, see Ap-pendix A.1.

Chapter 4

Sensitivity Analysis

4.1 Background

Mathematical models are built under uncertainties, including uncertainties in the parameter values and in the choice of model structure scenarios. Uncertainties can also be related to inherent randomness including the input parameters and output variables. It is important to have a clear understanding of the types and contributions of uncertainty in the model to give an elaborate interpretation of the model dynamics and results.

Sensitivity analysis techniques help in investigating the way uncertainties of "different orders" propagate on the output variables and on the inference or conclusions which the model is called to support. The analysis quantifies the variability of the outputs caused by uncertainties in parameter inputs and the incomplete knowledge of the model. Sensitivity analysis

aims at establishing the relative importance of the input factors (and in their ranges) involved in the model, answering questions such as:

1. Which of the uncertain input factors/parameters are most influential?

2. If the uncertainty of one of the inputs could be eliminated, which one should be chosen in order to reduce the variation without affecting the results?

3. Are there factors whose effect is so low that they can be confidently fixed anywhere in their ranges of variation without affecting the results?

Sensitivity analysis can help the modeler to measure model adequacy and relevance, to identify critical regions in the input space, to detect interactions between factors, to establish priorities for research, and to simplify the model structure. An ideal sensitivity analysis method should be sensitive to the range of variation of each input factor and to the shape of its probability distribution. It should also operate simultaneously on all uncertain inputs so that interactions among factors can be determined [1].

There are multiple types of sensitivity analyses. The local approach was utilized in this project. In this approach, the effect of the variation of a single factor is estimated while keeping all the others fixed at their nominal values. A more ideal approach would be to use the global method which estimates the effect of the output of a factor when all the others are varying, enabling the identification of interactions in non-linear and/or non-additive models. Generally, global approaches allow the use of the model-independent methods as they do not require assumptions of additivity or linearity. However, the reason that the global

approach was not utilized was due to the fact that they are computationally expensive to estimate, while local methods can be set to produce system derivatives with a number of model simulations much lower than the number of derivatives to be estimated.

This chapter shows estimations of the effect of each parameter on the system for any random variations in all factors. Advantages of this investigation include highlighting key parameters that are important to the model while considering imprecisions. The goals include determining values and roles of the independent parameters, quantifying the influential parameters, focusing data collection efforts to reduce uncertainty in those key parameters, and outlining strategies to reduce alfalfa damage and pest outbreaks.

4.2   Results

By introducing a small change on one parameter, we want to measure the change at the equilibria of the total alfalfa damage Z and then normalize these measures to determine the parameters that are most sensitive. Given the fixed set of eleven parameters (plant diversity q was excluded here), the sensitivity of the total alfalfa damage to a parameter y at a given value y_0 is given by the partial derivative $\frac{\partial (Z)}{\partial y}$ evaluated at y_0 and then normalized. Numerical approximations represent the summary results when each parameter y is increased or decreased slightly using a set of small percentage values. For methods, see Appendix A.2.

Based upon the sensitivity analysis (Figures 4.1-4.3), we conclude that parameters, d, PLH adult mortality rate; b, *Nabis* adult consumption rate of PLH adults; and c, *Nabis* adult
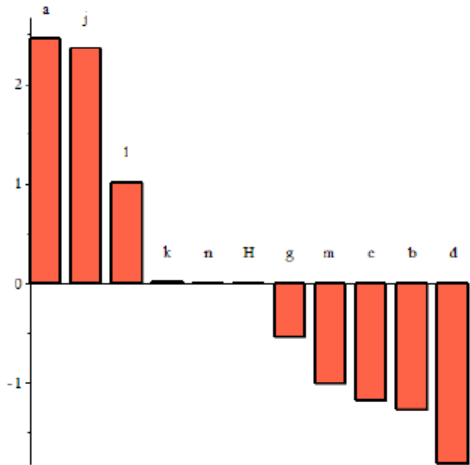
Figure 4.1: Monoculture - Normalized Sensitivity Analysis for the Base Case. Larger amplitude indicates higher sensitivity. Positive indicates an increase in damage for an increase in the parameter. Negative indicates a decrease in damage for an increase in the parameter.
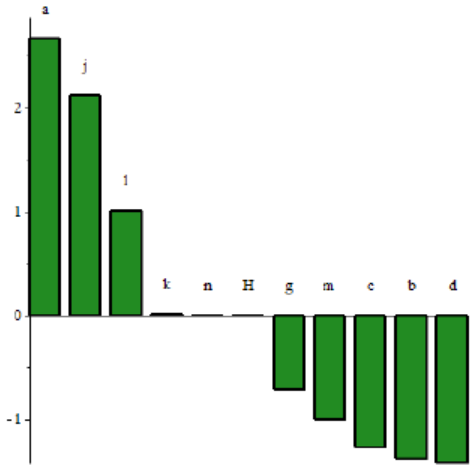
Figure 4.2: Polyculture - Normalized Sensitivity Analysis for the Base Case. Larger amplitude indicates higher sensitivity. Positive indicates an increase in damage for an increase in the parameter. Negative indicates a decrease in damage for an increase in the parameter.
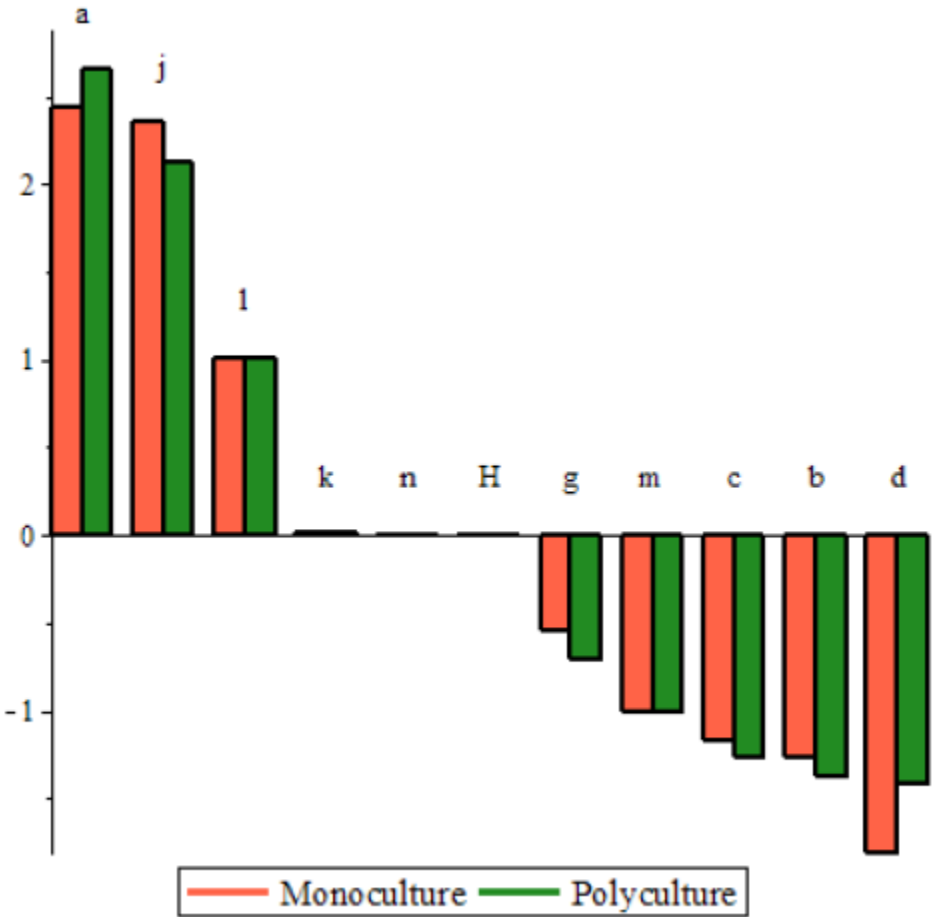
Figure 4.3: Monoculture and Polyculture Comparison - Normalized Sensitivity Analysis for the Base Case

population gain from nymphs have the largest effect in decreasing the plant damage when all of the parameters are at their mean values. It makes sense that the high PLH mortality rate, d, and high *Nabis* consumption rate of PLH, b, would cause a positive change in the ecosystem. Obviously, if PLH are dying due to *Nabis* consumption, there will be a decrease in alfalfa damage. The other parameter, c, *Nabis* adult population gain from nymphs is a little less trivial. When more adult *Nabis* mature, there will be more *Nabis* to consume PLH. Since more PLH will be consumed, there will be less damage. Thus, it is crucial to find ways to promote growth from nymphs to adults or at least to limit factors that can prevent that growth such as monoculture and pesticides.

On the other hand, parameters a, *Nabis* adult mortality rate and j, PLH adult population gain from nymphs have the largest effects in increasing the alfalfa damage. It is clear that as *Nabis* die, there will be less to control PLH, resulting in an increase in alfalfa damage. Finally, PLH adult consumption gain from nymphs, j, seems less trivial. As more PLH age, there will be more pests to cause more damage. Thus, it is critical to find a way to prevent PLH growth from nymphs to adults. However, if pesticides are that mean, this will also affect the *Nabis* population and especially the parameter c described above. It is thus crucial to evaluate the relative impact of such strategy (pesticides) on *Nabis* versus PLH in monoculture and polyculture field experiments.

This analysis suggests pesticides will be counterproductive. Indeed, on one hand the pesticides may kill both the PLH and the predator. On the other hand, the impact on the mortality rate of the predator is higher than the impact on the mortality rate of the PLH. Denote by A the increase in alfalfa damage for a slight increase in predator adult mortality

rate a. Denote by D the decrease in alfalfa damage for a slight increase in the PLH adult mortality rate d. It is clear from Figure 4.3 that A's magnitude is significantly larger than D's. Therefore, there will be a greater increase in alfalfa damage if pesticides are used since the predators' death will be impacted more strongly.

Moreover, a comparison of the changes in sensitivity of the parameters between monoculture and polyculture shows a higher sensitivity for the parameter a in polyculture fields. This means that for a similar increase in the predator adult mortality rate a (for example via pesticides), the alfalfa damage increase is worse in polyculture compared to monoculture fields. On the other hand, an increase of the mortality rate of the PLH d leads to a smaller decrease in alfalfa damage for polyculture versus monoculture fields. There is then more evidence indicating that the impact of the pesticides on alfalfa damage is worse for polyculture compared to monoculture fields.

In addition to the monetary, environmental and health costs and risks associated with pesticides, these counterproductive aspects of pesticide usage in alfalfa fields represent significant incentives for farmers to avoid or to reduce the use of pesticides. In a future work, field experiments could be conducted to confirm these findings.

Chapter 5

Optimal Control Theory

5.1   Background

Optimal control, also known as dynamic optimization, is used with mathematical modeling to find the best outcome using a predetermined goal or goals [6]. Many times, the problem will include tradeoffs between two competing factors and where parameters are transformed into control functions called control variables. Using optimal control, the state variable(s) and control variables(s) are used to maximize or minimize the objective function. Ultimately, optimal control determines the control(s) to optimize a specific set of goals in a dynamical system.

The behavior of the underlying dynamical system is described by a state variable(s). We assume that there is a way to steer the state by acting upon it with a suitable control

(a) Non-Singular Bang-Bang Opti-   (b) Singular Bang-Bang Op-

mal Control   timal Control
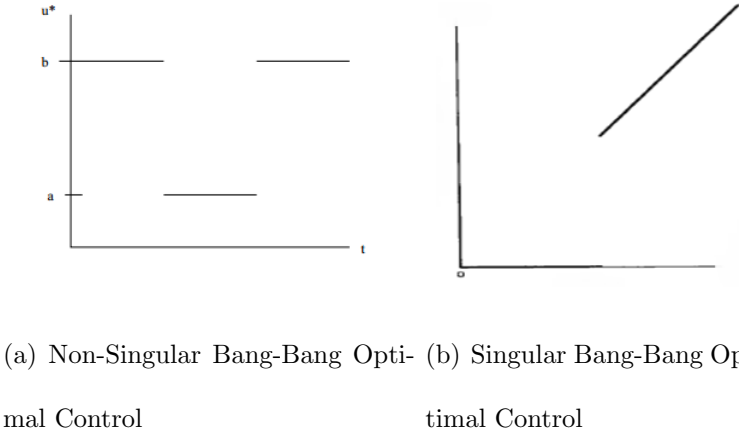
Figure 5.1: Types of Optimal Control

function(s). The control enters the system of ordinary differential equations and affects the

dynamics of the state system. The goal is to adjust the control in order to either maximize

or minimize a given objective functional. A functional refers to a map from a certain set

of functions to the real numbers. Often, this functional will balance judiciously the desired

goal with the required cost to reach it and other constraints. The cost may not always

represent money, but may include side effects or damages caused by the control. In general,

the objective functional depends on one or more of the state and the control variables [6].

There are different approaches to optimal control analysis. Two types explored in this project

are Bang-Bang Optimal Control and Singular Optimal Control. Bang-Bang Optimal Control

is a type of control in which the control variables are either at their maximum or at their

minimum values in a discontinuous fashion, like a chemotherapy treatment schedule like in

Figure 5.1(a). Singular Optimal Control is typically a function that does not necessarily

take just maximum or minimum values of the control like in Figure 5.1(b).

5.2   Optimal Control and Alfalfa Damage

Our goal is to minimize the alfalfa damage and pest abundance while also minimizing plant diversity and the loss of production of alfalfa over a period of T days. The reason is that farmers are reluctant to use polyculture farming approaches due to the belief of a loss in productivity and revenues as described in Chapter 1. The goal here is to consider the limitations of the benefit of polyculture and pesticide free alfalfa on increasing the revenues even with reduced alfalfa yield. For this case, the objective function is given by

$$\min[\int_0^T (A \cdot Z(t) + B \cdot X(t) + C \cdot (U(t) + Q(q(T))))] \tag{5.1}$$

The variables inside the integrand are alfalfa damage, $Z(t)$, and pest abundance, $X(t)$. There is one control, $U(t)$ which is the increase in plant diversity, bounded by $[0, 50]$ for 0% to 50% alfalfa. The loss of productivity, $Q(q(T))$, measures how the system will be affected by using a polyculture environment. The terminal time, T, is typically the time frame relevant to farmers, ranging from one year to the whole life of an alfalfa crop before replanting. The coefficients A, B, and C are relative costs associated with each respective variable.

The corresponding Hamiltonian, H, the integrand of the objective function plus the state equations multiplied by the adjoint, is given by

$$\Gamma = A \cdot Z(t) + B \cdot X(t) + C \cdot (U(t) + \tfrac{Q_0(T) - Q(q(T))}{T})$$

$$+ \lambda_1 \cdot (-aY + bX(\tfrac{Y}{1+Y})(1 - \tfrac{Y}{n}) + cY)$$

$$+ \lambda_2 \cdot (-dX - gY(\tfrac{X}{1+X}) + f(H)jX + k(X\tfrac{Z}{1+X})) \quad\quad (5.2)$$

$$+ \lambda_3 \cdot (lX - mZ)$$

Each of the three variables is paired with a corresponding adjoint, λ_i, and determined from

the costate equations and the transversality condition given by

$$
\begin{cases}
\dfrac{d\lambda_1}{dt} &= -\dfrac{\partial H}{\partial Z} \\[2mm]
\dfrac{d\lambda_2}{dt} &= -\dfrac{\partial H}{\partial X} \\[2mm]
\dfrac{d\lambda_3}{dt} &= -\dfrac{\partial H}{\partial U}
\end{cases}
\quad\quad (5.3)
$$

$$\lambda_i(T) = 0, \quad\quad (5.4)$$

where T is the final time and i is the index running through all three adjoints.

These adjoints can also be described through the following costate equations.

$$
\begin{cases}
\lambda_1' = -\lambda_1\left(-a + \frac{bX(t)(1-\frac{Y(t)}{n})}{1+Y(t)} + c\right) - \lambda_1 Y(t)\left(\frac{-bX(t)(1-\frac{Y(t)}{n})}{(1+Y(t))^2}\right. \\[2mm]
\left. - \frac{bX(t)}{(1+Y(t))n}\right) + \frac{\lambda_2 X(t)g}{1+X(t)} \\[2mm]
\lambda_2' = -B - \frac{\lambda_1 Y(t)b(1-\frac{Y(t)}{n})}{1+Y(t)} - \lambda_2\left(-d - \frac{gY(t)}{1+X(t)}\right. \\[2mm]
+(1 - 0.1546[(1 - q(1+U(t)))\ln(1 - q(1+U(t))) \\[2mm]
+q(1+U(t))\ln(q(1+U(t)))](1+U(t))j + \frac{kZ(t)}{1+X(t)}\right) \\[2mm]
-\lambda_2 X(t)\left(\frac{gY(t)-kZ(t)}{(1+X(t))^2}\right) - \lambda_3 l \\[2mm]
\lambda_3' = -A - \frac{\lambda_2 X(t)k}{1+X(t)} + \lambda_3 m
\end{cases}
\tag{5.5}
$$

We want to check if this leads to a Bang-Bang Optimal Control. Using the equation for the optimality condition which gives the boundaries of the constraints, u^*, we determine the switching function, Ψ, which is defined by

$$
\Psi = -\frac{\partial H}{\partial u}.
\tag{5.6}
$$

For this optimal control problem, Ψ is

$$
\Psi = C + 0.1546\lambda_2 jq \cdot \ln\left(\frac{1}{q(1+U(t))} - 1\right).
\tag{5.7}
$$

We need to know if Ψ can equal 0 at any point. In fact, we can prove that in our case, Ψ is always positive. Indeed,

$$
C + 0.1546\lambda_2 jq \cdot \ln\left(\frac{1}{q(1+U(t))} - 1\right) = 0
$$

$$
\implies U(t) = \frac{-C}{0.1545\lambda_2 jq \cdot \ln\left(\frac{1}{q(1+U(t))} - 1\right)}
$$

We see that the denominator is positive if $\ln(\frac{1}{q(1+U(t))} - 1) > 0$ since all of the other terms are positive parameters. Thus, we have

$$\frac{1}{q(1+U(t))} - 1 > 1$$

$$\implies \frac{1}{q(1+U(t))} > 2$$

$$\implies \frac{1}{q} > 2(1+U(t))$$

$$\implies \frac{1}{2q} - 1 > U(t)$$

Since q is the plant diversity, we know that it will be between 0% and 50%. Thus, we know that if $0.01 < q < 0.50$, then $0 < \frac{1}{2q} - 1 < 49$. We also know that $q(1+U(t))$ must be between the same constraints since it still applies to the plant diversity. Thus, we have proven our result and the optimality condition is reached when the plant diversity is at its maximum, 50% [6].

Using the above information and the computer simulation program, Matlab, three different optimal controls were simulated, with initial diversity of 1, 10, and 20% diversity respectively.

5.3   Results

For each of the three cases, simulations were run with and without the optimality condition. Figure 5.2 is a comparison of the 20% plant diversity with and without the control.

The benefit of optimal control is easily seen in these graphs. Using optimal control, the overall damage is reduced and the quantity of predators and pests does not spike. Instead,
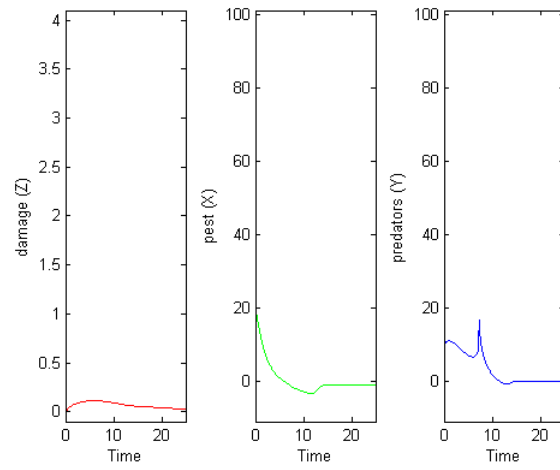
(a) With Optimal Control
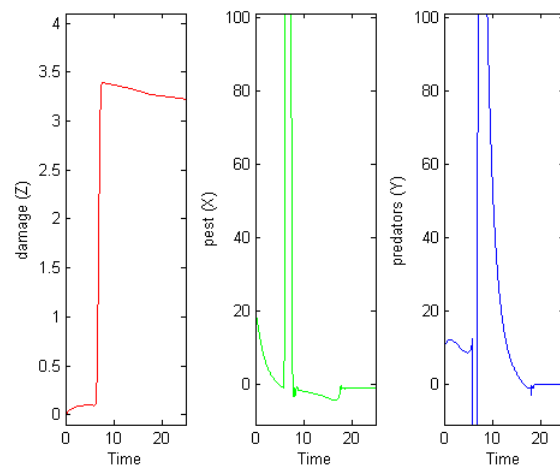
(b) Without Optimal Control

Figure 5.2: 20% Plant Diversity With and Without Optimal Control

the optimal control keeps everything to a minimum without pest outbreaks and spikes in alfalfa damage. This means that in order to reduce the damage done to the alfalfa, our best strategy would be to use plant diversity. Thus, for the remainder of this paper, the focus will be on utilizing the optimality condition with plant diversity.

The next comparison run was on the costs associated with optimal control. For the damage, pests, and increase in plant diversity, the associated costs, A, B, and C in equation (5.1), could be altered to reflect monetary, environmental, societal, and labor costs. It was found that regardless of the changes in the costs, the graphs remained identical. Thus, the only major alteration to the system happens when we alter the initial conditions of the predator, pest, and damage.

The final comparison is between varying plant diversities, 1%, 10%, and 20% as depicted in Figure 5.3. For each case, we can observe the changes in the plant damage, pest, and predator populations. The damage begins to increase, but eventually decreases in the end. Both the predator and pest tend toward an equilibrium. As a whole, these simulations can be compared to find an optimal solution that reduces plant damage. However, just looking at the amount of damage reduced in each case setting is not enough to tell a farmer that it will be the best strategy for his alfalfa field. Instead, we must compare this control of plant diversity and loss of productivity to the farmer's potential revenue.
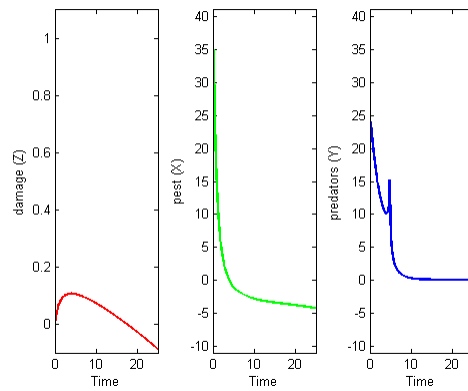
(a) 1% Plant Diversity
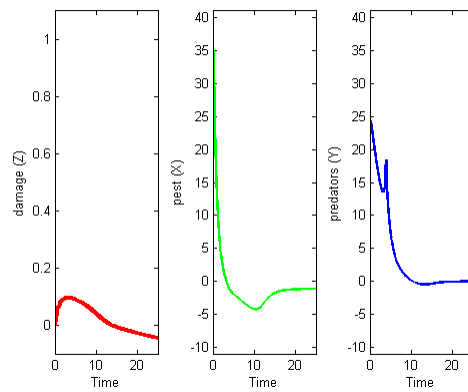
(b) 10% Plant Diversity
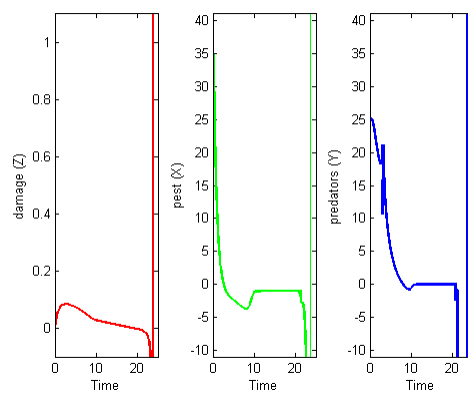
(c) 20% Plant Diversity

Figure 5.3: Comparison of Optimal Control in Varying Plant Diversities

5.4   Extended Optimal Control

The final optimal control considered the extension to the alfalfa cutting cycle and life cycle. Alfalfa is cut on average every twenty-five days during growth season. Each alfalfa planting can be cut four times a year for four consecutive years. Thus, every alfalfa crop planted can be cut approximately sixteen times. This optimal control looks into this sixteen cycle process.

Four different scenarios have been created and shown in Figure 5.4. The first in Figure 5.4(a) considers a system in which the alfalfa holds at a steady 10% for all four years. Thus, a farmer would need to go into his plot and remove the extra growth of grasses over that of 10%. Figure 5.4(b) considers a system of alfalfa where grasses increase from 10% to 20% incrementally over the course of the four years assuming the natural expansion of grasses and plants other than alfalfa do not limit the growth of alfalfa. In both cases, there is a near exponential growth of the pest population. Thus, the pests will continue to increase until carrying capacity is reached. In this case, the plant damage will also continue to increase. Thus, 10% plant diversity is not enough to limit pest damage.

In Figures 5.4(c) and (d), similar cases can be seen. Figure 5.4(c) considers a system in which alfalfa holds at a steady 20% for all four years. The second in Figure 5.4(d) considers a system of alfalfa where grasses increase from 20% to 40% incrementally over the course of the four years assuming the natural expansion of grasses and plants other than alfalfa do not limit the growth of alfalfa.
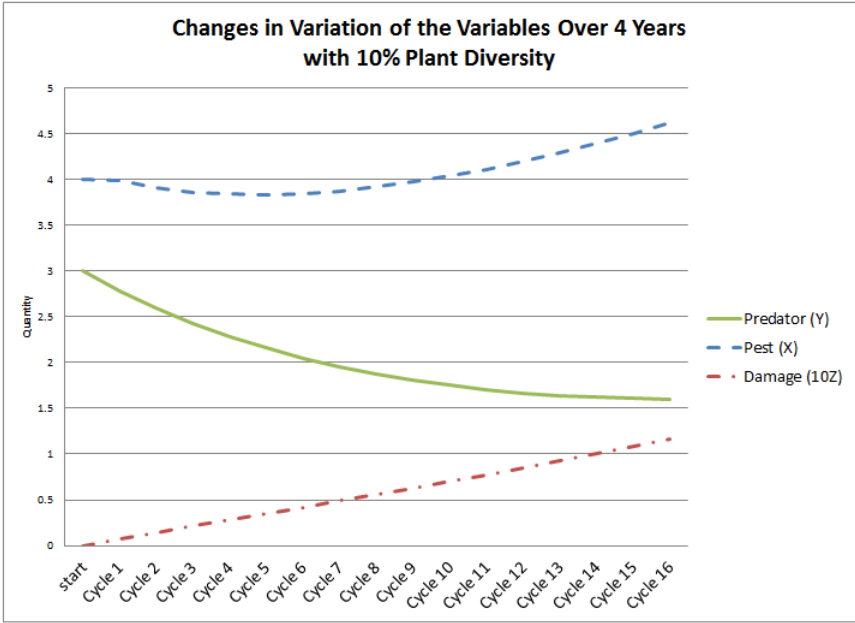
(a)
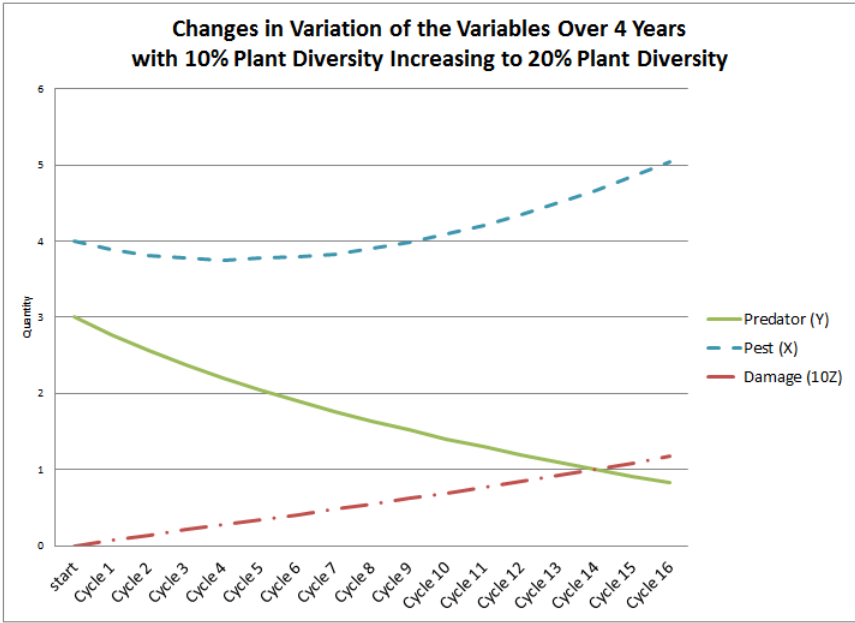
(b)
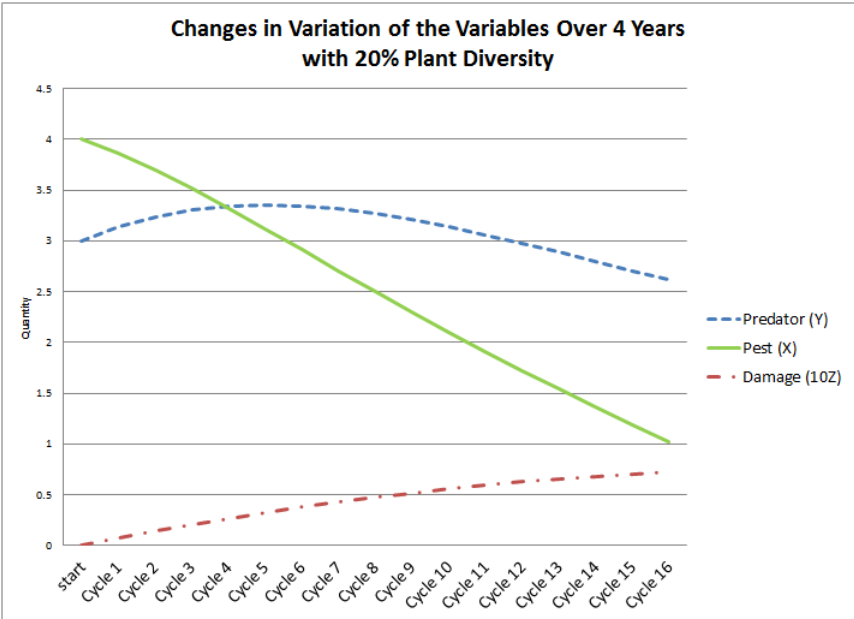
(c)
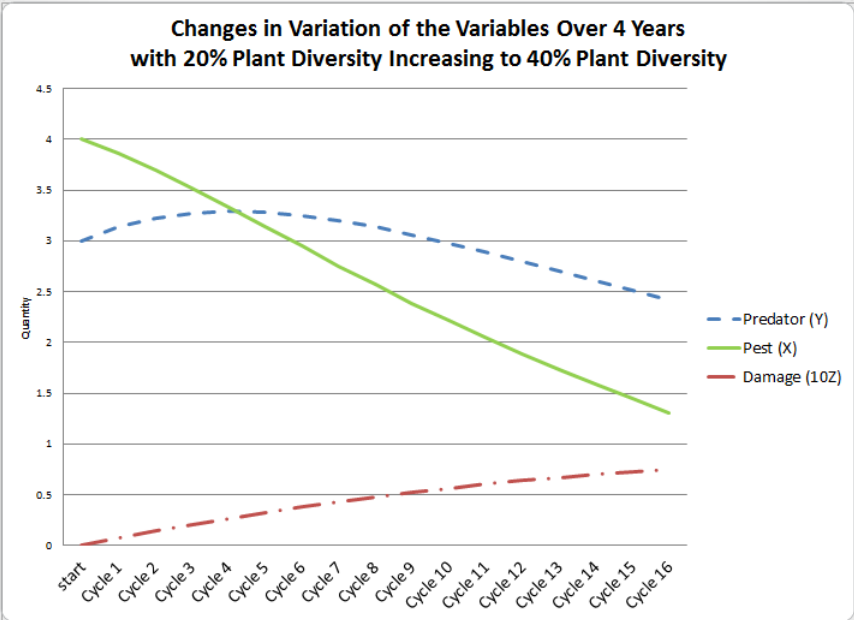
(d)

Figure 5.4: Variations of pest and predator populations and level of damage with extended optimal control theory to 4 years with 16 alfalfa cutting cycles

In both of these cases, the pest population continues to decrease over time. In addition, while the damage increases over the course of the four years, it begins to come to an equilibrium. Between the two cases beginning with 20%, it is found that there is not much of a difference. There is a slight benefit to allowing the other grasses to increase over the course of the four years. There is a 3.61% decrease in damage, 8.37% increase in predators, and 21.81% decrease in pests at the end of the four years. Thus, a farmer should allow these extra grasses to continue their growth provided they do not begin to stunt the growth of alfalfa.

Chapter 6

Cost-Benefit Analysis

Using a Cost-Benefit Analysis, the supposed revenue for farmers can be calculated depending on the planting strategies. As there is more plant diversity in the crops, there is a decrease in nutritional values of the alfalfa-dominant mixed legume strands. However, when planting alfalfa in a polyculture environment, there is less of a need to spray pesticides. If a farmer can refrain from spraying pesticides, the alfalfa can be considered organic. When this organic alfalfa is fed to cattle, along with other strategies (see [7]), the milk produced from those cattle can be sold at an organic cost. Since organic milk sells at a higher price than regular milk, there is still an increase in the revenue from selling less milk at a higher price. In 2013, the average price of organic milk was \$5.09 per gallon while the average for non-organic milk was \$3.46 per gallon in the United States.

In general, for $a\%$ of plants other than alfalfa, it is estimated that $\frac{a}{2}\%$ is the decrease in milk production per cow. Thus, the nutrition is cut approximately in half. Using this knowledge,

one can calculate the overall expected revenues for farmers depending on the percent alfalfa. We assume that regardless of the percent alfalfa the milk will be sold at an organic price.

On average, for 10% plant diversity, there is a 26.15% increase in revenue per each cutting cycle while for 20% plant diversity, there is a 5.27% increase in revenue per each cutting cycle. Figure 6.1 graphically shows the expected revenues for farmers (whether positive or negative) at the end of each cutting cycle. The cost-benefit analysis is limited in the fact that it only takes into consideration the revenue increases from selling the milk at an organic price. It is expected that farmers would actually have a greater revenue increase from not having to pay for pesticides or associated monetary costs. For more information regarding the calculations of the cost-benefit analysis, see Appendix C.

This analysis must be further applied in relation to the optimal controls in order to find out which planting strategy would increase the monetary gain the most in the agroecosystem while minimizing the plant damage. Depending on a farmer's crop and the initial amount of predators and pests, we have an ability to calculate and make predictions as to which planting strategy will reduce the most damage and increase revenue costs. Based on the cost-benefit analysis, we want to keep the plant diversity kept at a minimum, but must balance that strategy with the amount of plant diversity that will decrease the plant damage enough in order to not have to spray pesticides.
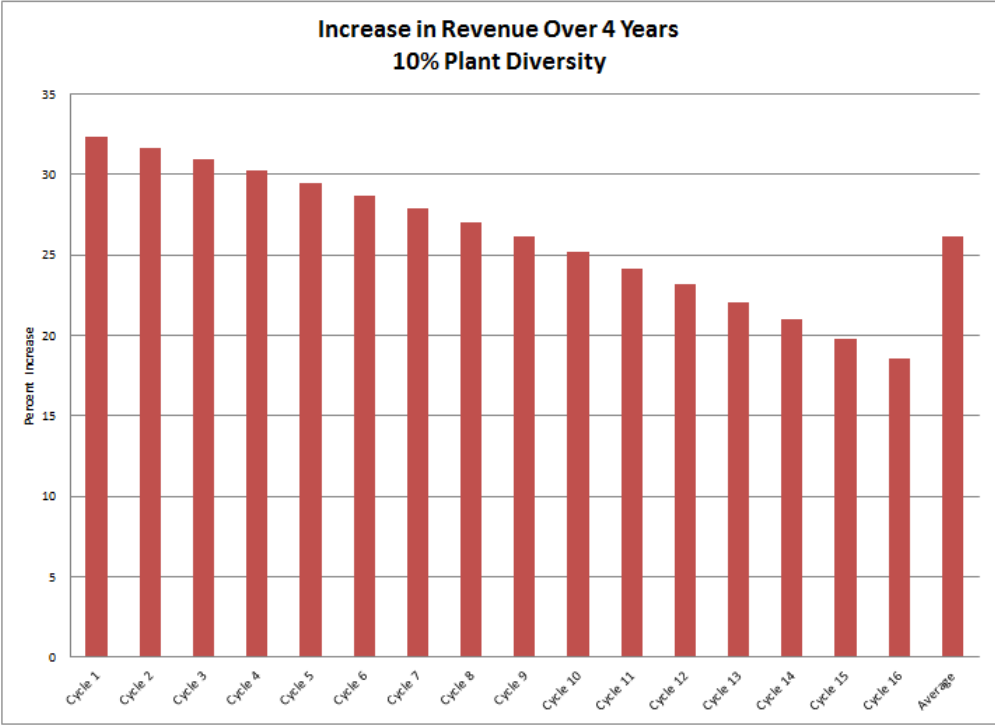
(a)
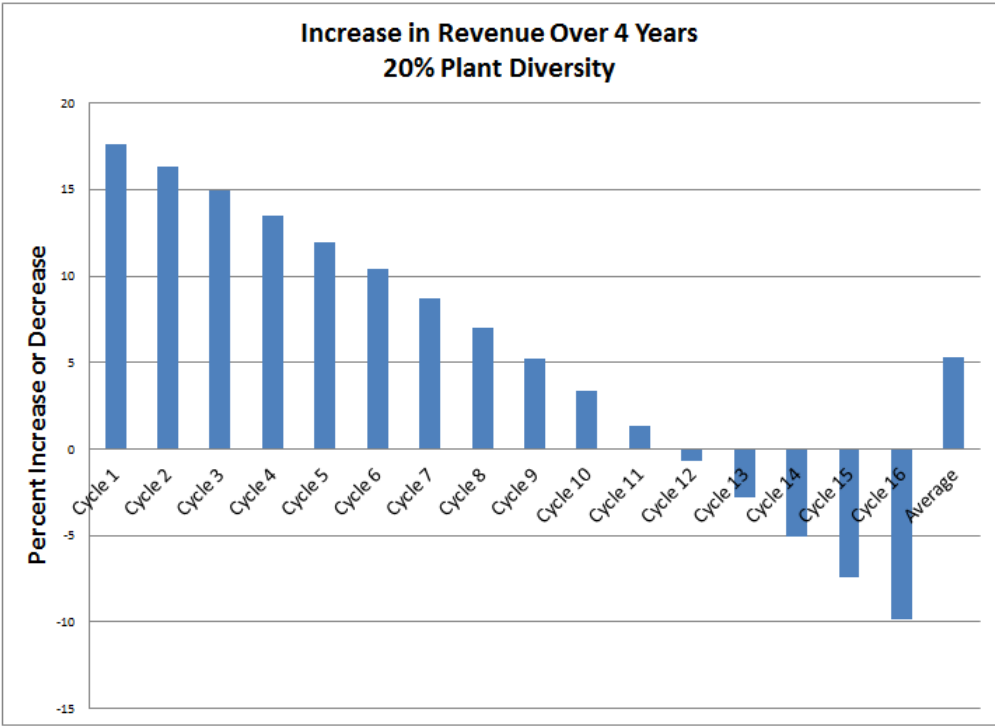
(b)

Figure 6.1: Cost-Benefit Analysis

Chapter 7

Conclusion

The synthesis of mathematics and science is not a new concept. Mathematics is found in the natural sciences, specifically physics and chemistry. This project shows the synthesis of mathematics and biology. The model and analyses illustrate how the union of mathematics and biology can solve complex problems.

With these results, our goals are fulfilled, the model was able to support biological findings that polyculture is indeed a planting strategy to reduce pest abundance and plant damage. In addition, the sensitivity analysis produced a new hypothesis that should be tested through field experimentation - that pesticides will cause more damage when sprayed in polyculture than in monoculture fields. This counterproductive use of pesticides stems from the fact that pesticides may kill both the PLH and the predator. Because impact on the mortality rate of the predator is higher than the impact on the mortality rate of the PLH, there will be a greater increase in alfalfa damage if pesticides are used since the predators' death

will be impacted more strongly. It was also predicted that pesticide use is more damaging in a polyculture field. These counterproductive aspects of pesticide usage in alfalfa fields represent significant incentives for farmers to avoid or to reduce the use of pesticides. In a future work, field experiments could be conducted to confirm these findings.

In addition, we found that it is critical to find a way to prevent PLH growth from nymphs to adults. However, if pesticides are that mean, this will also affect the *Nabis* population. It is thus crucial to evaluate the relative impact of such strategy (pesticides) on *Nabis* versus PLH in monoculture and polyculture field experiments.

Proving that the system indeed takes that of Bang-Bang Optimal Control allows the optimality condition will always be the maximum 50% plant diversity. In addition, using the optimal control does allow a decrease in plant damage. However, extended work showed that a farmer must balance the nutrition of the field from the plant diversity with expected revenues. To do this, we considered a Cost-Benefit Analysis. Depending on a farmer's crop and the initial amount of predators and pests, we have an ability to calculate and make predictions as to which planting strategy will reduce the most damage and increase revenue costs. Based on these, we want to keep the plant diversity kept at a minimum, but must balance that strategy with the amount of plant diversity that will decrease the plant damage enough in order to not have to spray pesticides. At this time, it is difficult to conclude one planting strategy that will work for all farmers because our model is limited to initial population of the pests. Given this initial condition, one can use the model and analyses to conclude the best strategy for a specific farmer and field.

The results can be applied to different agroecosystems due to the inclusion of ranges of the parameters and the interactive simulations. To determine accurate simulations of an agroecosystem, specific data would need to be gathered in that system. With the sensitivity analysis results, it is possible to focus data collection efforts to reduce uncertainty in key parameters and outline strategies to reduce alfalfa damage and to prevent PLH outbreaks.

In conclusion, the project provides a framework for designing cost-effective and environmentally-safe strategies to minimize alfalfa damage, determine critical parameters, and utilize the enemies hypothesis and polyculture diversity. This model can be extended as more accurate data has been collected.

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Appendix A

Maple Code

A.1 Steady State Analysis

```
f[1] := Y*(-a+b*X*(1-1/n)/(1+Y)+c);
f[2] := X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X));
f[3] := l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);

Case 1:
Y1 := 0;
X1 := 0;
Z1 := 0;

Case 2:
Y2 := 0;
X2 := -m*(-d+e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);
Z2 := -l*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);

monoculture substitution:
MX2 := subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, X2);
MZ2 := subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, Z2);

polyculture substitution:
PX2 := simplify(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, X2));
PZ2 := simplify(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, Z2));

Case 3:
Y3 := -(m*b*n*d+m*d*a*n-m*d*c*n+b*m*exp(-.1882*H)*j-l*k*a*n+l*k*c*n-exp(-.1882*H)*j*b*n*m-a*n*m*exp(-.1882*H)*j+c*n*m*exp(-.1882*H)*j-m*b*d)
    /(m*d*a*n-a*n*m*exp(-.1822*H)*j-l*k*a*n+m*b*n*g-m*b*g-m*d*c*n+c*n*m*exp(-.1882*H)*j+l*k*c*n);
X3 := -n*m*(-a*g+a*d-a*exp(-.1882*H)*j+c*g-c*d+c*exp(-.1882*H)*j)/(m*d*a*n-a*n*m*exp(-.1882*H)*j-l*k*a*n+m*b*n*g-m*b*g-m*d*c*n+c*n*m*exp(-.1882*H)
    *j+l*k*c*n);
Z3 := -l*n*(-a*g+a*d-a*exp(-.1882*H)*j+c*g-c*d+c*exp(-.1882*H)*j)/(m*d*a*n-a*n*m*exp(-.1882*H)*j-l*k*a*n+m*b*n*g-m*d*c*n+c*n*m*exp(-.1882*H)*j+l*k*c*n);

monoculture substitution:
MY3 := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, Y3));
MX3 := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, X3));
MZ3 := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, Z3));

polyculture substitution:
PY3 := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, Y3));
PX3 := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, X3));
PZ3 := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, Z3));
```

```
GetPalette("HTML");
with(Statistics);

ColumnGraph([MY3, MX3, (1/10)*MZ3], title = "Case 3 - Monoculture", datasetlabels = ["Predator", "Pest", "Damage"], color = "Tomato");
ColumnGraph([PY3, PX3, (1/10)*PZ3], title = "Case 3 - Polyculture", datasetlabels = ["Predator", "Pest", "Damage"], color = "ForestGreen");

Array3 := Array([MY3, MX3, (1/10)*MZ3]);
Array4 := Array([PY3, PX3, (1/10)*PZ3]);
ColumnGraph([Array3, Array4], title = "Case 3", legend = ["Monoculture", "Polyculture"], datasetlabels = ["Predator", "Pest", "Damage"],
    color = "Tomato" .. "ForestGreen");

Array5 := Array([MY3, PY3]);
Array6 := Array([MX3, PX3]);
Array7 := Array([(1/10)*MZ3, (1/10)*PZ3]);
ColumnGraph([Array5, Array6, Array7], legend = ["Predator", "Pest", "Damage"], datasetlabels = ["Monoculture", "Polyculture"],
    color = "Purple" .. "Coral");

R := evalf((1/10)*PZ3/PX3);
S := (1/10)*MZ3/MX3;
(S-R)/R;

RR := evalf((1/10)*PY3/PX3);
SS := evalf((1/10)*MY3/MX3);
(SS-RR)/RR;
```

A.2   Sensitivity Analysis

```
For a:
a1 := Y*(-1.01*a+b*X*(1-1/n)/(1+Y)+c);
a2 := X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X));
a3 := l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X)) = 0, Y*(-1.01*a+b*X*(1-1/n)/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);

Case 1:
Ya1 := 0;
Xa1 := 0;
Za1 := 0;

Case 2:
Ya2 := 0;
Xa2 := -m*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);
Za2 := -l*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);

Case 3:
Ya3 := -(100*m*b*n*d+101*m*d*a*n-100*m*d*c*n+100*e^(-.1882*H)*j*b*m-101*l*k*a*n+100*l*k*c*n-100*m*b*d-100*e^(-.1882*H)*j*b*n*m-101*m*e^(-.1882*H)*j*a*n
    +100*m*e^(-.1882*H)*j*c*n)/(101*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-101*l*k*a*n+100*m*b*n*g-100*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H)*j*c*n+100*l*k*c*n);
Xa3 := -n*m*(-101*a*g+101*a*d-101*e^(-.1882*H)*j*a+100*c*g-100*c*d+100*e^(-.1882*H)*j*c)/(101*m*d*a*n-101*m*e^(-.1882*H)*j*a*n-101*l*k*a*n+100*m*b*n*g
    -100*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H)*j*c*n+100*l*k*c*n);
Za3 := -l*m*n*(-101*a*g+101*a*d-101*e^(-.1882*H)*j*a+100*c*g-100*c*d+100*e^(-.1882*H)*j*c)/(101*m*d*a*n-101*m*e^(-.1882*H)*j*a*n-101*l*k*a*n+100*m*b*n*g
    -100*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H)*j*c*n+100*l*k*c*n);

Sensitivity Analysis for Case 3 wrt a:
YA := simplify((Ya3-Y3)/Y3);
XA := simplify((Xa3-X3)/X3);
ZA := simplify((Za3-Z3)/Z3);

For b:
b1 := Y*(-a+1.01*b*X*(1-1/n)/(1+Y)+c);
b2 := X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X));
b3 := l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X)) = 0, Y*(-a+1.01*b*X*(1-1/n)/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);

Case 1:
Yb1 := 0;
Xb1 := 0;
Zb1 := 0;

Case 2:
Yb2 := 0;
Xb2 := -m*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);
Zb2 := -l*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);

Case 3:
Yb3 := -(-100*m*e^(-.1882*H)*j*a*n+101*m*e^(-.1882*H)*j*c*n+101*m*b*n*d+101*e^(-.1882*H)*j*b*m-101*m*b*d-101*e^(-.1882*H)*j*b*n*m+100*m*d*a*n-100*m*d*c*n
    -101*l*k*a*n+100*l*k*c*n)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+101*m*b*n*g-101*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H)*j*c*n+100*l*k*c*n);
Xb3 := -100*n*m*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+101*m*b*n*g-101*m*b*g-100*m*d*c*n
```

```
    +100*m*e^(-.1882*H)*j*c*n+100*l*k*c*n);
Zb3 := -100*l*n*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+101*m*b*n*g-101*m*b*g-100*m*d*c*n
    +100*m*e^(-.1882*H)*j*c*n+100*l*k*c*n);


Sensitivity Analysis for Case 3 wrt b:
YB := simplify((Yb3-Y3)/Y3);
XB := simplify((Xb3-X3)/X3);
ZB := simplify((Zb3-Z3)/Z3);


For n:
n1 := Y*(-a+b*X*(1-1/(1.01*n)))/(1+Y)+c);
n2 := X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X));
n3 := l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/(1.01*n)))/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);


Case 1:
Yn1 := 0;
Xn1 := 0;
Zn1 := 0;


Case 2:
Yn2 := 0;
Xn2 := -m*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);
Zn2 := -l*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);


Case 3:
Yn3 := -(-9.900990099*10^9*d*b*m-10^10*e^(-.1882*H)*j*b*n*m-10^10*a*n*e^(-.1882*H)*j*m+10^10*c*n*e^(-.1882*H)*j*m-10^10*a*n*k*l+10^10*c*n*k*l
    +10^10*d*b*n*m+10^10*a*n*d*m-10^10*c*n*d*m+9.900990099*10^9*e^(-.1882*H)*j*b*m)/(10^10*a*n*d*m-10^10*a*n*e^(-.1882*H)*j*m-10^10*a*n*k*l
    -10^10*c*n*d*m+10^10*c*n*e^(-.1882*H)*j*m+10^10*c*n*k*l+10^10*b*n*g*m-9.900990099*10^9*b*g*m);
Xn3 := -10^10*n*m*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(10^10*a*n*d*m-10^10*a*n*e^(-.1882*H)*j*m-10^10*a*n*k*l-10^10*c*n*d*m
    +10^10*c*n*e^(-.1882*H)*j*m+10^10*c*n*k*l+10^10*b*n*g*m-9.900990099*10^9*b*g*m);
Zn3 := -10^10*l*n*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(10^10*a*n*d*m-10^10*a*n*e^(-.1882*H)*j*m
    -10^10*a*n*k*l-10^10*c*n*d*m+10^10*c*n*e^(-.1882*H)*j*m+10^10*c*n*k*l+10^10*b*n*g*m-9.900990099*10^9*b*g*m);


Sensitivity Analysis for Case 3 wrt n:
YN := simplify((Yn3-Y3)/Y3);
XN := simplify((Xn3-X3)/X3);
ZN := simplify((Zn3-Z3)/Z3);


For c:
c1 := Y*(-a+b*X*(1-1/n)/(1+Y)+1.01*c);
c2 := X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X));
c3 := l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+1.01*c) = 0, l*X-m*Z = 0}, [Y, X, Z]);


Case 1:
Yc1 := 0;
Xc1 := 0;
Zc1 := 0;


Case 2:
Yc2 := 0;
Xc2 := -m*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);
Zc2 := -l*(d-e^(-.1882*H)*j)/(-l*k+m*d-m*e^(-.1882*H)*j);


Case 3:
Yc3 := -(-100*m*e^(-.1882*H)*j*a*n+101*m*e^(-.1882*H)*j*c*n+100*m*b*n*d+100*e^(-.1882*H)*j*b*m-100*m*b*d-100*e^(-.1882*H)*j*b*n*m+100*m*d*a*n-101*m*d*c*n
    -100*l*k*a*n+101*l*k*c*n)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+100*m*b*n*g-100*m*b*g-101*m*d*c*n+101*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);
Xc3 := -n*m*(-100*a*g+100*a*d-100*e^(-.1882*H)*j*a+101*c*g-101*c*d+101*e^(-.1882*H)*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+100*m*b*n*g
    -100*m*b*g-101*m*d*c*n+101*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);
Zc3 := -l*n*(-100*a*g+100*a*d-100*e^(-.1882*H)*j*a+101*c*g-101*c*d+101*e^(-.1882*H)*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+100*m*b*n*g
    -100*m*b*g-101*m*d*c*n+101*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);


Sensitivity Analysis for Case 3 wrt c:
YC := simplify((Yc3-Y3)/Y3);
XC := simplify((Xc3-X3)/X3);
ZC := simplify((Zc3-Z3)/Z3);


For d:
d1 := Y*(-a+b*X*(1-1/n)/(1+Y)+c);
d2 := X*(-1.01*d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X));
d3 := l*X-m*Z;

solve({X*(-1.01*d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);


Case 1:
Yd1 := 0;
Xd1 := 0;
Zd1 := 0;


Case 2:
Yd2 := 0;
Xd2 := -m*(101*d-100*e^(-.1882*H)*j)/(101*m*d-100*m*e^(-.1882*H)*j-100*l*k);
Zd2 := -l*(101*d-100*e^(-.1882*H)*j)/(101*m*d-100*m*e^(-.1882*H)*j-100*l*k);


Case 3:
Yd3 := -(101*m*b*n*d+101*m*d*a*n-101*m*d*c*n+100*e^(-.1882*H)*j*b*m-100*l*k*a*n+100*l*k*c*n-101*m*b*d-100*e^(-.1882*H)*j*b*n*m-100*m*e^(-.1882*H)*j*a*n
    +100*m*e^(-.1882*H)*j*c*n)/(101*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+100*m*b*n*g-100*m*b*g-101*m*d*c*n+100*m*e^(-.1882*H)*j*c*n+100*l*k*c*n);
Xd3 := -n*m*(-100*a*g+101*a*d-100*e^(-.1882*H)*j*a+100*c*g-101*c*d+100*e^(-.1882*H)*j*c)/(101*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+100*m*b*n*g
    -100*m*b*g-101*m*d*c*n+100*m*e^(-.1882*H)*j*c*n+100*l*k*c*n);
```

```
Zd3 := -l*n*(-100*a*g+101*a*d-100*e^(-.1882*H))*j*a+100*c*g-101*c*d+100*e^(-.1882*H))*j*c)/(101*m*d*a*n-100*m*e^(-.1882*H))*j*a*n-100*l*k*a*n+100*m*b*n*g
    -100*m*b*g-101*m*d*c*n+100*m*e^(-.1882*H))*j*c*n+100*l*k*c*n);


Sensitivity Analysis for Case 3 wrt d:
YD := simplify((Yd3-Y3)/Y3);
XD := simplify((Xd3-X3)/X3);
ZD := simplify((Zd3-Z3)/Z3);


For g:
g1 := Y*(-a+b*X*(1-1/n)/(1+Y)+c);
g2 := X*(-d-1.01*g*Y/(1+X)+e^(-.1882*H))*j+k*Z/(1+X));
g3 := l*X-m*Z;

solve({X*(-d-1.01*g*Y/(1+X)+e^(-.1882*H))*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);


Case 1:
Yg1 := 0;
Xg1 := 0;
Zg1 := 0;


Case 2:
Yg2 := 0;
Xg2 := -m*(d-e^(-.1882*H))*j)/(-l*k+m*d-m*e^(-.1882*H))*j);
Zg2 := -l*(d-e^(-.1882*H))*j)/(-l*k+m*d-m*e^(-.1882*H))*j);


Case 3:
Yg3 := -(100*(m*b*n*d+m*d*a*n+m*d*c*n+e^(-.1882*H))*j*b*m-l*k*a*n+l*k*c*n-m*b*d-e^(-.1882*H))*j*b*n*m-m*e^(-.1882*H))*j*a*n+m*e^(-.1882*H))*j*c*n))/(100*m*d*a*n
    -100*m*e^(-.1882*H))*j*a*n-100*l*k*a*n+101*m*b*n*g-101*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H))*j*c*n+100*l*k*c*n);
Xg3 := -n*m*(-101*a*g+100*a*d-100*e^(-.1882*H))*j*a+101*c*g-100*c*d+100*e^(-.1882*H))*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H))*j*a*n-100*l*k*a*n+101*m*b*n*g
    -101*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H))*j*c*n+100*l*k*c*n);
Zg3 := -l*n*(-101*a*g+100*a*d-100*e^(-.1882*H))*j*a+101*c*g-100*c*d+100*e^(-.1882*H))*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H))*j*a*n-100*l*k*a*n+101*m*b*n*g
    -101*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H))*j*c*n+100*l*k*c*n);


Sensitivity Analysis for Case 3 wrt g:
YG := simplify((Yg3-Y3)/Y3);
XG := simplify((Xg3-X3)/X3);
ZG := simplify((Zg3-Z3)/Z3);


For H:
h1 := Y*(-a+b*X*(1-1/n)/(1+Y)+c);
h2 := X*(-d-g*Y/(1+X)+e^(-(.1882*1.01)*H))*j+k*Z/(1+X));
h3 := l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-(.1882*1.01)*H))*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);


Case 1:
Yh1 := 0;
Xh1 := 0;
Zh1 := 0;


Case 2:
Yh2 := 0;
Xh2 := -m*(-d+e^(-.1900820001*H))*j)/(l*k-m*d+m*e^(-.1900820001*H))*j);
Zh2 := -l*(-d+e^(-.1900820001*H))*j)/(l*k-m*d-m*e^(-.1900820001*H))*j);


Case 3:
Yh3 := -(l*k*a*n+m*e^(-.1900820001*H))*j*a*n-l*k*c*n-m*e^(-.1900820001*H))*j*c*n+m*b*d+m*b*n*e^(-.1900820001*H))*j-m*b*e^(-.1900820001*H))*j-m*b*n*d+m*d*c*n
    -m*d*a*n)/(m*b*g-m*d*a*n-m*b*n*g-m*d*c*n+l*k*a*n+m*e^(-.1900820001*H))*j*a*n-l*k*c*n-m*e^(-.1900820001*H))*j*c*n);
Xh3 := -m*n*(a*e^(-.1900820001*H))*j-c*e^(-.1900820001*H))*j+a*g-c*g-a*d+c*d)/(m*b*g-m*d*a*n-m*b*n*g+m*d*c*n+l*k*a*n+m*e^(-.1900820001*H))*j*a*n-l*k*c*n
    -m*e^(-.1900820001*H))*j*c*n);
Zh3 := -l*n*(a*e^(-.1900820001*H))*j-c*e^(-.1900820001*H))*j+a*g-c*g+a*d+c*d)/(m*b*g-m*d*a*n-m*b*n*g+m*d*c*n+l*k*a*n+m*e^(-.1900820001*H))*j*a*n-l*k*c*n
    -m*e^(-.1900820001*H))*j*c*n);

Sensitivity Analysis for Case 3 wrt H
YH := simplify((Yh3-Y3)/Y3);
XH := simplify((Xh3-X3)/X3);
ZH := simplify((Zh3-Z3)/Z3);


For j:
j1 := Y*(-a+b*X*(1-1/n)/(1+Y)+c);
j2 := X*(-d-g*Y/(1+X)+1.01*e^(-.1882*H))*j+k*Z/(1+X));
j3 := l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+1.01*e^(-.1882*H))*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);


Case 1:
Yj1 := 0;
Xj1 := 0;
Zj1 := 0;


Case 2:
Yj2 := 0;
Xj2 := -m*(100*d-101*e^(-.1882*H))*j)/(100*m*d-101*m*e^(-.1882*H))*j-100*l*k);
Zj2 := -l*(100*d-101*e^(-.1882*H))*j)/(100*m*d-101*m*e^(-.1882*H))*j-100*l*k);


Case 3:
Yh3 := -(-100*l*k*a*n+100*l*k*c*n+100*m*b*n*d+100*m*d*a*n-100*m*d*c*n+101*e^(-.1882*H))*j*b*m-101*e^(-.1882*H))*j*b*n*m-101*m*e^(-.1882*H))*j*a*n+101*m
    *e^(-.1882*H))*j*c*n-100*m*b*d)/(100*m*d*a*n-101*m*e^(-.1882*H))*j*a*n-100*l*k*a*n+100*m*b*n*g-100*m*b*g-101*m*d*c*n+101*m*e^(-.1882*H))*j*c*n+100*l*k*c*n);
Xj3 := -n*m*(-100*a*g+100*a*d-101*e^(-.1882*H))*j*a+100*c*g-100*c*d+101*e^(-.1882*H))*j*c)/(101*m*d*a*n-100*m*e^(-.1882*H))*j*a*n-100*l*k*a*n+100*m*b*n*g
    -100*m*b*g-100*m*d*c*n+101*m*e^(-.1882*H))*j*c*n+100*l*k*c*n);
Zj3 := -l*n*(-100*a*g+100*a*d-101*e^(-.1882*H))*j*a+100*c*g-100*c*d+101*e^(-.1882*H))*j*c)/(100*m*d*a*n-101*m*e^(-.1882*H))*j*a*n-100*l*k*a*n+100*m*b*n*g
    -100*m*b*g-100*m*d*c*n+101*m*e^(-.1882*H))*j*c*n+100*l*k*c*n);
```

```
Sensitivity Analysis for Case 3 wrt j:
YJ := simplify((Yj3-Y3)/Y3);
XJ := simplify((Xj3-X3)/X3);
ZJ := simplify((Zj3-Z3)/Z3);


For k:
k1 := Y*(-a+b*X*(1-1/n)/(1+Y)+c);
k2 := X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+1.01*k*Z/(1+X));
k3 := l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+1.01*k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+c) = 0, l*X-m*Z = 0}, [Y, X, Z]);


Case 1:
Yk1 := 0;
Xk1 := 0;
Zk1 := 0;


Case 2:
Yk2 := 0;
Xk2 := -100*m*(d-e^(-.1882*H)*j)/(100*m*d-100*m*e^(-.1882*H)*j-101*l*k);
Zk2 := -100*l*(d-e^(-.1882*H)*j)/(100*m*d-100*m*e^(-.1882*H)*j-101*l*k);


Case 3:
Yk3 := -(-101*l*k*a*n+101*l*k*c*n+100*m*b*n*d+100*m*d*a*n-100*m*d*c*n+100*e^(-.1882*H)*j*b*m-100*e^(-.1882*H)*j*b*n*m-100*m*e^(-.1882*H)*j*a*n+100*m
      *e^(-.1882*H)*j*c*n-100*m*b*d)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-101*l*k*a*n+100*m*b*n*g-100*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);
Xk3 := -100*n*m*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-101*l*k*a*n+100*m*b*n*g-100*m*b*g-100*m*d*c*n
      +100*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);
Zk3 := -100*l*n*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-101*l*k*a*n+100*m*b*n*g-100*m*b*g-100*m*d*c*n
      +100*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);

Sensitivity Analysis for Case 3 wrt k:
YK := simplify((Yk3-Y3)/Y3);
XK := simplify((Xk3-X3)/X3);
ZK := simplify((Zk3-Z3)/Z3);


For l:
l1 := Y*(-a+b*X*(1-1/n)/(1+Y)+c);
l2 := X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X));
l3 := 1.01*l*X-m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+c) = 0, 1.01*l*X-m*Z = 0}, [Y, X, Z]);


Case 1:
Yl1 := 0;
Xl1 := 0;
Zl1 := 0;


Case 2:
Yl2 := 0;
Xl2 := -100*m*(d-e^(-.1882*H)*j)/(100*m*d-100*m*e^(-.1882*H)*j-101*l*k);
Zl2 := -101*l*(d-e^(-.1882*H)*j)/(100*m*d-100*m*e^(-.1882*H)*j-101*l*k);


Case 3:
Yl3 := -(-101*l*k*a*n+101*l*k*c*n+100*m*b*n*d+100*m*d*a*n-100*m*d*c*n+100*e^(-.1882*H)*j*b*m-100*e^(-.1882*H)*j*b*n*m-100*m*e^(-.1882*H)*j*a*n+100*m
      *e^(-.1882*H)*j*c*n-100*m*b*d)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-101*l*k*a*n+100*m*b*n*g-100*m*b*g-100*m*d*c*n+100*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);
Xl3 := -100*n*m*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(101*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-100*l*k*a*n+100*m*b*n*g-100*m*b*g-100*m*d*c*n
      +101*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);
Zl3 := -101*l*n*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(100*m*d*a*n-100*m*e^(-.1882*H)*j*a*n-101*l*k*a*n+100*m*b*n*g-100*m*b*g-100*m*d*c*n
      +100*m*e^(-.1882*H)*j*c*n+101*l*k*c*n);

Sensitivity Analysis for Case 3 wrt l:
YL := simplify((Yl3-Y3)/Y3);
XL := simplify((Xl3-X3)/X3);
ZL := simplify((Zl3-Z3)/Z3);


For m:
m1 := Y*(-a+b*X*(1-1/n)/(1+Y)+c);
m2 := X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X));
m3 := l*X-1.01*m*Z;

solve({X*(-d-g*Y/(1+X)+e^(-.1882*H)*j+k*Z/(1+X)) = 0, Y*(-a+b*X*(1-1/n)/(1+Y)+c) = 0, l*X-1.10*m*Z = 0}, [Y, X, Z]);


Case 1:
Ym1 := 0;
Xm1 := 0;
Zm1 := 0;


Case 2:
Ym2 := 0;
Xm2 := -11*m*(d-e^(-.1882*H)*j)/(11*m*d-11*m*e^(-.1882*H)*j-10*l*k);
Zm2 := -10*l*(d-e^(-.1882*H)*j)/(11*m*d-11*m*e^(-.1882*H)*j-10*l*k);


Case 3:
Ym3 := -(-10*l*k*a*n+10*l*k*c*n+11*m*b*n*d+11*m*d*a*n-11*m*d*c*n+11*e^(-.1882*H)*j*b*m-11*e^(-.1882*H)*j*b*n*m-11*m*e^(-.1882*H)*j*a*n+11*m*e^(-.1882*H)*j*c*n
      -11*m*b*d)/(11*m*d*a*n-11*m*e^(-.1882*H)*j*a*n-10*l*k*a*n+11*m*b*n*g-11*m*b*g-11*m*d*c*n+11*m*e^(-.1882*H)*j*c*n+10*l*k*c*n);
Xm3 := -11*n*m*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(11*m*d*a*n-11*m*e^(-.1882*H)*j*a*n-10*l*k*a*n+11*m*b*n*g-11*m*b*g-11*m*d*c*n
      +11*m*e^(-.1882*H)*j*c*n+10*l*k*c*n);
Zm3 := -10*l*n*(-a*g+a*d-e^(-.1882*H)*j*a+c*g-c*d+e^(-.1882*H)*j*c)/(11*m*d*a*n-11*m*e^(-.1882*H)*j*a*n-10*l*k*a*n+11*m*b*n*g-11*m*b*g-11*m*d*c*n
      +11*m*e^(-.1882*H)*j*c*n+10*l*k*c*n);

Sensitivity Analysis for Case 3 wrt m:
```

```
YM := simplify((Ym3-Y3)/Y3);
XM := simplify((Xm3-X3)/X3);
ZM := simplify((Zm3-Z3)/Z3);

with(stats);
with(stats[statplots]);
with(Statistics);
with(plots);

For Monoculture: Numerical Analysis
MZA := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZA));
MZB := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZB));
MZN := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZN));
MZC := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZC));
MZD := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZD));
MZG := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZG));
MZH := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZH));
MZJ := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZJ));
MZK := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZK));
MZL := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZL));
MZM := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, ZM));
ColumnGraph([(1/10)*MZH, MZJ, MZA, MZL, MZK, MZN, MZG, MZC, MZB, MZD, MZM], title = "Monoculture - Numerical Analysis",
    datasetlabels = ["H", "j", "a", "l", "k", "n", "g", "c", "b", "d", "m"], color = "Tomato");

For Polyculture: Numerical Analysis
PZA := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZA));
PZB := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZB));
PZN := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZN));
PZC := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZC));
PZD := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZD));
PZG := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZG));
PZH := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZH));
PZJ := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZJ));
PZK := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZK));
PZL := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZL));
PZM := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, ZM));
ColumnGraph([(1/10)*PZH, PZJ, PZA, PZL, PZK, PZN, PZG, PZC, PZB, PZD, PZM], title = "Polyculture - Numerical Analysis",
    datasetlabels = ["H", "j", "a", "l", "k", "n", "g", "c", "b", "d", "m"], color = "ForestGreen");

Array8 := Array([(1/10)*MZH, MZJ, MZA, MZL, MZK, MZN, MZG, MZC, MZB, MZD, MZM]);
Array9 := Array([(1/10)*PZH, PZJ, PZA, PZL, PZK, PZN, PZG, PZC, PZB, PZD, PZM]);
ColumnGraph([Array8, Array9], title = "Numerical Analysis", legend = ["Monoculture", "Polyculture"],
    datasetlabels = ["H", "j", "a", "l", "k", "n", "g", "c", "b", "d", "m"], color = ["Tomato", "ForestGreen"]);

Analytical Analysis:
DA := diff(Z3, a);
DB := diff(Z3, b);
DN := diff(Z3, n);
DC := diff(Z3, c);
DD := diff(Z3, d);
DG := diff(Z3, g);
DH := diff(Z3, h);
DJ := diff(Z3, j);
DK := diff(Z3, k);
DL := diff(Z3, l);
DM := diff(Z3, m);

For Monoculture:
MDA := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DA*a/Z3));
MDB := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DB*b/Z3));
MDN := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DN*n/Z3));
MDC := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DC*c/Z3));
MDD := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DD*d/Z3));
MDG := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DG*g/Z3));
MDH := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DH*H/Z3));
MDJ := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DJ*j/Z3));
MDK := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DK*k/Z3));
MDL := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DL*l/Z3));
MDM := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .385, g = .3, H = 0, j = .5, k = 0.1e-3, l = 0.27e-2, m = 0.1e-3, DM*m/Z3));
ColumnGraph([MDA, MDJ, MDL, MDK, MDN, MDH, MDG, MDM, MDC, MDB, MDD], title = "Monoculture - Analytical Analysis",
    datasetlabels = ["a", "j", "l", "k", "n", "H", "g", "m", "c", "b", "d"], color = "Tomato");

For Polyculture:
PDA := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DA*a/Z3));
PDB := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DB*b/Z3));
PDN := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DN*n/Z3));
PDC := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DC*c/Z3));
PDD := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DD*d/Z3));
PDG := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DG*g/Z3));
PDH := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DH*h/Z3));
PDJ := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DJ*j/Z3));
PDK := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DK*k/Z3));
PDL := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DL*l/Z3));
PDM := evalf(subs(a = .25, b = .24, n = 4, c = .12, d = .3, g = .3, H = ln(2), j = .512, k = 0.1e-3, l = 0.3e-2, m = 0.1e-3, DM*m/Z3));
ColumnGraph([PDA, PDJ, PDL, PDK, PDN, PDH, PDG, PDM, PDC, PDB, PDD], title = "Polyculture - Analytical Analysis",
    datasetlabels = ["a", "j", "l", "k", "n", "H", "g", "m", "c", "b", "d"], color = "ForestGreen");

Array10 := [MDA, MDJ, MDL, MDK, MDN, MDH, MDG, MDM, MDC, MDB, MDD];
Array11 := [PDA, PDJ, PDL, PDK, PDN, PDH, PDG, PDM, PDC, PDB, PDD];
ColumnGraph([Array10, Array11], title = "Analytical Analysis", legend = ["Monoculture", "Polyculture"],
    datasetlabels = ["a", "j", "l", "k", "n", "H", "g", "m", "c", "b", "d"], color = ["Tomato", "ForestGreen"]);
```

A.3   Optimal Control Theory

```
f[1] := Y(t)*(-a+b*X(t)*(1-Y(t)/n)/(1+Y(t))+c);
f[2] := X(t)*(-d-g*Y(t)/(1+X(t))+(1+(.1546*(-1))*[(1-.5)*ln(1-.5)+.5*ln(.5)]*(1+U(t)))*j+k*Z(t)/(1+X(t)));
f[3] := l*X(t)-m*Z(t);

minimize[int(A*Z(t)+B*X(t)+C*U(t), t = 0 .. T)+Q(.5), U];

State Variables - damage and pest:
Z(t), X(t);

Control Function - effect of plant diversity on system:
U(t);

Terminal Time:
T;

State Equations - differential equations:
f[1], f[2], f[3];

Q(U(t)*q(T)) := -0.1e-1*(.8712*(1+U(t)))+73.982;

Hamiltonian:
HL := A*Z(t)+B*X(t)+C*U(t)+(73.982-73.54640)/T+l[1]*f[1]+l[2]*f[2]+l[3]*f[3];

Co-State Equations: (these are really derivatives)
CCS := A*Z+B*X+C*U+(73.982-73.5464)/T+l[1]*Y*(-a+b*X*(1-Y/n)/(1+Y)+c)+l[2]*X*(-d-g*Y/(1+X)+
(1+(.1546*(-1))*[(1-.5)*ln(1-.5)+.5*ln(.5)]*(1+U))*j+k*Z/(1+X))+l[3]*(X*l-Z*m);

dl[L] := -(diff(CCS, Y));
dl[2*L] := -(diff(CCS, X));
dl[3*L] := -(diff(CCS, Z));

Optimality Condition:
OCL := diff(CCS, U);
      C + l[2] X (-0.1546(1 + U) + [0.1071605541]) j

XUL := solve(OCL = 0, U); optimality condition
```

Appendix B

MATLAB Code - Optimal Control Theory

B.1   1% Plant Diversity

```
control: 1 + U(t) plus addition of loss of alfalfa productivity
using exponential Q_0 from mono and poly combined
q = 10%,  U = 4, 0.5

test = -1;
parameters from Open Field Experiment - Polyculture
a=0.25;
b=0.24;
n=4;
c=0.12;
d=0.3;
g=0.3;
P=0.3250829734; for q = 10% where P = -[(1-q)ln(1-q)+qln(q)]
j=0.512;
k=0.0001;
l=0.003;
m=0.001;

T=25; terminal time
Delta = T/1000; how often to change
M = 1000;
t=linspace(0,T,M+1);
h=T/M;
h2 = h/2;

Y=zeros(1,M+1);
X=zeros(1,M+1);
Z=zeros(1,M+1);

initial values - can be altered
Y0=25; predator
X0=40; pest
Z0=0; damage

costs - can be altered
A=1;  for predator
B=1;  for pest
C=1; for damage

Y(1)=Y0;
X(1)=X0;
Z(1)=Z0;

lambda1=zeros(1,M+1);
lambda2=zeros(1,M+1);
lambda3=zeros(1,M+1);

u=zeros(1,M+1);
```

```
while(test < 0)

    oldu = u;
    oldY = Y;
    oldX = X;
    oldZ = Z;

    oldlambda1 = lambda1;
    oldlambda2 = lambda2;
    oldlambda3 = lambda3;

    for i = 1:M state equations
        m11 = Y(i)*(-a+((b*X(i))/(1+Y(i)))*(1-(Y(i)/n))+c);
        m12 = X(i)*(-d-((g*Y(i))/(1+X(i)))+(1-0.1546*P*(1+u(i)))*j+((k*Z(i))/(1+X(i))));
        m13 = l*X(i)-m*Z(i);

        m21 = (Y(i)+h2*m11)*(-a+((b*(X(i)+h2*m12))/(1+(Y(i)+h2*m11)))*(1-((Y(i)+h2*m11)/n))+c);
        m22 = (X(i)+h2*m12)*(-d-((g*(Y(i)+h2*m11))/(1+(X(i)+h2*m12)))+(1-0.1546*P*(1+u(i)))*j+((k*(Z(i)+h2*m13))/(1+(X(i)+h2*m12)))));
        m23 = l*(X(i)+h2*m12)-m*(Z(i)+h2*m13);

        m31 = (Y(i)+h2*m21)*(-a+((b*(X(i)+h2*m22))/(1+(Y(i)+h2*m21)))*(1-((Y(i)+h2*m21)/n))+c);
        m32 = (X(i)+h2*m22)*(-d-((g*(Y(i)+h2*m21))/(1+(X(i)+h2*m22)))+(1-0.1546*P*(1+u(i)))*j+((k*(Z(i)+h2*m23))/(1+(X(i)+h2*m22)))));
        m33 = l*(X(i)+h2*m22)-m*(Z(i)+h2*m23);

        m41 = (Y(i)+h*m31)*(-a+((b*(X(i)+h*m32))/(1+(Y(i)+h*m31)))*(1-((Y(i)+h*m31)/n))+c);
        m42 = (X(i)+h*m32)*(-d-((g*(Y(i)+h*m31))/(1+(X(i)+h*m32)))+(1-0.1546*P*(1+u(i)))*j+((k*(Z(i)+h*m33))/(1+(X(i)+h*m32)))));
        m43 = l*(X(i)+h*m32)-m*(Z(i)+h*m33);

        X(i+1) = X(i) + (h/6)*(m11 + 2*m21 + 2*m31 + m41);
        Y(i+1) = Y(i) + (h/6)*(m12 + 2*m22 + 2*m32 + m42);
        Z(i+1) = Z(i) + (h/6)*(m13 + 2*m23 + 2*m33 + m43);
    end

    for i = 1:M costate equations
        J = M+2-i;

        m11 = -lambda1(J)*(-a+(b*X(J)*(1-(Y(J)/n)))/(1+Y(J))+c)-(lambda1(J)*Y(J))*((-b*X(J)*(1-(Y(J)/n)))/((1+Y(J))^2))-((b*X(J))/((1+Y(J))*n))
        +((lambda2(J)*X(J)*g)/(1+X(J)));
        m12 = -B-((lambda1(J)*Y(J)*b*(1-(Y(J)/n)))/(1+Y(J))-lambda2(J)*(-d-((g*Y(J))/(1+X(J)))+(1-0.1546*(-0.6931471806)*(1+u(J))*j
        +((k*Z(J))/(1+X(J)))))-lambda2(J)*X(J)*(((g*Y(J))/((1+X(J))^2))-((k*Z(J))/((1+X(J))^2)))-lambda3(J)*l;
        m13 = -A-((lambda2(J)*X(J)*k)/(1+X(J)))+lambda3(J)*m;

        m21 = -(lambda1(J)-h2*m11)*(-a+(b*((0.5)*(X(J)+X(J-1)))*(1-(((0.5)*(Y(J)-Y(J-1)))/n)))/(1+((0.5)*(Y(J)-Y(J-1))))+c)-((lambda1(J)-h2*m11)
        *((0.5)*(Y(J)+Y(J-1)))*((-b*((0.5)*(X(J)+X(J-1)))*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/((1+((0.5)*(Y(J)+Y(J-1)))^2))
        -((b*((0.5)*(X(J)+X(J-1))))/((1+((0.5)*(Y(J)+Y(J-1)))*n))+(((lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1)))*g)/(1+((0.5)*(X(J)+X(J-1))))));
        m22 = -B-(((lambda1(J)-h2*m11)*((0.5)*(Y(J)+Y(J-1)))*b*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/(1+((0.5)*(Y(J)+Y(J-1))))-(lambda2(J)-h2*m12)
        *(-d-((g*((0.5)*(Y(J)+Y(J-1))))/(1+((0.5)*(X(J)+X(J-1))))+(1-0.1546*(-0.6931471806)*(1+u(J))*j+((k*((0.5)*(Z(J)+Z(J-1))))/(1+((0.5)
        *(X(J)+X(J-1))))))-(lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1))*(((g*((0.5)*(Y(J)+Y(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2))
        -((k*((0.5)*(Z(J)+Z(J-1))))/((1+((0.5)*(X(J)+X(J-1)))^2)))-(lambda3(J)-h2*m13)*l;
        m23 = -A-(((lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1)))*k)/(1+((0.5)*(X(J)+X(J-1)))))+(lambda3(J)-h2*m13)*m;

        m31 = -(lambda1(J)-h2*m21)*(-a+(b*((0.5)*(X(J)+X(J-1)))*(1-(((0.5)*(Y(J)-Y(J-1)))/n)))/(1+((0.5)*(Y(J)-Y(J-1))))+c)-((lambda1(J)-h2*m21)
        *((0.5)*(Y(J)+Y(J-1)))*((-b*((0.5)*(X(J)+X(J-1)))*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/((1+((0.5)*(Y(J)+Y(J-1)))^2))
        -((b*((0.5)*(X(J)+X(J-1))))/((1+((0.5)*(Y(J)+Y(J-1)))*n))+(((lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1)))*g)/(1+((0.5)*(X(J)+X(J-1))))));
        m32 = -B-(((lambda1(J)-h2*m21)*((0.5)*(Y(J)+Y(J-1)))*b*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/(1+((0.5)*(Y(J)+Y(J-1))))-(lambda2(J)-h2*m22)
        *(-d-((g*((0.5)*(Y(J)+Y(J-1))))/(1+((0.5)*(X(J)+X(J-1))))+(1-0.1546*(-0.6931471806)*(1+u(J))*j+((k*((0.5)*(Z(J)+Z(J-1))))/(1+((0.5)
        *(X(J)+X(J-1))))))-(lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1))*(((g*((0.5)*(Y(J)+Y(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2))-((k*((0.5)
        *(Z(J)+Z(J-1))))/((1+((0.5)*(X(J)+X(J-1)))^2)))-(lambda3(J)-h2*m23)*l;
        m33 = -A-(((lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1)))*k)/(1+((0.5)*(X(J)+X(J-1)))))+(lambda3(J)-h2*m23)*m;

        m41 = -(lambda1(J)-h2*m31)*(-a+(b*X(J)*(1-(Y(J)/n)))/(1+Y(J))+c)-((lambda1(J)-h2*m31)*Y(J))*((-b*X(J)*(1-(Y(J)/n)))/((1+Y(J))^2))-((b*X(J))
        /((1+Y(J))*n))+(((lambda2(J)-h2*m32)*X(J)*g)/(1+X(J)));
        m42 = -B-(((lambda1(J)-h2*m31)*Y(J)*b*(1-(Y(J)/n)))/(1+Y(J))-(lambda2(J)-h2*m32)*(-d-((g*Y(J))/(1+X(J)))+(1-0.1546*(-0.6931471806)*(1+u(J))*j
        +((k*Z(J))/(1+X(J)))))-(lambda2(J)-h2*m32)*X(J)*(((g*Y(J))/((1+X(J))^2))-((k*Z(J))/((1+X(J))^2)))-(lambda3(J)-h2*m33)*l;
        m43 = -A-(((lambda2(J)-h2*m32)*X(J)*k)/(1+X(J)))+(lambda3(J)-h2*m33)*m;

        lambda1(J-1) = lambda1(J) - (h/6)*(m11 + 2*m21 + 2*m31 + m41);
        lambda2(J-1) = lambda2(J) - (h/6)*(m12 + 2*m22 + 2*m32 + m42);
        lambda3(J-1) = lambda3(J) - (h/6)*(m13 + 2*m23 + 2*m33 + m43);
    end

    temp = 4; optimality condition
    u1 = min(49, max(0, abs(temp)));
    u = 0.5*(u1 + oldu);

    temp1  = Delta*sum(abs(u)) - sum(abs(oldu - u));
    temp2  = Delta*sum(abs(Y)) - sum(abs(oldY - Y));
    temp3  = Delta*sum(abs(X)) - sum(abs(oldX - X));
    temp4  = Delta*sum(abs(Z)) - sum(abs(oldZ - Z));
    temp5  = Delta*sum(abs(lambda1)) - sum(abs(oldlambda1 - lambda1));
    temp6  = Delta*sum(abs(lambda2)) - sum(abs(oldlambda2 - lambda2));
    temp7  = Delta*sum(abs(lambda3)) - sum(abs(oldlambda3 - lambda3));
    test   = min(temp1, min(temp2, min(temp3, min(temp4, min(temp5, min(temp6, temp7))))));
end

y(1,:) = t;
y(2,:) = Y;
y(3,:) = X;
y(4,:) = Z;
y(5,:) = lambda1;
y(6,:) = lambda2;
y(7,:) = lambda3;
```

```
y(8,:) = 1+u;

     subplot(1,3,1);plot(y(1,:),y(4,:),'r')
         subplot(1,3,1);xlabel('Time')
         subplot(1,3,1);ylabel('damage (Z)')
         subplot(1,3,1);axis([0 T -0.1 1.1])

         subplot(1,3,2);plot(y(1,:),y(3,:),'g')
         subplot(1,3,2);xlabel('Time')
         subplot(1,3,2);ylabel('pest (X)')
         subplot(1,3,2);axis([0 T -11 41])

         subplot(1,3,3);plot( y(1,:),y(2,:),'b')
         subplot(1,3,3);xlabel('Time')
         subplot(1,3,3);ylabel('predators (Y)')
         subplot(1,3,3);axis([0 T -11 41])
```

B.2   10% Plant Diversity

```
%control: 1 + U(t) plus addition of loss of alfalfa productivity
%using exponential Q_0 from mono and poly combined
%q = 1%,  U = 49, 0.5

%function y = OCT_Summer_2014(A,B,C,Y0,X0,Z0)

test = -1;
%parameters from Open Field Experiment - Polyculture
a=0.25;
b=0.24;
n=4;
c=0.12;
d=0.3;
g=0.3;
P=0.05600153435; %for q = 1% where P = -[(1-q)ln(1-q)+qln(q)]
j=0.512;
k=0.0001;
l=0.003;
m=0.001;

T=25; %terminal time
Delta = T/1000; %how often to change
M = 1000;
t=linspace(0,T,M+1);
h=T/M;
h2 = h/2;

Y=zeros(1,M+1);
X=zeros(1,M+1);
Z=zeros(1,M+1);

%initial values - can be altered
Y0=25; %predator
X0=40; %pest
Z0=0; %damage

%costs - can be altered
A=1;  %for predator
B=1;  %for pest
C=1; %for damage

%change the starting point of predator, pest, damage, and costs to
%determine different situations

Y(1)=Y0;
X(1)=X0;
Z(1)=Z0;

lambda1=zeros(1,M+1);
lambda2=zeros(1,M+1);
lambda3=zeros(1,M+1);

u=zeros(1,M+1);

while(test < 0)

    oldu = u;
    oldY = Y;
    oldX = X;
    oldZ = Z;

    oldlambda1 = lambda1;
    oldlambda2 = lambda2;
    oldlambda3 = lambda3;

    for i = 1:M %state equations
```

```
    m11 = Y(i)*(-a+((b*X(i))/(1+Y(i)))*(1-(Y(i)/n))+c);
    m12 = X(i)*(-d-((g*Y(i))/(1+X(i)))+(1-0.1546*P*(1+u(i)))*j+((k*Z(i))/(1+X(i))));
    m13 = l*X(i)-m*Z(i);

    m21 = (Y(i)+h2*m11)*(-a+((b*(X(i)+h2*m12))/(1+(Y(i)+h2*m11)))*(1-((Y(i)+h2*m11)/n))+c);
    m22 = (X(i)+h2*m12)*(-d-((g*(Y(i)+h2*m11))/(1+(X(i)+h2*m12)))+(1-0.1546*P*(1+u(i)))*j+((k*(Z(i)+h2*m13))/(1+(X(i)+h2*m12))));
    m23 = l*(X(i)+h2*m12)-m*(Z(i)+h2*m13);

    m31 = (Y(i)+h2*m21)*(-a+((b*(X(i)+h2*m22))/(1+(Y(i)+h2*m21)))*(1-((Y(i)+h2*m21)/n))+c);
    m32 = (X(i)+h2*m22)*(-d-((g*(Y(i)+h2*m21))/(1+(X(i)+h2*m22)))+(1-0.1546*P*(1+u(i)))*j+((k*(Z(i)+h2*m23))/(1+(X(i)+h2*m22))));
    m33 = l*(X(i)+h2*m22)-m*(Z(i)+h2*m23);

    m41 = (Y(i)+h*m31)*(-a+((b*(X(i)+h*m32))/(1+(Y(i)+h*m31)))*(1-((Y(i)+h*m31)/n))+c);
    m42 = (X(i)+h*m32)*(-d-((g*(Y(i)+h*m31))/(1+(X(i)+h*m32)))+(1-0.1546*P*(1+u(i)))*j+((k*(Z(i)+h*m33))/(1+(X(i)+h*m32))));
    m43 = l*(X(i)+h*m32)-m*(Z(i)+h*m33);

    X(i+1) = X(i) + (h/6)*(m11 + 2*m21 + 2*m31 + m41);
    Y(i+1) = Y(i) + (h/6)*(m12 + 2*m22 + 2*m32 + m42);
    Z(i+1) = Z(i) + (h/6)*(m13 + 2*m23 + 2*m33 + m43);
end

for i = 1:M %costate equations
    J = M+2-i;

    m11 = -lambda1(J)*(-a+(b*X(J)*(1-(Y(J)/n)))/(1+Y(J))+c)-(lambda1(J)*Y(J))*((-b*X(J)*(1-(Y(J)/n)))/((1+Y(J))^2))-((b*X(J))/((1+Y(J))*n))
    +((lambda2(J)*X(J)*g)/(1+Y(J)));
    m12 = -B-((lambda1(J)*Y(J)*b*(1-(Y(J)/n)))/(1+Y(J))-lambda2(J)*(-d-((g*Y(J))/(1+X(J)))+(1-0.1546*(-0.6931471806)*(1+u(J))*j+((k*Z(J))/(1+X(J)))))
    -lambda2(J)*X(J)*(((g*Y(J))/((1+X(J))^2))-((k*Z(J))/((1+X(J))^2)))-lambda3(J)*l;
    m13 = -A-((lambda2(J)*X(J)*k)/(1+X(J)))+lambda3(J)*m;

    m21 = -(lambda1(J)-h2*m11)*(-a+(b*((0.5)*(X(J)+X(J-1)))*(1-(((0.5)*(Y(J)-Y(J-1)))/n)))/(1+((0.5)*(Y(J)-Y(J-1))))+c)-((lambda1(J)-h2*m11)
    *((0.5)*(Y(J)+Y(J-1))))*((-b*((0.5)*(X(J)+X(J-1)))*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/((1+((0.5)*(Y(J)+Y(J-1)))))^2))-((b*((0.5)*(X(J)+X(J-1))))
    /((1+((0.5)*(Y(J)+Y(J-1))))*n))+(((lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1)))*g)/(1+((0.5)*(X(J)+X(J-1)))));
    m22 = -B-(((lambda1(J)-h2*m11)*((0.5)*(Y(J)+Y(J-1)))*b*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/(1+((0.5)*(Y(J)+Y(J-1))))-(lambda2(J)-h2*m12)
    *(-d-((g*((0.5)*(Y(J)+Y(J-1))))/(1+((0.5)*(X(J)+X(J-1))))+(1-0.1546*(-0.6931471806)*(1+u(J))*j+((k*((0.5)*(Z(J)+Z(J-1))))/(1+((0.5)
    *(X(J)+X(J-1))))))))-(lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1)))*(((g*((0.5)*(Y(J)+Y(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2))-((k*((0.5)
    *(Z(J)+Z(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2)))-(lambda3(J)-h2*m13)*l;
    m23 = -A-(((lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1)))*k)/(1+((0.5)*(X(J)+X(J-1)))))+(lambda3(J)-h2*m13)*m;

    m31 = -(lambda1(J)-h2*m21)*(-a+(b*((0.5)*(X(J)+X(J-1)))*(1-(((0.5)*(Y(J)-Y(J-1)))/n)))/(1+((0.5)*(Y(J)-Y(J-1))))+c)-((lambda1(J)-h2*m21)
    *((0.5)*(Y(J)+Y(J-1))))*((-b*((0.5)*(X(J)+X(J-1)))*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/((1+((0.5)*(Y(J)+Y(J-1)))))^2))-((b*((0.5)*(X(J)+X(J-1))))
    /((1+((0.5)*(Y(J)+Y(J-1))))*n))+(((lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1)))*g)/(1+((0.5)*(X(J)+X(J-1)))));
    m32 = -B-(((lambda1(J)-h2*m21)*((0.5)*(Y(J)+Y(J-1)))*b*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/(1+((0.5)*(Y(J)+Y(J-1))))-(lambda2(J)-h2*m22)
    *(-d-((g*((0.5)*(Y(J)+Y(J-1))))/(1+((0.5)*(X(J)+X(J-1))))+(1-0.1546*(-0.6931471806)*(1+u(J))*j+((k*((0.5)*(Z(J)+Z(J-1))))/(1+((0.5)
    *(X(J)+X(J-1))))))))-(lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1)))*(((g*((0.5)*(Y(J)+Y(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2))-((k*((0.5)
    *(Z(J)+Z(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2)))-(lambda3(J)-h2*m23)*l;
    m33 = -A-(((lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1)))*k)/(1+((0.5)*(X(J)+X(J-1)))))+(lambda3(J)-h2*m23)*m;

    m41 = -(lambda1(J)-h2*m31)*(-a+(b*X(J)*(1-(Y(J)/n)))/(1+Y(J))+c)-((lambda1(J)-h2*m31)*Y(J))*((-b*X(J)*(1-(Y(J)/n)))/((1+Y(J))^2))
    -((b*X(J))/((1+Y(J))*n))+(((lambda2(J)-h2*m32)*X(J)*g)/(1+X(J)));
    m42 = -B-(((lambda1(J)-h2*m31)*Y(J)*b*(1-(Y(J)/n)))/(1+Y(J))-(lambda2(J)-h2*m32)*(-d-(g*Y(J))/(1+X(J))+(1-0.1546*(-0.6931471806)
    *(1+u(J))*j+((k*Z(J))/(1+X(J)))))-(lambda2(J)-h2*m32)*X(J)*(((g*Y(J))/((1+X(J))^2))-((k*Z(J))/((1+X(J))^2)))-(lambda3(J)-h2*m33)*l;
    m43 = -A-(((lambda2(J)-h2*m32)*X(J)*k)/(1+X(J)))+(lambda3(J)-h2*m33)*m;

    lambda1(J-1) = lambda1(J) - (h/6)*(m11 + 2*m21 + 2*m31 + m41);
    lambda2(J-1) = lambda2(J) - (h/6)*(m12 + 2*m22 + 2*m32 + m42);
    lambda3(J-1) = lambda3(J) - (h/6)*(m13 + 2*m23 + 2*m33 + m43);
end

temp = 49;
u1 = min(49, max(0, abs(temp)));
u = 0.5*(u1 + oldu);

temp1 = Delta*sum(abs(u)) - sum(abs(oldu - u));
temp2 = Delta*sum(abs(Y)) - sum(abs(oldY - Y));
temp3 = Delta*sum(abs(X)) - sum(abs(oldX - X));
temp4 = Delta*sum(abs(Z)) - sum(abs(oldZ - Z));
temp5 = Delta*sum(abs(lambda1)) - sum(abs(oldlambda1 - lambda1));
temp6 = Delta*sum(abs(lambda2)) - sum(abs(oldlambda2 - lambda2));
temp7 = Delta*sum(abs(lambda3)) - sum(abs(oldlambda3 - lambda3));
test  = min(temp1, min(temp2, min(temp3, min(temp4, min(temp5, min(temp6, temp7))))));
end

y(1,:) = t;
y(2,:) = Y;
y(3,:) = X;
y(4,:) = Z;
y(5,:) = lambda1;
y(6,:) = lambda2;
y(7,:) = lambda3;
y(8,:) = 1+u;

    subplot(1,3,1);plot(y(1,:),y(4,:),'r')
    subplot(1,3,1);xlabel('Time')
    subplot(1,3,1);ylabel('damage (Z)')
    subplot(1,3,1);axis([0 T -0.1 1.1])

    subplot(1,3,2);plot(y(1,:),y(3,:),'g')
    subplot(1,3,2);xlabel('Time')
    subplot(1,3,2);ylabel('pest (X)')
    subplot(1,3,2);axis([0 T -11 41])
```

```
    subplot(1,3,3);plot( y(1,:),y(2,:),'b')
    subplot(1,3,3);xlabel('Time')
    subplot(1,3,3);ylabel('predators (Y)')
    subplot(1,3,3);axis([0 T -11 41])
```

B.3 20% Plant Diversity

```
%control: 1 + U(t) plus addition of loss of alfalfa productivity
%using exponential Q_0 from mono and poly combined
%q = 20%,  U = 1.5, 0.5

%function y = OCT_Summer_2014(A,B,C,Y0,X0,Z0)

test = -1;
%parameters from Open Field Experiment - Polyculture
a=0.25;
b=0.24;
n=4;
c=0.12;
d=0.3;
g=0.3;

j=0.512;
k=0.0001;
l=0.003;
m=0.001;

T=25; %terminal time
Delta = T/1000; %how often to change
M = 1000;
t=linspace(0,T,M+1);
h=T/M;
h2 = h/2;

Y=zeros(1,M+1);
X=zeros(1,M+1);
Z=zeros(1,M+1);

P=0.67478408; %for q = 20% where P = -[(1-q)ln(1-q)+qln(q)]

%initial values - can be altered
Y0=2.5069; %predator
X0=1.4412; %pest
Z0=0.0724; %damage

%costs - can be altered
A=1;  %for predator
B=1;  %for pest
C=1; %for damage

%change the starting point of predator, pest, damage, and costs to
%determine different situations

Y(1)=Y0;
X(1)=X0;
Z(1)=Z0;

lambda1=zeros(1,M+1);
lambda2=zeros(1,M+1);
lambda3=zeros(1,M+1);

u=zeros(1,M+1);

while(test < 0)

    oldu = u;
    oldY = Y;
    oldX = X;
    oldZ = Z;

    oldlambda1 = lambda1;
    oldlambda2 = lambda2;
    oldlambda3 = lambda3;

    for i = 1:M %state equations
        m11 = Y(i)*(-a+((b*X(i))/(1+Y(i)))*(1-(Y(i)/n))+c);
        m12 = X(i)*(-d-((g*Y(i))/(1+X(i)))+(1-0.1546*P*(1+u(i)))*j+((k*Z(i))/(1+X(i))));
        m13 = l*X(i)-m*Z(i);

        m21 = (Y(i)+h2*m11)*(-a+((b*(X(i)+h2*m12))/(1+(Y(i)+h2*m11)))*(1-((Y(i)+h2*m11)/n))+c);
        m22 = (X(i)+h2*m12)*(-d-((g*(Y(i)+h2*m11))/(1+(X(i)+h2*m12)))+(1-0.1546*P*(1+u(i)))+((k*(Z(i)+h2*m13))/(1+(X(i)+h2*m12))));
        m23 = l*(X(i)+h2*m12)-m*(Z(i)+h2*m13);

        m31 = (Y(i)+h2*m21)*(-a+((b*(X(i)+h2*m22))/(1+(Y(i)+h2*m21)))*(1-((Y(i)+h2*m21)/n))+c);
        m32 = (X(i)+h2*m22)*(-d-((g*(Y(i)+h2*m21))/(1+(X(i)+h2*m22)))+(1-0.1546*P*(1+u(i)))*j+((k*(Z(i)+h2*m23))/(1+(X(i)+h2*m22))));
```

```matlab
        m33 = l*(X(i)+h2*m22)-m*(Z(i)+h2*m23);

        m41 = (Y(i)+h*m31)*(-a+((b*(X(i)+h*m32))/(1+(Y(i)+h*m31)))*(1-((Y(i)+h*m31)/n))+c);
        m42 = (X(i)+h*m32)*(-d-((g*(Y(i)+h*m31))/(1+(X(i)+h*m32)))+(1-0.1546*P*(1+u(i)))*j+((k*(Z(i)+h*m33))/(1+(X(i)+h*m32))));
        m43 = l*(X(i)+h*m32)-m*(Z(i)+h*m33);

        X(i+1) = X(i) + (h/6)*(m11 + 2*m21 + 2*m31 + m41);
        Y(i+1) = Y(i) + (h/6)*(m12 + 2*m22 + 2*m32 + m42);
        Z(i+1) = Z(i) + (h/6)*(m13 + 2*m23 + 2*m33 + m43);
    end

    for i = 1:M %costate equations
        J = M+2-i;

        m11 = -lambda1(J)*(-a+(b*X(J)*(1-(Y(J)/n)))/(1+Y(J))+c)-(lambda1(J)*Y(J))*((-b*X(J)*(1-(Y(J)/n)))/((1+Y(J))^2))-((b*X(J))/((1+Y(J))*n))
        +((lambda2(J)*X(J)*g)/(1+X(J)));
        m12 = -B-((lambda1(J)*Y(J)*b*(1-(Y(J)/n)))/(1+Y(J)))-lambda2(J)*(-d-((g*Y(J))/(1+X(J)))+(1-0.1546*(-0.6931471806)*(1+u(J))*j+((k*Z(J))/(1+X(J)))))
        -lambda2(J)*X(J)*(((g*Y(J))/((1+X(J))^2))-((k*Z(J))/((1+X(J))^2)))-lambda3(J)*l;
        m13 = -A-((lambda2(J)*X(J)*k)/(1+X(J)))+lambda3(J)*m;

        m21 = -(lambda1(J)-h2*m11)*(-a+(b*((0.5)*(X(J)+X(J-1))*(1-(((0.5)*(Y(J)-Y(J-1)))/n)))/(1+((0.5)*(Y(J)-Y(J-1))))+c)-((lambda1(J)-h2*m11)
        *((0.5)*(Y(J)+Y(J-1)))*((-b*((0.5)*(X(J)+X(J-1))*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/((1+((0.5)*(Y(J)+Y(J-1))))^2))-((b*((0.5)*(X(J)+X(J-1))))
        /((1+((0.5)*(Y(J)+Y(J-1))))*n))+(((lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1))*g)/(1+((0.5)*(X(J)+X(J-1)))));
        m22 = -B-(((lambda1(J)-h2*m11)*((0.5)*(Y(J)+Y(J-1)))*b*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/(1+((0.5)*(Y(J)+Y(J-1)))))-(lambda2(J)-h2*m12)
        *(-d-((g*((0.5)*(Y(J)+Y(J-1))))/(1+((0.5)*(X(J)+X(J-1)))))+(1-0.1546*(-0.6931471806)*(1+u(J))*j+((k*((0.5)*(Z(J)+Z(J-1)))/(1+((0.5)
        *(X(J)+X(J-1))))))-(lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1)))*(((g*((0.5)*(Y(J)+Y(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2))-((k*((0.5)
        *(Z(J)+Z(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2)))-(lambda3(J)-h2*m13)*l;
        m23 = -A-(((lambda2(J)-h2*m12)*((0.5)*(X(J)+X(J-1)))*k)/(1+((0.5)*(X(J)+X(J-1)))))+(lambda3(J)-h2*m13)*m;

        m31 = -(lambda1(J)-h2*m21)*(-a+(b*((0.5)*(X(J)+X(J-1))*(1-(((0.5)*(Y(J)-Y(J-1)))/n)))/(1+((0.5)*(Y(J)-Y(J-1))))+c)-((lambda1(J)-h2*m21)
        *((0.5)*(Y(J)+Y(J-1)))*((-b*((0.5)*(X(J)+X(J-1))*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/((1+((0.5)*(Y(J)+Y(J-1))))^2))-((b*((0.5)*(X(J)+X(J-1))))
        /((1+((0.5)*(Y(J)+Y(J-1))))*n))+(((lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1))*g)/(1+((0.5)*(X(J)+X(J-1)))));
        m32 = -B-(((lambda1(J)-h2*m21)*((0.5)*(Y(J)+Y(J-1)))*b*(1-(((0.5)*(Y(J)+Y(J-1)))/n)))/(1+((0.5)*(Y(J)+Y(J-1)))))-(lambda2(J)-h2*m22)
        *(-d-((g*((0.5)*(Y(J)+Y(J-1))))/(1+((0.5)*(X(J)+X(J-1)))))+(1-0.1546*(-0.6931471806)*(1+u(J))*j+((k*((0.5)*(Z(J)+Z(J-1)))/(1+((0.5)
        *(X(J)+X(J-1))))))-(lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1)))*(((g*((0.5)*(Y(J)+Y(J-1))))/((1+((0.5)*(X(J)+X(J-1))))^2))-((k*((0.5)*(Z(J)+Z(J-1))))
        /((1+((0.5)*(X(J)+X(J-1))))^2)))-(lambda3(J)-h2*m23)*l;
        m33 = -A-(((lambda2(J)-h2*m22)*((0.5)*(X(J)+X(J-1)))*k)/(1+((0.5)*(X(J)+X(J-1)))))+(lambda3(J)-h2*m23)*m;

        m41 = -(lambda1(J)-h2*m31)*(-a+(b*X(J)*(1-(Y(J)/n)))/(1+Y(J))+c)-((lambda1(J)-h2*m31)*Y(J))*((-b*X(J)*(1-(Y(J)/n)))/((1+Y(J))^2))-((b*X(J))
        /((1+Y(J))*n))+(((lambda2(J)-h2*m32)*X(J)*g)/(1+X(J)));
        m42 = -B-(((lambda1(J)-h2*m31)*Y(J)*b*(1-(Y(J)/n)))/(1+Y(J)))-(lambda2(J)-h2*m32)*(-d-((g*Y(J))/(1+X(J)))+(1-0.1546*(-0.6931471806)*(1+u(J))*j
        +((k*Z(J))/(1+X(J)))))-(lambda2(J)-h2*m32)*X(J)*(((g*Y(J))/((1+X(J))^2))-((k*Z(J))/((1+X(J))^2)))-(lambda3(J)-h2*m33)*l;
        m43 = -A-(((lambda2(J)-h2*m32)*X(J)*k)/(1+X(J)))+(lambda3(J)-h2*m33)*m;

        lambda1(J-1) = lambda1(J) - (h/6)*(m11 + 2*m21 + 2*m31 + m41);
        lambda2(J-1) = lambda2(J) - (h/6)*(m12 + 2*m22 + 2*m32 + m42);
        lambda3(J-1) = lambda3(J) - (h/6)*(m13 + 2*m23 + 2*m33 + m43);
    end

    temp = 1.5; %optimality condition
    u1 = min(49, max(0, abs(temp)));
    u = 0.5*(u1 + oldu);

    temp1 = Delta*sum(abs(u)) - sum(abs(oldu - u));
    temp2 = Delta*sum(abs(Y)) - sum(abs(oldY - Y));
    temp3 = Delta*sum(abs(X)) - sum(abs(oldX - X));
    temp4 = Delta*sum(abs(Z)) - sum(abs(oldZ - Z));
    temp5 = Delta*sum(abs(lambda1)) - sum(abs(oldlambda1 - lambda1));
    temp6 = Delta*sum(abs(lambda2)) - sum(abs(oldlambda2 - lambda2));
    temp7 = Delta*sum(abs(lambda3)) - sum(abs(oldlambda3 - lambda3));
    test  = min(temp1, min(temp2, min(temp3, min(temp4, min(temp5, min(temp6, temp7))))));
end

y(1,:) = t;
y(2,:) = Y;
y(3,:) = X;
y(4,:) = Z;
y(5,:) = lambda1;
y(6,:) = lambda2;
y(7,:) = lambda3;
y(8,:) = 1+u;

        subplot(1,3,1);plot(y(1,:),y(4,:),'r')
        subplot(1,3,1);xlabel('Time')
        subplot(1,3,1);ylabel('damage (Z)')
        subplot(1,3,1);axis([0 T -0.1 4.1])

        subplot(1,3,2);plot(y(1,:),y(3,:),'g')
        subplot(1,3,2);xlabel('Time')
        subplot(1,3,2);ylabel('pest (X)')
        subplot(1,3,2);axis([0 T -11 101])

        subplot(1,3,3);plot( y(1,:),y(2,:),'b')
        subplot(1,3,3);xlabel('Time')
        subplot(1,3,3);ylabel('predators (Y)')
        subplot(1,3,3);axis([0 T -11 101])
```

Appendix C

Excel Calculations - Cost-Benefit Analysis

```
To calculate, use: ((price per gallon of organic milk)*(1-(% plant diversity)*1.045)-(price per gallon of regular milk))/(price per gallon of regular milk)
Note: The 1.045 signifies an increase in plant diversity of 4.5% for each cycle. This was used in the extended optimal control as well.


20% to 40.44740306
using 4.5%
20% Plant Diversity

Cycle 1      0.176198729      17.61987291
Cycle 2      0.162966493      16.29664934
Cycle 3      0.149138807      14.9138807
Cycle 4      0.134688875      13.46888749
Cycle 5      0.119588696      11.95886957
Cycle 6      0.103809009      10.38090085
Cycle 7      0.087319235      8.731923539
Cycle 8      0.070087422      7.008742247
Cycle 9      0.052080178      5.208017797
Cycle 10 0.033262607      3.326260747
Cycle 11 0.013598246      1.35982463
Cycle 12 -0.006951011      -0.695101113
Cycle 13 -0.028424985 -2.842498514
Cycle 14 -0.050865288 -5.086528798
Cycle 15 -0.074315404 -7.431540445
Cycle 16 -0.098820776 -9.882077616
Average 5.271005208
        0.843360833      84.33608333 0.052710052
5.271005208


10% to 20.22370153
using 4.5%
10% Plant Diversity
Cycle 1      0.32322357      32.32235702
Cycle 2      0.316607452      31.66074523
Cycle 3      0.309693609      30.96936092
Cycle 4      0.302468643      30.24686431
Cycle 5      0.294918554      29.49185535
Cycle 6      0.28702871      28.70287099
Cycle 7      0.278783823      27.87838234
Cycle 8      0.270167917      27.01679169
Cycle 9      0.261164295      26.11642946
Cycle 10 0.251755509      25.17555094
Cycle 11 0.241923329      24.19233288
Cycle 12 0.2316487      23.16487001
Cycle 13 0.220911713      22.09117131
Cycle 14 0.209691562      20.96915617
Cycle 15 0.197966503      19.79665034
Cycle 16 0.185713818      18.57138176
Average 26.14792317
        4.183667707      418.3667707      0.261479232
26.14792317
```