2017

# Euler's Rediscovery of e With Instructor Notes

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# Euler's Rediscovery of $e$
# With Instructor Notes

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November 21, 2016

## 1 Introduction

The famous constant $e$ appears occasionally in the history of mathematics. In 1683 Jacob Bernoulli essentially found $e$ while studying compound interest and evaluating the sequence $(1+1/j)^j$ as $j \to \infty$. By 1697 Johann Bernoulli was working with the calculus of exponential functions [B1]. However, a full understanding was missing. Mathematicians couldn't agree on how to define logarithms of negative numbers, and the connection between logarithms and exponential functions was still not well understood. Leonard Euler would later clear up the confusion on logarithms of negative numbers, and clarify the idea of a logarithmic *function* [E1]. In 1748, Euler published one of his most influential works, *Introductio in Analysin Infinitorum* [E]. This was translated into English by John Blanton and we shall quote his translation with a few minor changes. In Chapter VI, Euler discusses logarithms for various bases and their properties. Logarithms were well known in Euler's day, and tables of common logarithms (base 10) had been compiled, as no scientific calculators were available in 1748. In Chapter VII of his *Introductio*, Euler wants to examine exponential and logarithmic functions, especially as infinite series. We are particularly interested in how $e$ appears naturally in his development of these functions.

## 2 Euler's Definition of $e$

Part of Euler's challenge for working with logarithmic functions is to find a logarithmic base $a$ for which infinite series expansions are convenient. It is here that that Euler derives $e$, both as the limiting value of $(1+1/j)^j$ and as the infinite series $1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots$. As was common in his day, Euler worked with *infinitely small and large numbers,* a practice that has largely been abandoned with the modern definition of limit. Nevertheless, Euler used his infinitely small and large numbers with great skill, as we shall see.

Section 114. Since $a^0 = 1$, when the exponent on $a$ increases, the power itself increases, provided $a$ is greater than 1. It follows that if the exponent is infinitely small and positive, then the power also exceeds 1 by an infinitely small number. Let $\omega$ be an infinitely small number, ..., $a^\omega = 1+\psi$ where $\psi$ is also an infinitely small number. ... we let $\psi = k\omega$. Then we have $a^\omega = 1 + k\omega$, and with $a$ as the base for logarithms, we have $\omega = \log(1 + k\omega)$.

EXAMPLE

In order that it may be clearer how the number $k$ depends on $a$, let $a = 10$. From the table of common logarithms[1], we look for the logarithm of a number which exceeds 1 by the smallest possible amount, for instance, $1 + \frac{1}{1000000}$, so that $k\omega = \frac{1}{1000000}$.
Then $\log\left(1 + \frac{1}{1000000}\right) = \log \frac{1000001}{1000000} = 0.00000043429 = \omega$. Since $k\omega = 0.00000100000$, it follows that $\frac{1}{k} = \frac{43429}{100000}$ and $k = \frac{100000}{43429} = 2.30258$. We see that $k$ is a finite number which depends on the value of the base $a$. If a different base had been chosen, then the logarithm of the same number $1 + k\omega$ will differ from the logarithm already given. It follows that a different value of $k$ will result.

**Exercise 1** *To gain some visual insight into what Euler is doing, plot $y = a^\omega$ and $y = 1 + k\omega$ for $a = 10$ and $k = \frac{100000}{43429} \approx 2.30258$. Euler claims these quantities $a^\omega$ and $1 + k\omega$ should be identical for "infinitely small" $\omega$. Would changing the $k$ value to something else, say $-3$, change anything about this claim?*

**Exercise 2** *Use Euler's ideas and a scientific calculator to estimate $k$ for $a = 2$. Get a visual check by plotting $y = 2^\omega$ and $y = 1 + k\omega$ together.*

Euler is interested in finding an $a$ value for which exponential and logarithmic expansions are nice and easy to work with. He derives a first series expansion in his Section 115.

Section 115. Since $a^\omega = 1 + \psi$, we have $a^{j\omega} = (1 + \psi)^j$, whatever value we assign to $j$. It follows that

$$a^{j\omega} = 1 + \frac{j}{1}k\omega + \frac{j(j-1)}{1 \cdot 2}k^2\omega^2 + \frac{j(j-1)(j-2)}{1 \cdot 2 \cdot 3}k^3\omega^3 + \cdots \quad (1)$$

If now we let $j = \dfrac{z}{\omega}$, where $z$ denotes any finite number, since $\omega$ is infinitely small, then $j$ is infinitely large ...

$$a^z = (1 + kz/j)^j = 1 + \frac{1}{1}kz + \frac{1(j-1)}{1 \cdot 2 \cdot j}k^2z^2 + \frac{1(j-1)(j-2)}{1 \cdot 2j \cdot 3j}k^3z^3 + \frac{1(j-1)(j-2)(j-3)}{1 \cdot 2j \cdot 3j \cdot 4j}k^4z^4 + \cdots. \quad (2)$$

We would like to capture the spirit of Euler's ideas but put his work on modern foundations by avoiding infinitely small numbers.

[1] As recently as the 1970's, most students used tables rather than calculators to find logarithms. Such a table would have an entry that exceeds 1 by the "smallest possible amount" for that table.

**Exercise 3** *Assume $a, \omega$ are small, positive finite numbers defined by $a^\omega = 1 + \psi$ and $k = \psi/\omega$.*

(a) *What theorem is Euler using to obtain (1)? For what $\psi$ values is this series known to converge?*

(b) *Verify the algebraic details needed to obtain (1).*

**Exercise 4** *Assume $a, \omega$ are small, positive finite numbers defined by $a^\omega = 1 + \psi$ and $k = \psi/\omega$ and $j = z/\omega$.*

(a) *What is the general $n$th term in the series (2)?*

(b) *Verify the algebraic details needed to obtain (2) from $j = z/\omega$ and (1).*

Euler next uses his infinitely large numbers to produce an infinite series expression for his ideal logarithm base $a$. In a much later section Euler will be want to let $z$ vary to discuss the function $e^z$. For now, he is going to set $z = 1$ while finding his special value for $a$.

Section 116. Since $j$ is infinitely large, $\frac{j-1}{j} = 1$, ... , $\frac{j-2}{j} = 1$, $\frac{j-3}{j} = 1$, and so forth. It follows that $\frac{j-1}{2j} = \frac{1}{2}$, ... , $\frac{j-2}{3j} = \frac{1}{3}$, $\frac{j-3}{4j} = \frac{1}{4}$, and so forth. When we we substitute these values, we obtain $1 + \frac{kz}{1} + \frac{k^2z^2}{1\cdot 2} + \frac{k^3z^3}{1\cdot 2\cdot 3} + \frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$. This equation expresses a relationship between the numbers $a$ and $k$, since when we let $z = 1$, we have

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \cdots. \tag{3}$$

Section 122. Since we are free to choose the base $a$ for the system of logarithms, we now choose $a$ in such a way that $k = 1$. Then the series found above in Section 116,

$$1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots \tag{4}$$

is equal to $a$. If the terms are represented as decimal fractions and summed, we obtain the value $a = 2.71828182845904523536028...$. When this base is chosen, the logarithms are called natural or hyperbolic. The latter name is used since the quadrature of a hyperbola[2] can be expressed through these logarithms. For the sake of brevity for this number $2.718281828459$ ... we will use the symbol $e$, which will denote the base for the natural or hyperbolic logarithms.

Why do you think Euler chose $a$ "in such a way that $k = 1$" in his series (3)?

We can obtain an expression for Euler's special $a$ value as the limit of a *sequence*, and then use modern methods with Euler's ideas to prove this sequence converges. To justify Euler's work from a modern point of view, let's look at the key equation (2)

$$(1 + kz/j)^j = 1 + \frac{1}{1}kz + \frac{1\,(j-1)}{1\cdot 2\cdot j}k^2z^2 + \frac{1\,(j-1)\,(j-2)}{1\cdot 2j\cdot 3j}k^3z^3 + \cdots$$

and set $k = 1, z = 1$ as Euler does, but suppose $j$ is a *natural number.*

[2] The area of the region between the $x$-axis and hyperbola $y = 1/x$

**Exercise 5** *Apply the finite Binomial Theorem for natural number $j$ to expand $(1+1/j)^j$ as a finite series and show that*

$$(1+1/j)^j = 1 + \frac{1}{1} + \frac{1\,(j-1)}{1\cdot 2\cdot j} + \frac{1\,(j-1)\,(j-2)}{1\cdot 2j\cdot 3j} + \cdots + \frac{1\,(j-1)\,(j-2)\cdots(j-(j-1))}{1\cdot 2j\cdot 3j\cdot\cdots\cdot(jj)}$$

Now we form a sequence $(b_j)_{j=1}^{\infty}$ with $b_j = (1+1/j)^j$. If we can justify taking the limit of this sequence, we should obtain Euler's number $e$.

**Exercise 6** *Show that*

$$b_j = 1 + \frac{1}{1} + \frac{(1-1/j)}{1\cdot 2} + \frac{(1-1/j)\,(1-2/j)}{1\cdot 2\cdot 3} + \cdots + \frac{(1-1/j)\,(1-2/j)\cdots(1-(j-1)/j)}{1\cdot 2\cdot 3\cdot\cdots\cdot j}$$

**Exercise 7** *Write out $b_{j+1}$ and compare it term-by-term with $b_j$. What can you conclude about the sequence $(b_j)_{j=1}^{\infty}$?*

**Exercise 8** *Compare $b_j$ to a geometric series to show $(b_j)$ is bounded.*

**Exercise 9** *Apply the Monotone Convergence Theorem to give a modern proof that $\lim_{j\to\infty}(1+1/j)^j$ exists.*

We now have a modern justification of the constant $e$ as $e = \lim_{j\to\infty}(1+1/j)^j$. You may recall from Introductory Calculus courses that Euler is correct with the series expansion (4) for $e$. A modern justification of this series expansion for $e$ is beyond the scope of this project.

**Exercise 10** *To gain some appreciation for Euler's work to obtain the series (4), use technology to find $b_{10} = (1+1/10)^{10}$. How close is this to Euler's Section 122 decimal approximation to $e$? How does $b_{10}$ compare to Euler's partial sum $\sum_{k=0}^{10}\frac{1}{k!}$ from his Section 122? Are you surprised by the difference?*

**Exercise 11** *Euler wanted to use his work to express the function $e^z$ as an infinite series. Use Euler's (2) and his Section 116 infinitesimal methods to find an infinite series expression for $e^z$. How does this series compare with the Taylor series for $e^z$ you learned about in Introductory Calculus?*

Recall that Euler was trying to find a logarithm base $a$ for which $\omega = \log_a(1+k\omega)$ with "infinitely small" $\omega$. We can try to interpret this claim in terms of the limit

$$\lim_{\omega\to 0}\frac{\omega}{\log_a(1+k\omega)}. \tag{5}$$

**Exercise 12** *Use Introductory Calculus techniques to find the limit in (5). In particular, what do you obtain when $k=1$ and $a=e$? Then use this limit (5) to reflect on the decimal values for $a$ you found in Exercises 1 and 2.*

# 3 Instructor Notes

The heart of this project for an Analysis course is giving a modern justification of $e = \lim_{j\to\infty}(1+1/j)^j$ using Euler's ideas along with some modern theory. The approach using Monotone Convergence Theorem, as outlined in Exercises 5-9, is a common approach in current Analysis textbooks. Reading about it via Euler gives context to the exercise and some appreciation of his dexterity with infinitesimals and series, as well as the close connection with $e$ as a logarithm base to motivate the definition. This series development of $e^z$ is a interesting alternative to the Taylor series approach students have seen in Introductory Calculus courses.

In Exercises 1 and 2, it is interesting to note that if students try to approximate $k$ better by using smaller values of $k\omega$, they may run into technology problems. For example, a TI-84 evaluates $\dfrac{10^{-10}}{\log_{10}(1+10^{-10})}$ to be 2.302585093, but Mathematica 10 does not fare so well, producing 2.30258490259. This is likely the case because the TI calculator uses base 10 floating point arithmetic, while Mathematica uses base 2. Using $k\omega = 10^{-6}$ accomplishes the main goal while avoiding technology problems. These $k$ values come back to students in the last exercise of the project.

**Assumptions.** Analysis students have studied sequences and are familiar with the Monotone Convergence Theorem.

# References


[E] Euler, L. 1748, *Introductio in Analysin Infinitorum,* St. Petersburg; tr. J Blanton.

[B1] Bernoulli, Johann I, 1697, *Principia calculi exponentialium seu percurrentium*, Acta Eruditorium, March, 125-133.

[E1] Euler, L. 1749, *De la controverse entre Mrs. Leibniz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires*, Mem. Acad. Sci. Berlin 5, 1751, 139-179.